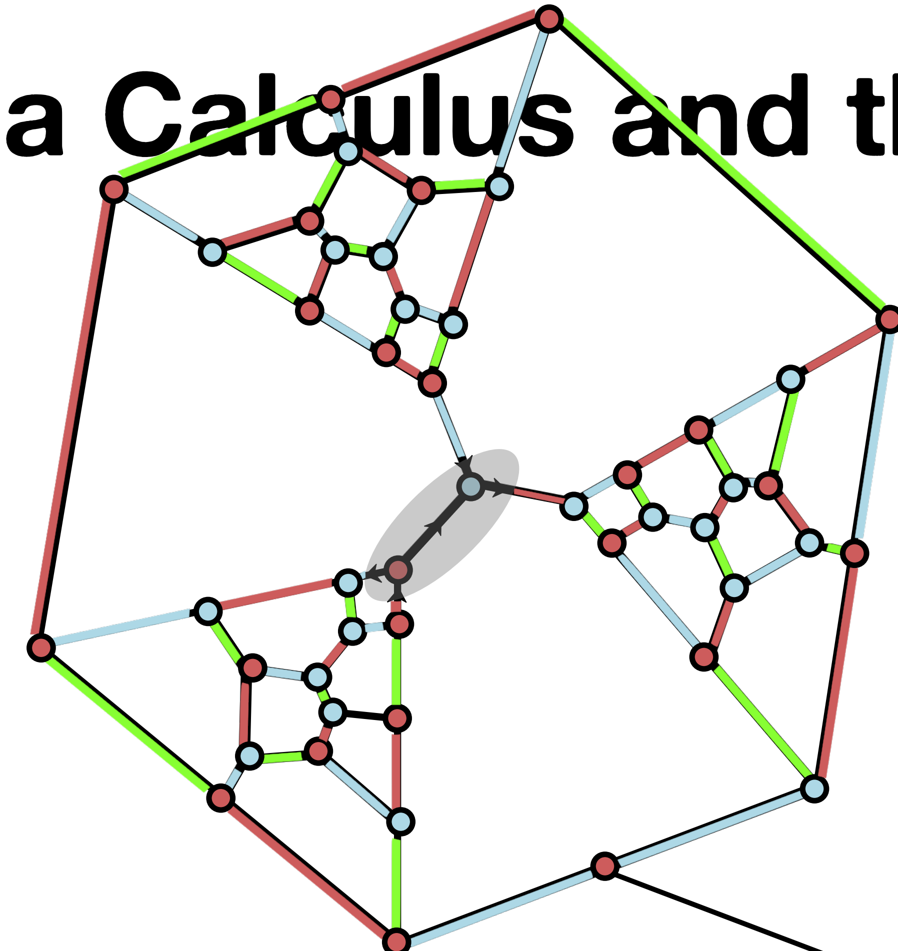
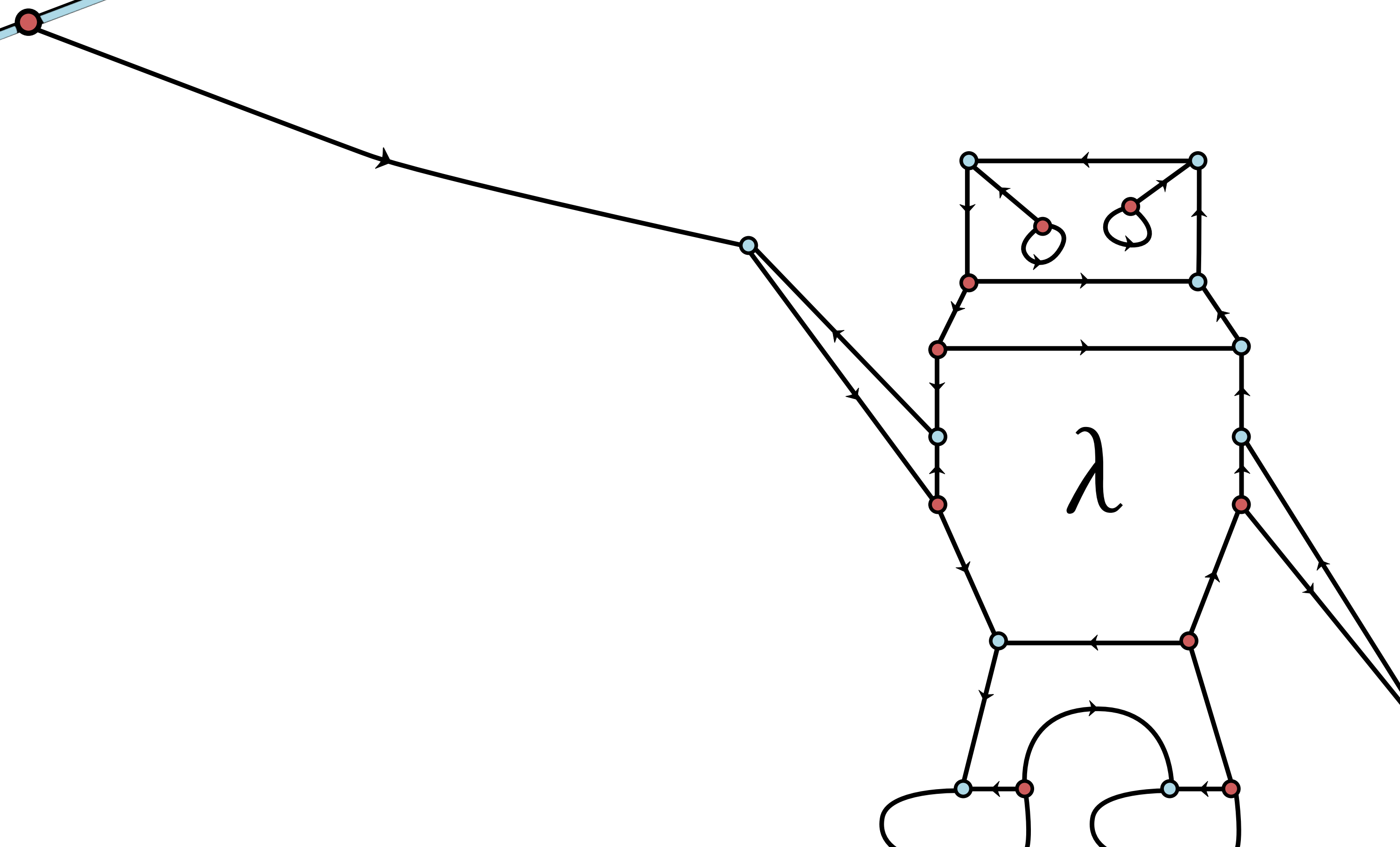
Lambda Calculus and the Four Colour Theorem

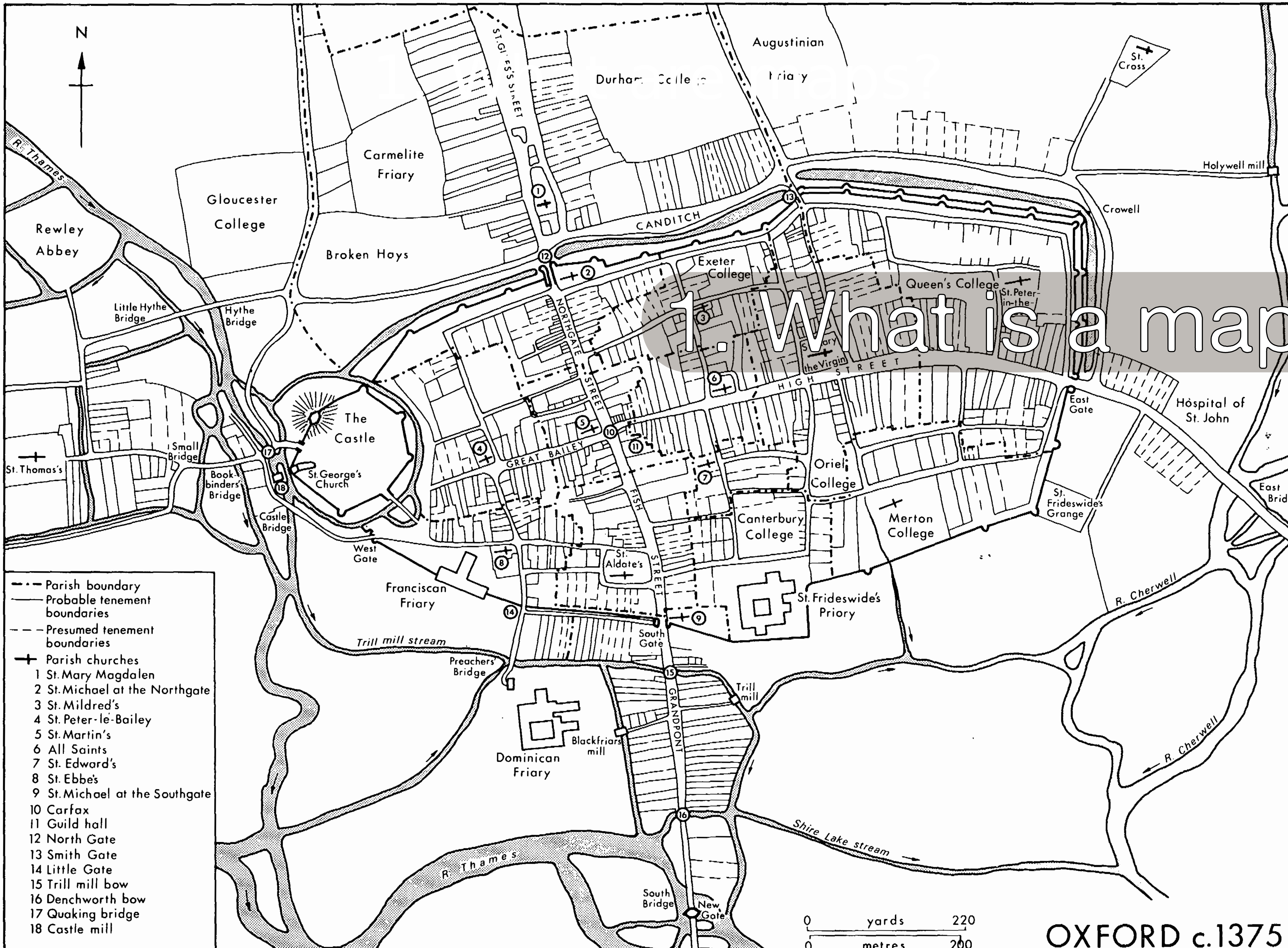


Noam Zeilberger
School of Computer Science
University of Birmingham



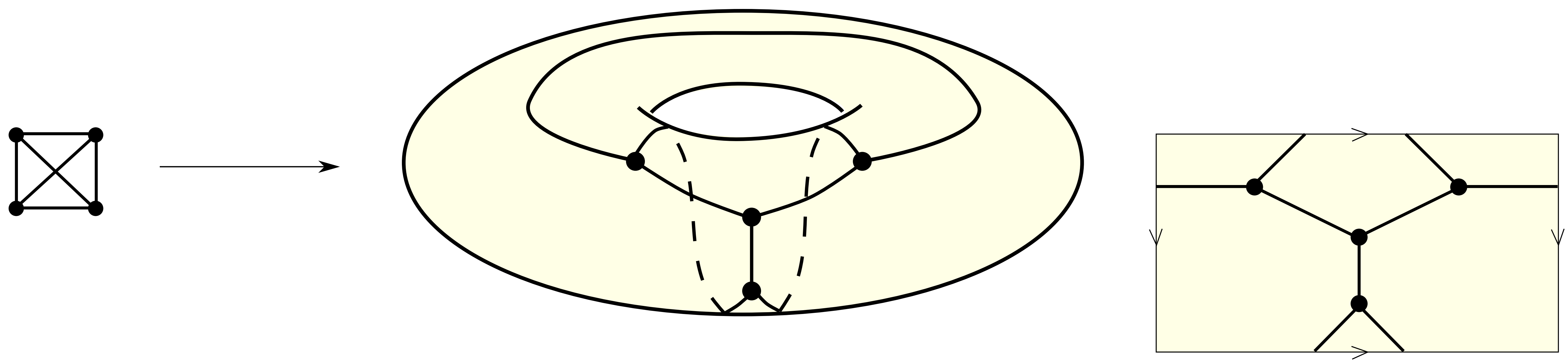
OASIS Seminar
22 June 2018
University of Oxford

1. What is a map?



Topological definition

map = 2-cell embedding of a graph into a surface*



considered up to deformation of the underlying surface.

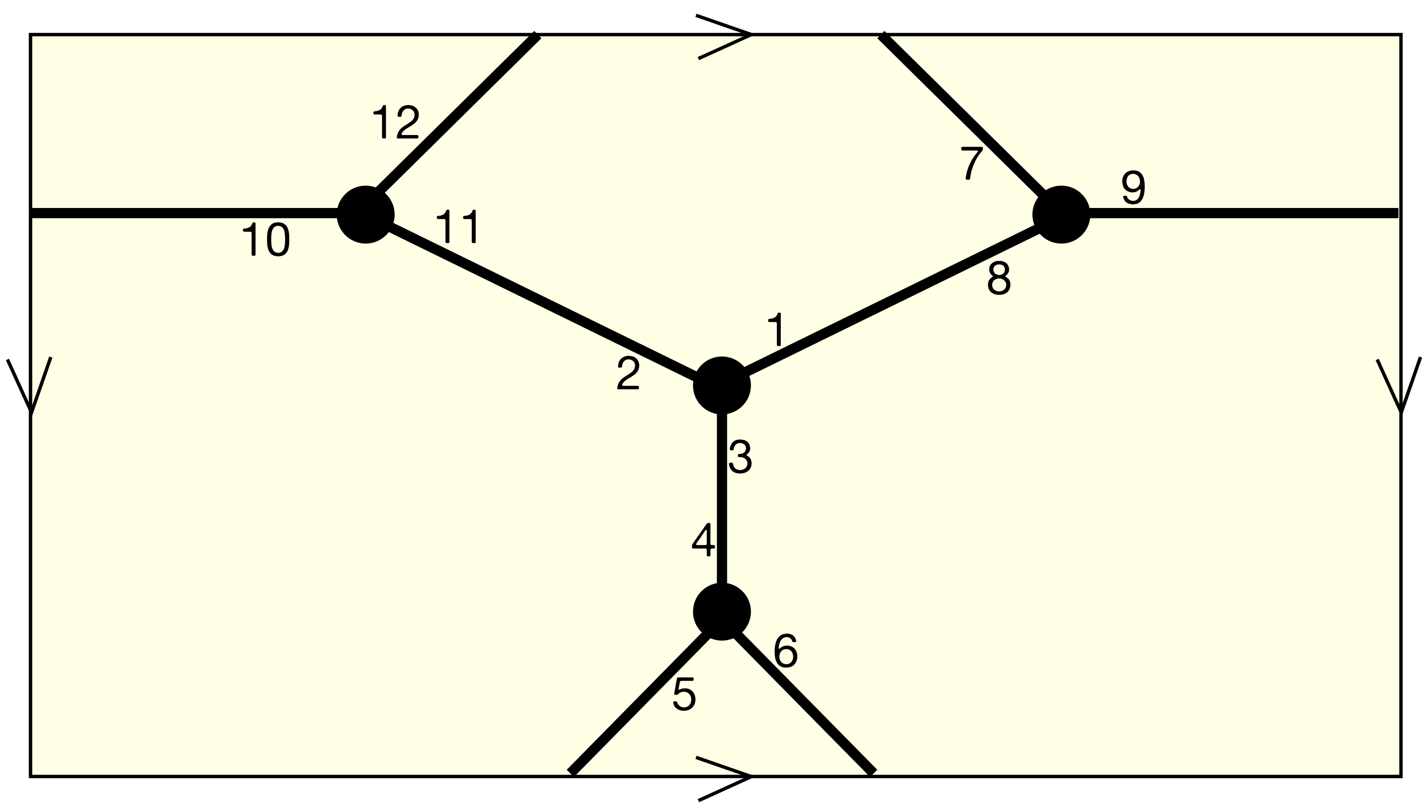
*All surfaces are assumed to be connected and oriented throughout this talk

Algebraic definition

map = transitive permutation representation of the group

$$G = \langle v, e, f \mid e^2 = vef = 1 \rangle$$

considered up to G-equivariant isomorphism.



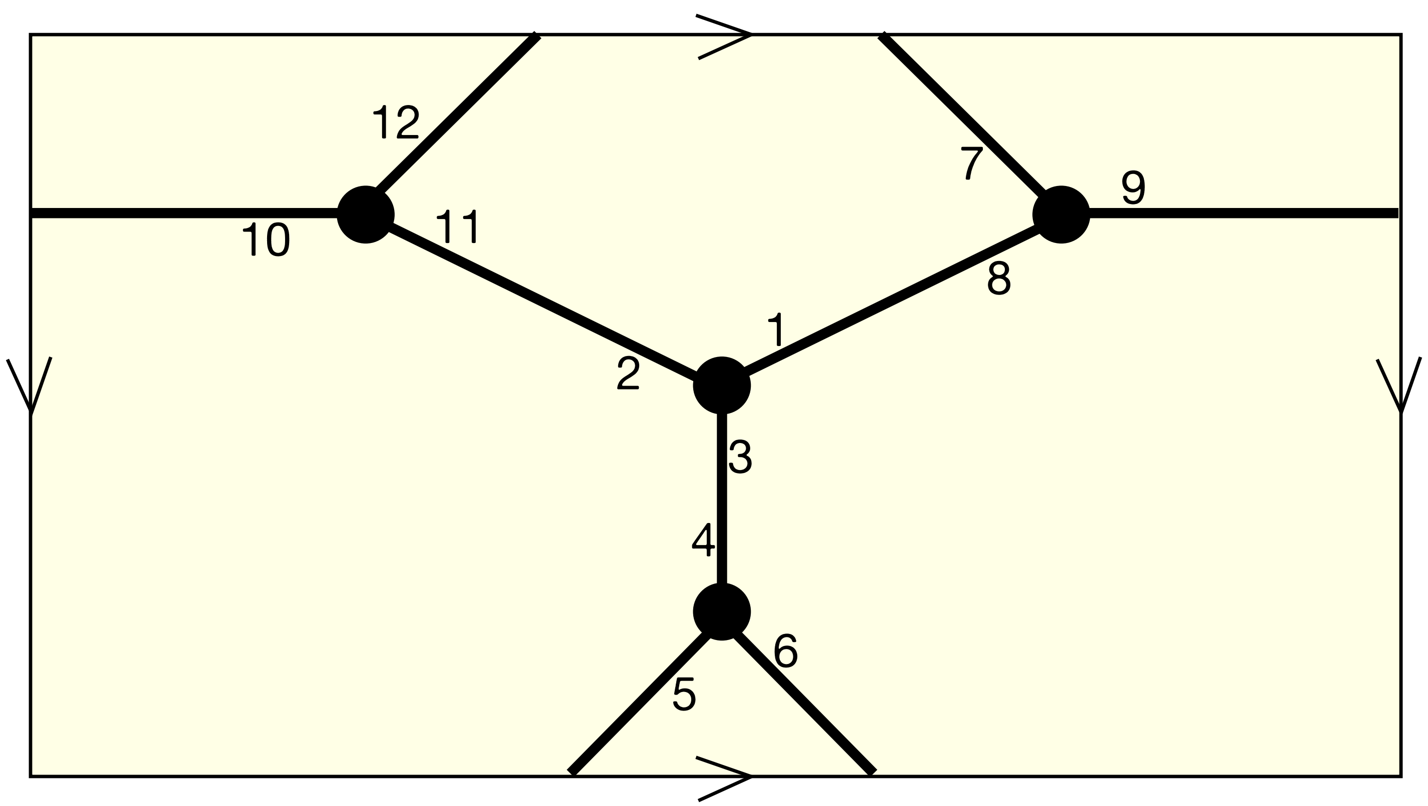
$$v = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)$$
$$e = (1\ 8)(2\ 11)(3\ 4)(5\ 12)(6\ 7)(9\ 10)$$
$$f = (1\ 7\ 5\ 11)(2\ 10\ 8\ 3\ 6\ 9\ 12\ 4)$$

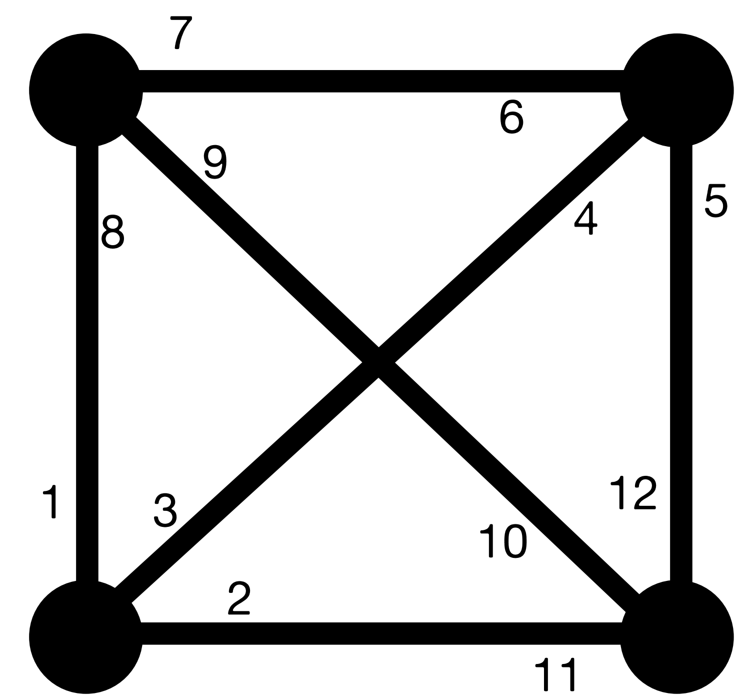
Note: can compute genus from Euler characteristic

$$c(v) - c(e) + c(f) = 2 - 2g$$

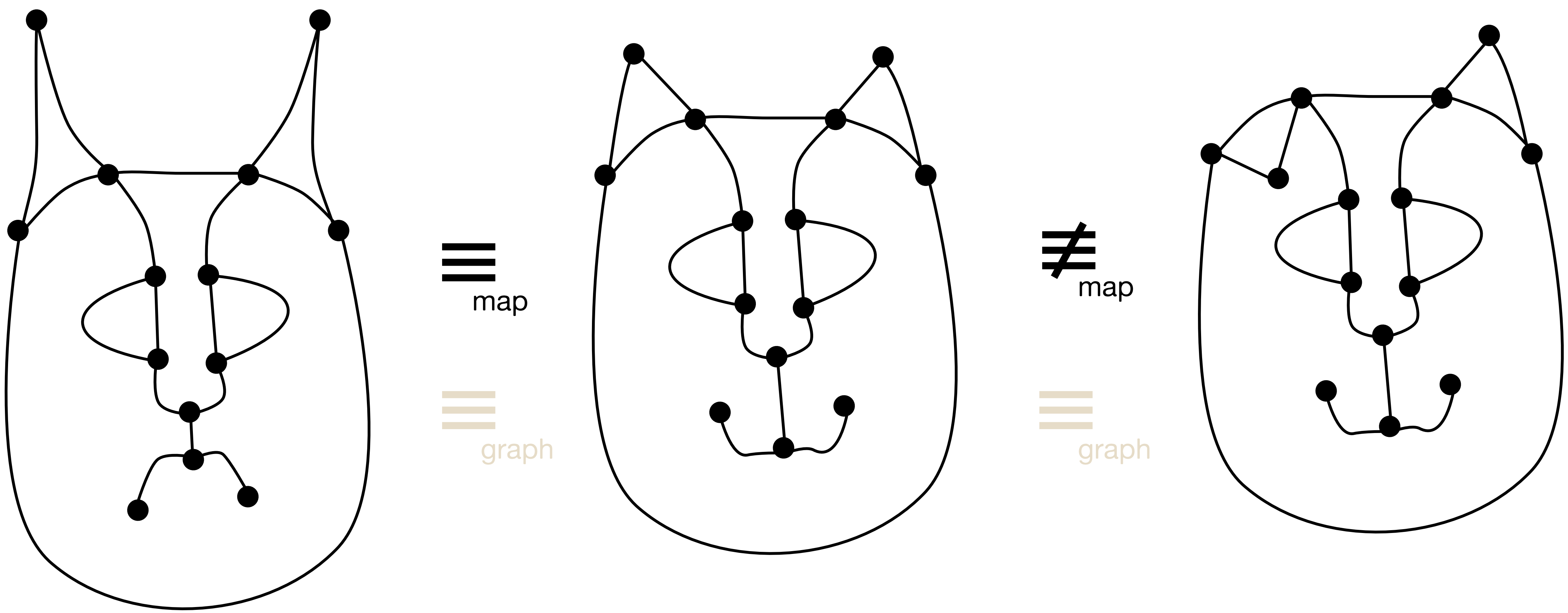
Combinatorial definition

map = connected graph + cyclic ordering of the half-edges around each vertex (say, as given by a drawing with "virtual crossings").

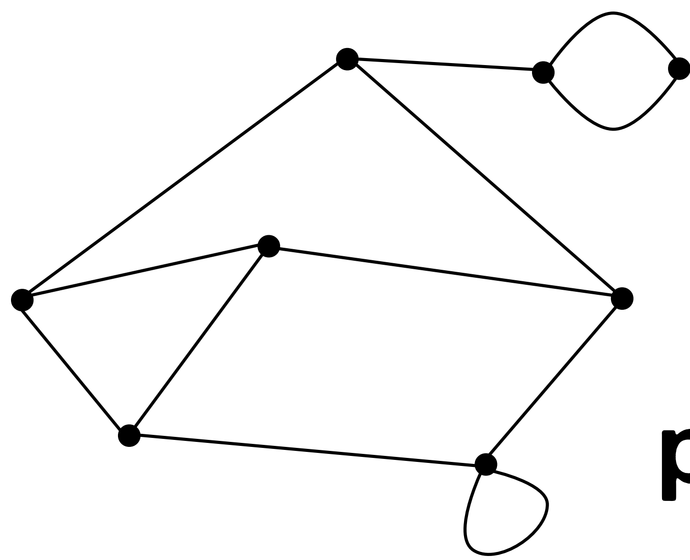




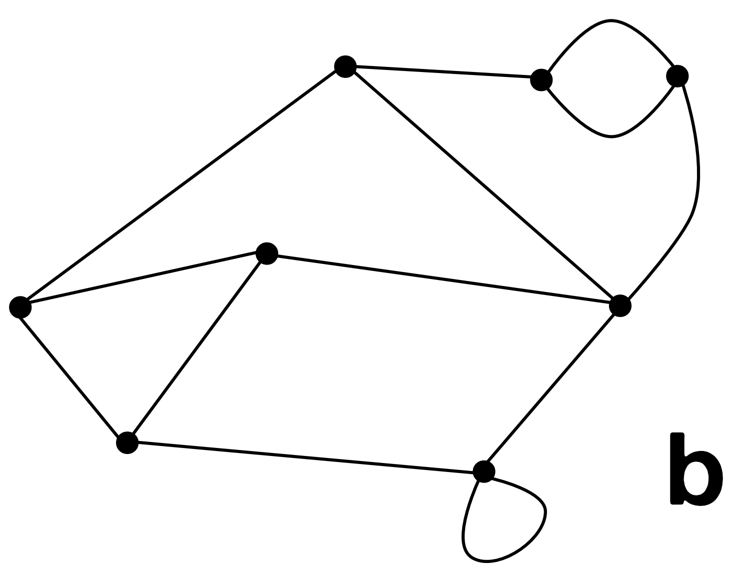
Graph versus Map



Some special kinds of maps

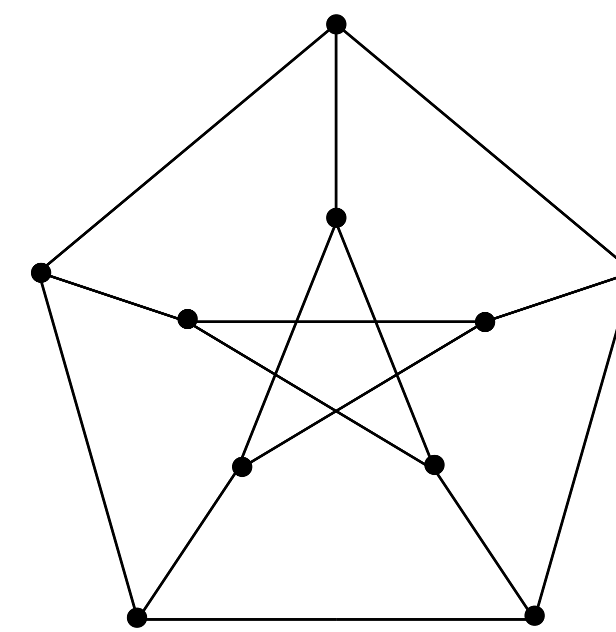


planar

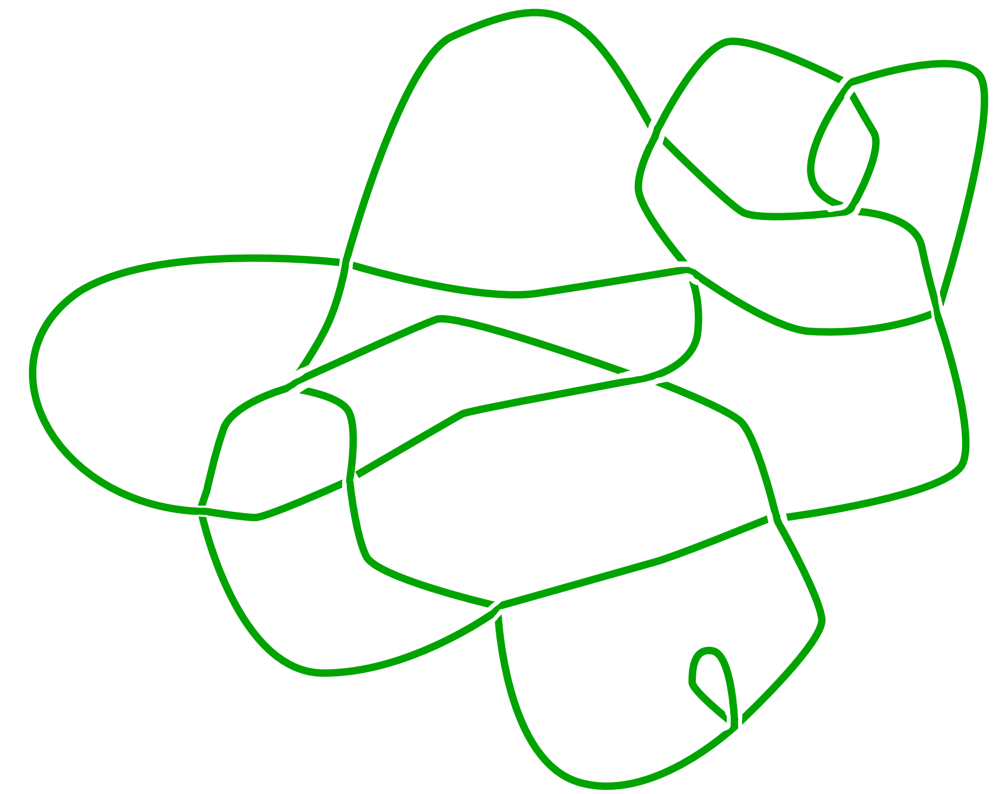


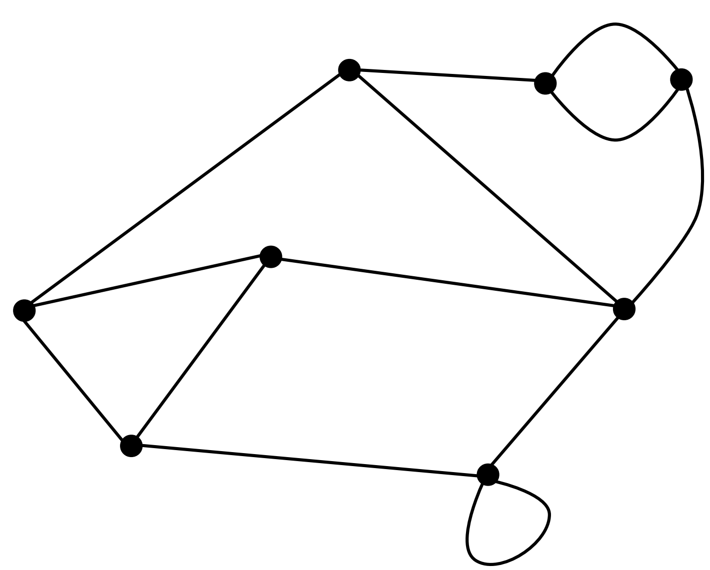
bridgeless

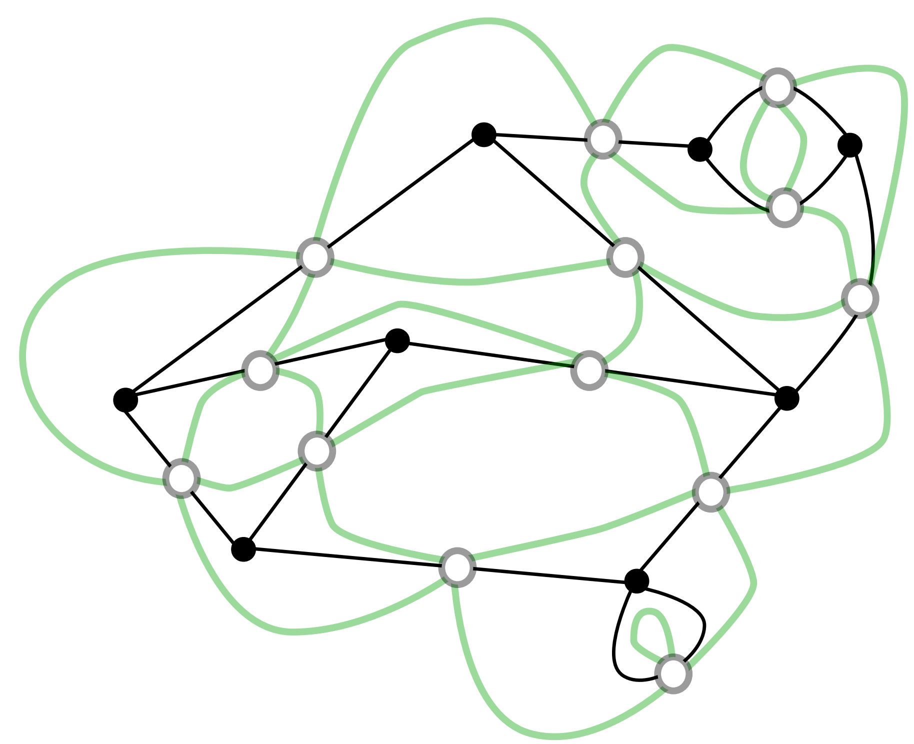
3-valent

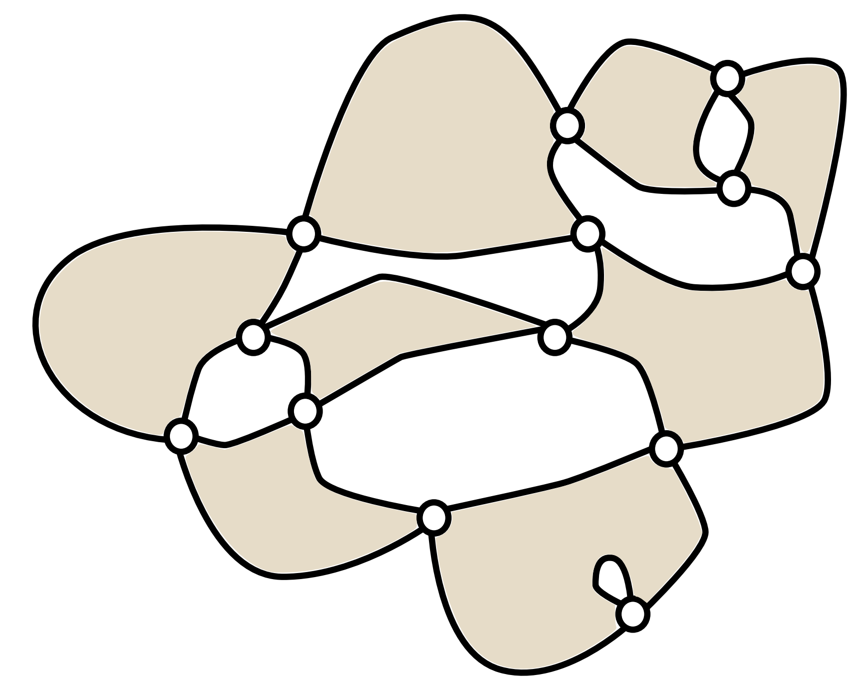


Aside: close connections to knot theory via the **medial map** construction



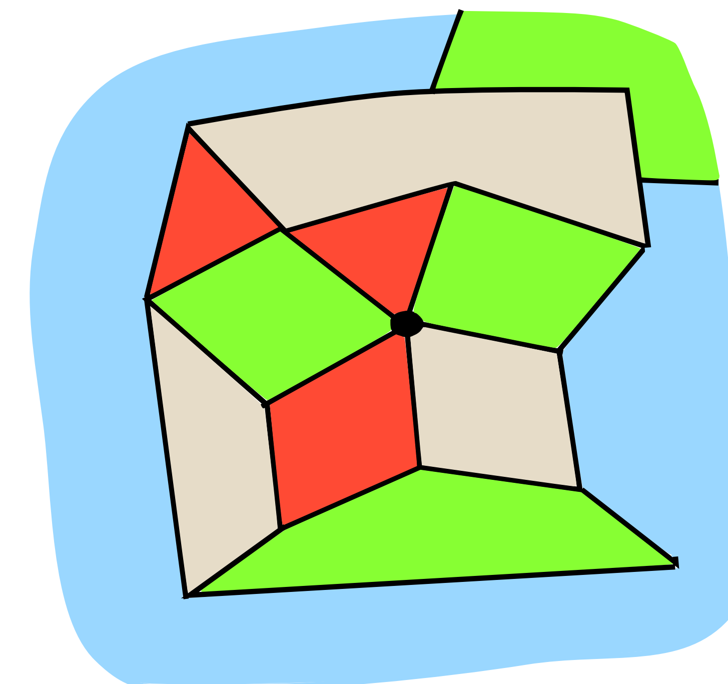






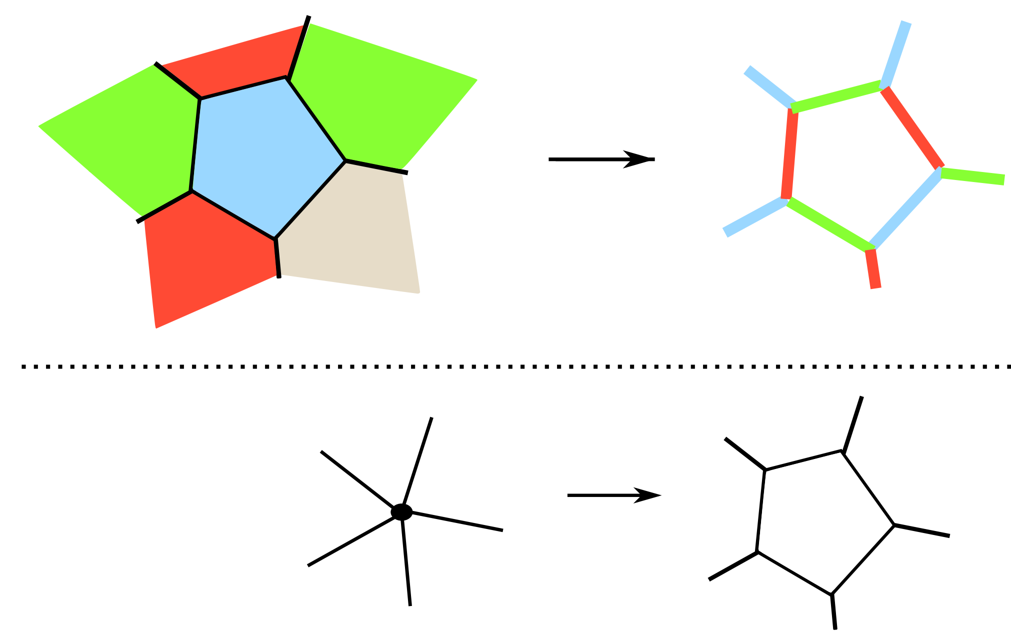
Four Colour Theorem

The 4CT is a statement about maps.



every bridgeless planar map has a proper face 4-coloring

By a well-known reduction (Tait 1880), 4CT is equivalent to a statement about 3-valent maps



every bridgeless planar 3-valent map has a proper edge 3-coloring

Map enumeration

From time to time in a graph-theoretical career one's thoughts turn to the Four Colour Problem. It occurred to me once that it might be possible to get results of interest in the theory of map-colourings without actually solving the Problem. For example, it might be possible to find the average number of colourings on vertices, for planar triangulations of a given size.

One would determine the number of triangulations of 2n faces, and then the number of 4-coloured triangulations of 2n faces. Then one would divide the second number by the first to get the required average. I gathered that this sort of retreat from a difficult problem to a related average was not unknown in other branches of Mathematics, and that it was particularly common in Number Theory.

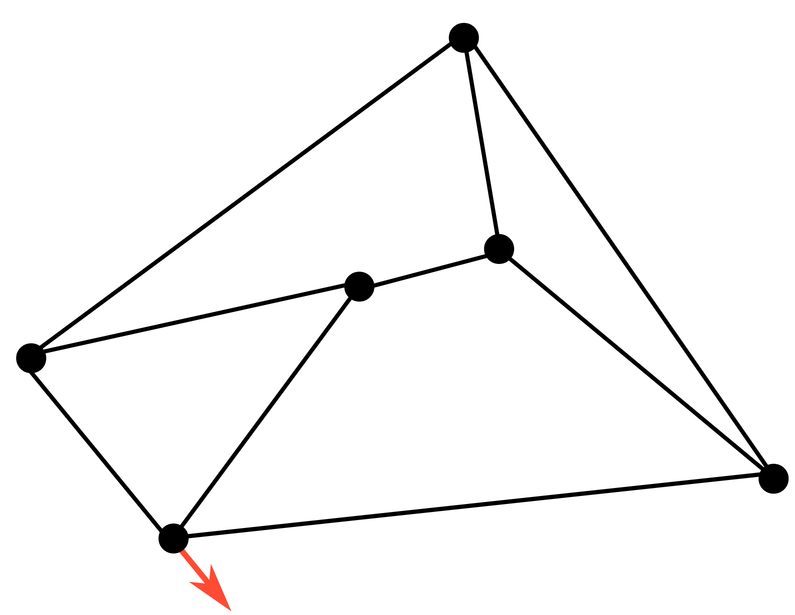
W. T. Tutte, Graph Theory as I Have Known It

Map enumeration

Tutte wrote a pioneering series of papers (1962-1969)

W. T. Tutte (1962), A census of planar triangulations. Canadian Journal of Mathematics 14:21–38
W. T. Tutte (1962), A census of Hamiltonian polygons. Can. J. Math. 14:402–417
W. T. Tutte (1962), A census of slicings. Can. J. Math. 14:708–722
W. T. Tutte (1963), A census of planar maps. Can. J. Math. 15:249–271
W. T. Tutte (1968), On the enumeration of planar maps. Bulletin of the American Mathematical Society 74:64–74
W. T. Tutte (1969), On the enumeration of four-colored maps. SIAM Journal on Applied Mathematics 17:454–460

One of his insights was to consider **rooted** maps



Key property: rooted maps have no non-trivial automorphisms

Map enumeration

Ultimately, Tutte obtained some remarkably simple formulas for counting different families of rooted planar maps.

(5.1) *The number* a_n *of rooted maps with* n *edges is*

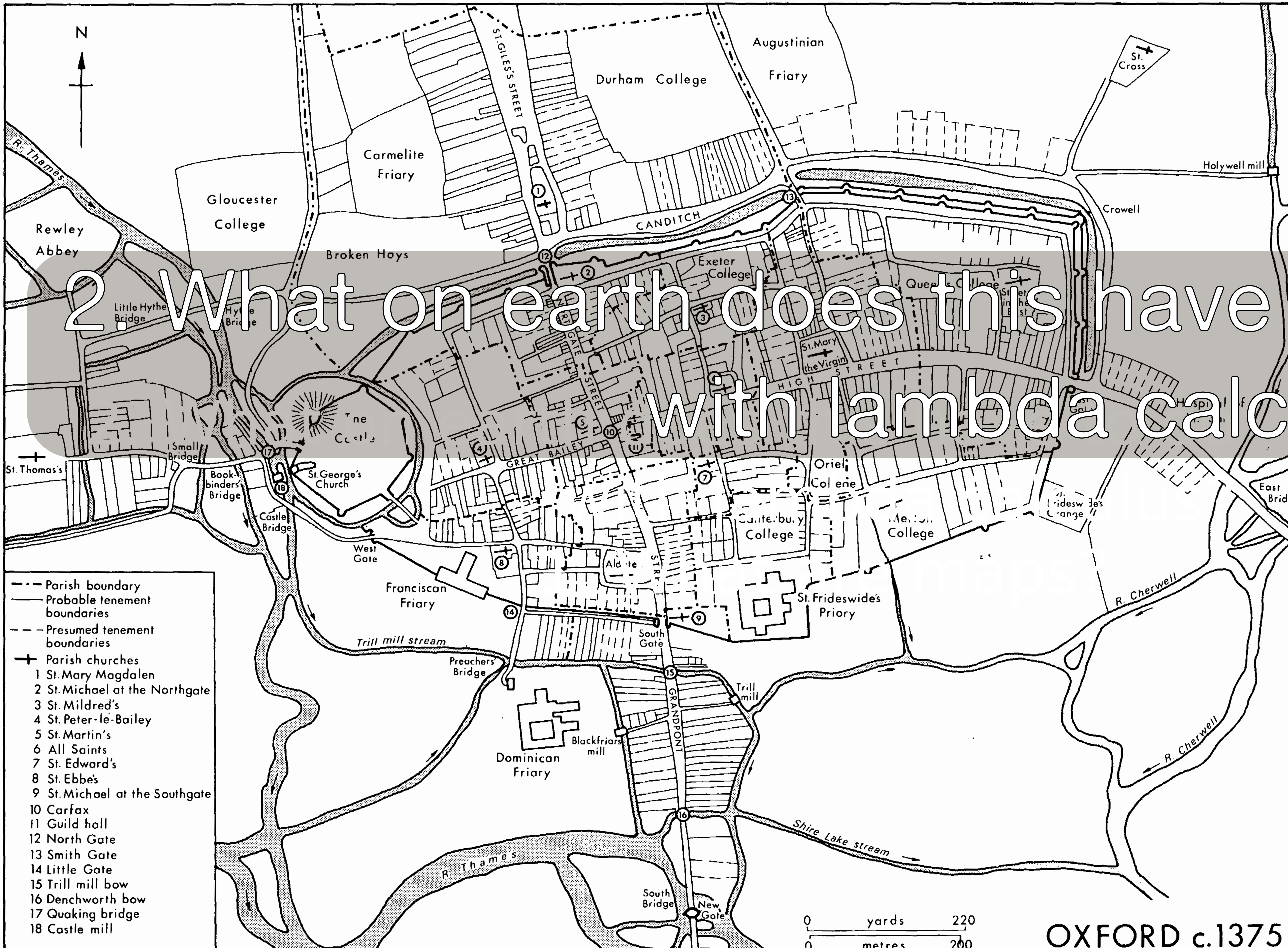
$$\frac{2(2n)!\,3^n}{n!\,(n+2)!}.$$

We write

$$A(x) = \sum_{n=1}^{\infty} a_n x^n.$$

Thus $A(x) = 2x + 9x^2 + 54x^3 + 378x^4 + \ldots$. Figure 2 shows the 2 rooted maps with 1 edge, and Figure 3 the 9 rooted maps with 2 edges.

2. What on earth does this have to do with lambda calculus?



Some enumerative connections

family of rooted maps	family of lambda terms	sequence	OEIS
trivalent maps (genus g≥0)	linear terms	1,5,60,1105,27120,...	A062980

1. O. Bodini, D. Gardy, A. Jacquot (2013), Asymptotics and random sampling for BCI and BCK lambda terms, TCS 502: 227-238

OEIS = Online Encyclopedia of Integer Sequences (oeis.org)

Some enumerative connections

family of rooted maps	family of lambda terms	sequence	OEIS
trivalent maps (genus g≥0)	linear terms	1,5,60,1105,27120,...	A062980
planar maps	normal planar terms	1,2,9,54,378,2916,...	A000168

1. O. Bodini, D. Gardy, A. Jacquot (2013), Asymptotics and random sampling for BCI and BCK lambda terms, TCS 502: 227-238
2. Z, A. Giorgetti (2015), A correspondence between rooted planar maps and normal planar lambda terms, LMCS 11(3:22): 1-39

OEIS = Online Encyclopedia of Integer Sequences (oeis.org)

Some enumerative connections

family of rooted maps	family of lambda terms	sequence	OEIS
trivalent maps (genus g≥0)	linear terms	1,5,60,1105,27120,...	A062980
planar trivalent maps	planar terms	1,4,32,336,4096,...	A002005
bridgeless trivalent maps	unitless linear terms	1,2,20,352,8624,...	A267827
bridgeless planar trivalent maps	unitless planar terms	1,1,4,24,176,1456,...	A000309
maps (genus g≥0)	normal linear terms (mod ~)	1,2,10,74,706,8162,...	A000698
planar maps	normal planar terms	1,2,9,54,378,2916,...	A000168
bridgeless maps	normal unitless linear terms (mod ~)	1,1,4,27,248,2830,...	A000699
bridgeless planar maps	normal unitless planar terms	1,1,3,13,68,399,...	A000260

1. O. Bodini, D. Gardy, A. Jacquot (2013), Asymptotics and random sampling for BCI and BCK lambda terms, TCS 502: 227-238
2. Z, A. Giorgetti (2015), A correspondence between rooted planar maps and normal planar lambda terms, LMCS 11(3:22): 1-39
3. Z (2015), Counting isomorphism classes of beta-normal linear lambda terms, arXiv:1509.07596
4. Z (2016), Linear lambda terms as invariants of rooted trivalent maps, J. Functional Programming 26(e21)
5. J. Courtiel, K. Yeats, Z (2016), Connected chord diagrams and bridgeless maps, arXiv:1611.04611
6. Z (2017), A sequent calculus for a semi-associative law, FSCD

OEIS = Online Encyclopedia of Integer Sequences (oeis.org)

Some enumerative connections

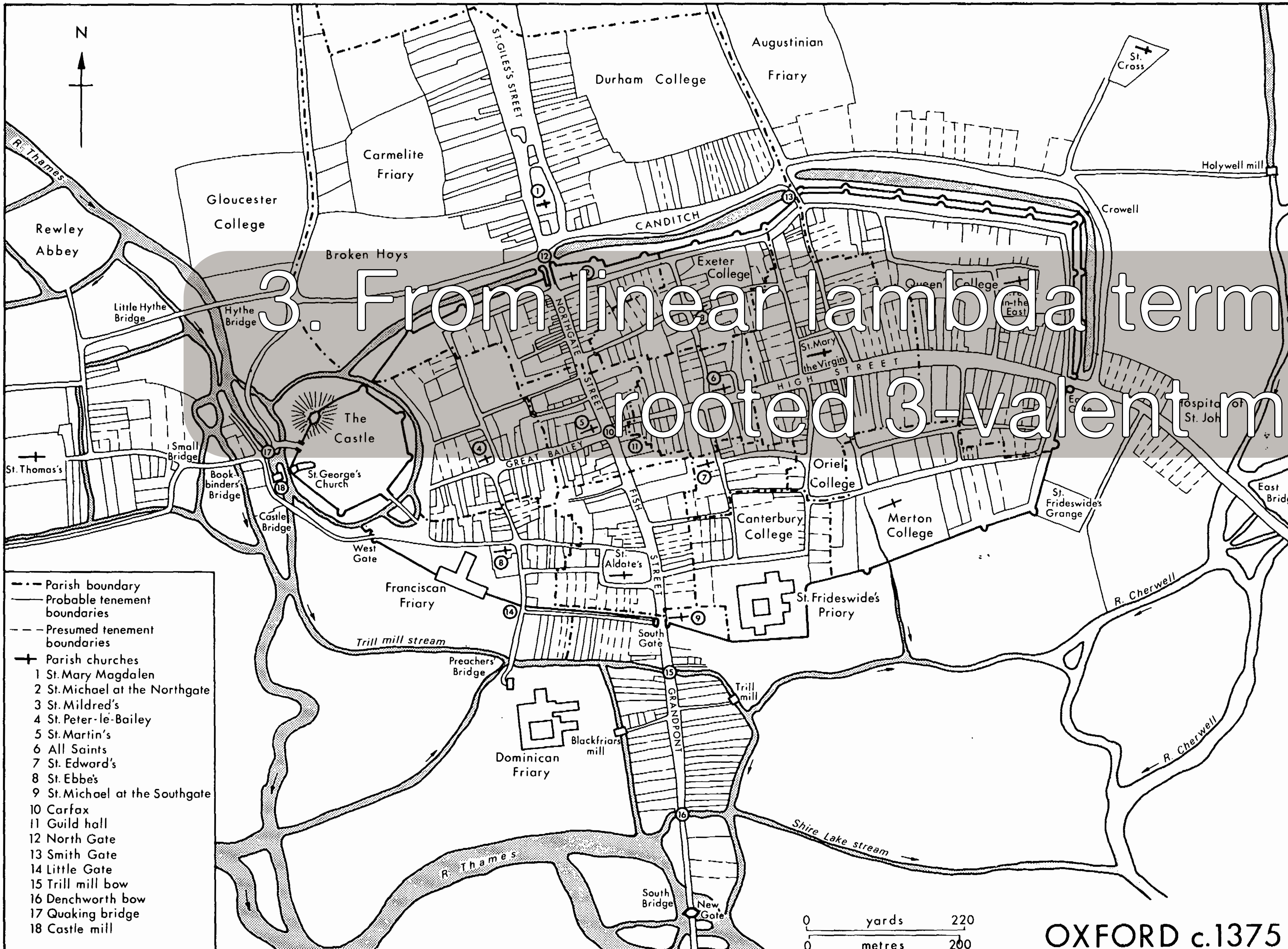
(technical focus of today's talk)

family of rooted maps	family of lambda terms	sequence	OEIS
trivalent maps (genus g≥0)	linear terms	1,5,60,1105,27120,...	A062980
planar trivalent maps	planar terms	1,4,32,336,4096,...	A002005
bridgeless trivalent maps	unitless linear terms	1,2,20,352,8624,...	A267827
bridgeless planar trivalent maps	unitless planar terms	1,1,4,24,176,1456,...	A000309
maps (genus g≥0)	normal linear terms (mod ~)	1,2,10,74,706,8162,...	A000698
planar maps	normal planar terms	1,2,9,54,378,2916,...	A000168
bridgeless maps	normal unitless linear terms (mod ~)	1,1,4,27,248,2830,...	A000699
bridgeless planar maps	normal unitless planar terms	1,1,3,13,68,399,...	A000260

1. O. Bodini, D. Gardy, A. Jacquot (2013), Asymptotics and random sampling for BCI and BCK lambda terms, TCS 502: 227-238
2. Z, A. Giorgetti (2015), A correspondence between rooted planar maps and normal planar lambda terms, LMCS 11(3:22): 1-39
3. Z (2015), Counting isomorphism classes of beta-normal linear lambda terms, arXiv:1509.07596
4. Z (2016), Linear lambda terms as invariants of rooted trivalent maps, J. Functional Programming 26(e21)
5. J. Courtiel, K. Yeats, Z (2016), Connected chord diagrams and bridgeless maps, arXiv:1611.04611
6. Z (2017), A sequent calculus for a semi-associative law, FSCD

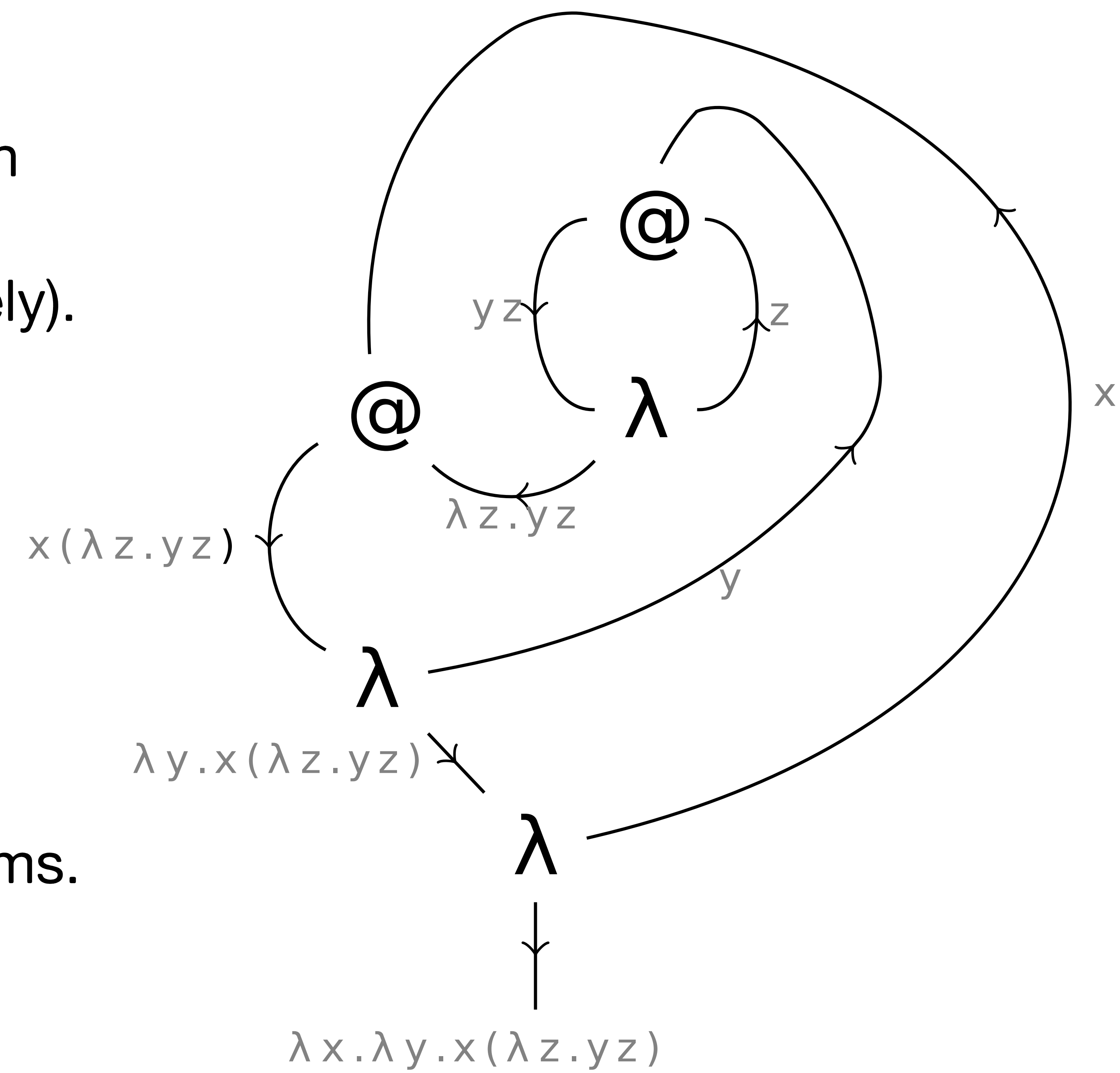
OEIS = Online Encyclopedia of Integer Sequences (oeis.org)

3. From linear lambda terms to rooted 3-valent maps



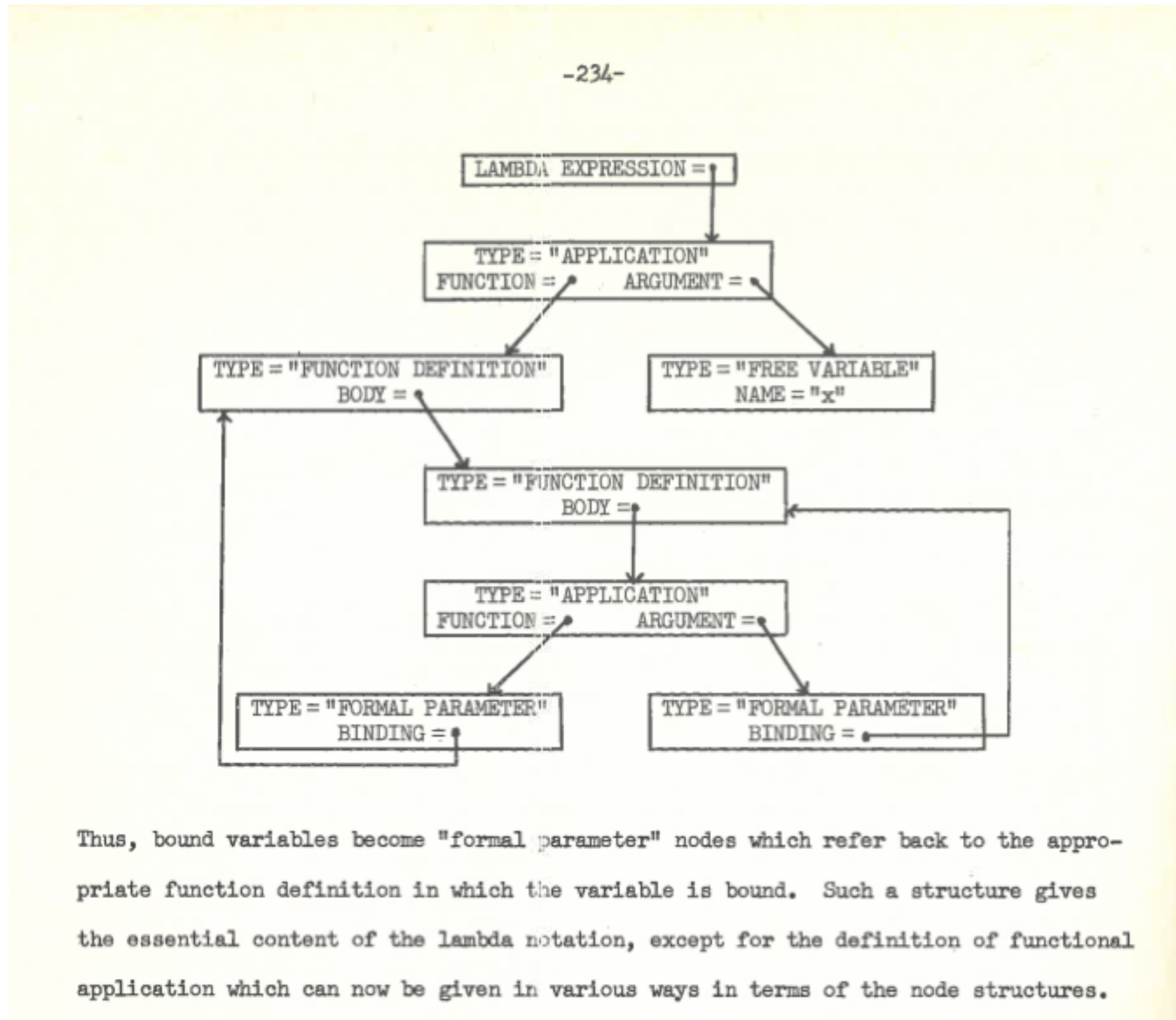
Representing terms as graphs
(an idea from the folklore)

Represent a term as a "tree with pointers", with lambda nodes pointing to the occurrences of the corresponding bound variable (or conversely).



This old idea is especially natural for **linear** terms.

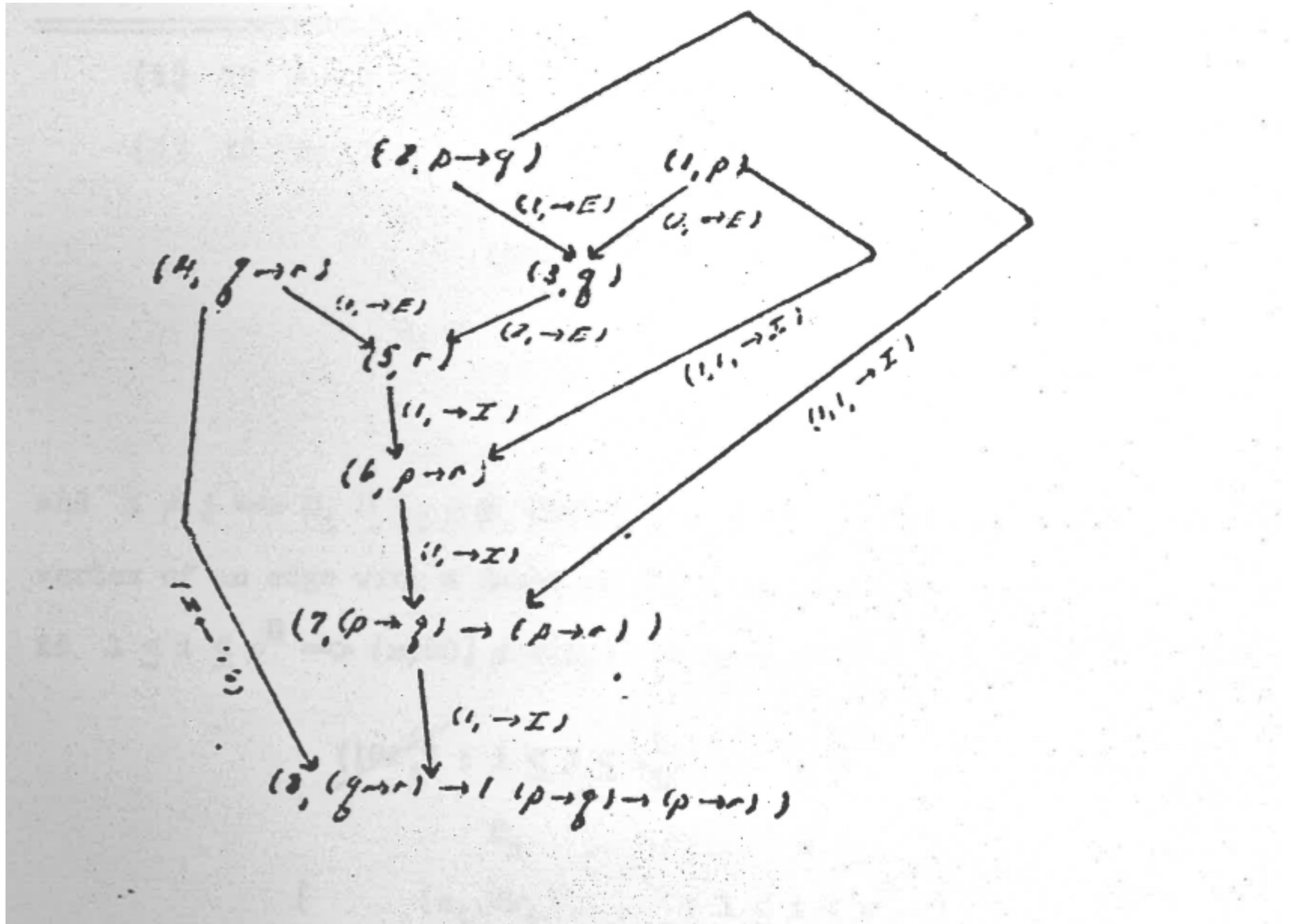
Representing terms as graphs (an idea from the folklore)


-234-

Thus, bound variables become "formal parameter" nodes which refer back to the appropriate function definition in which the variable is bound. Such a structure gives the essential content of the lambda notation, except for the definition of functional application which can now be given in various ways in terms of the node structures.

D. E. Knuth (1970), "Examples of formal semantics", in Symposium on Semantics of Algorithmic Languages.

Representing proofs as graphs
(a closely related idea)



A derivation in the System I (Section 3)

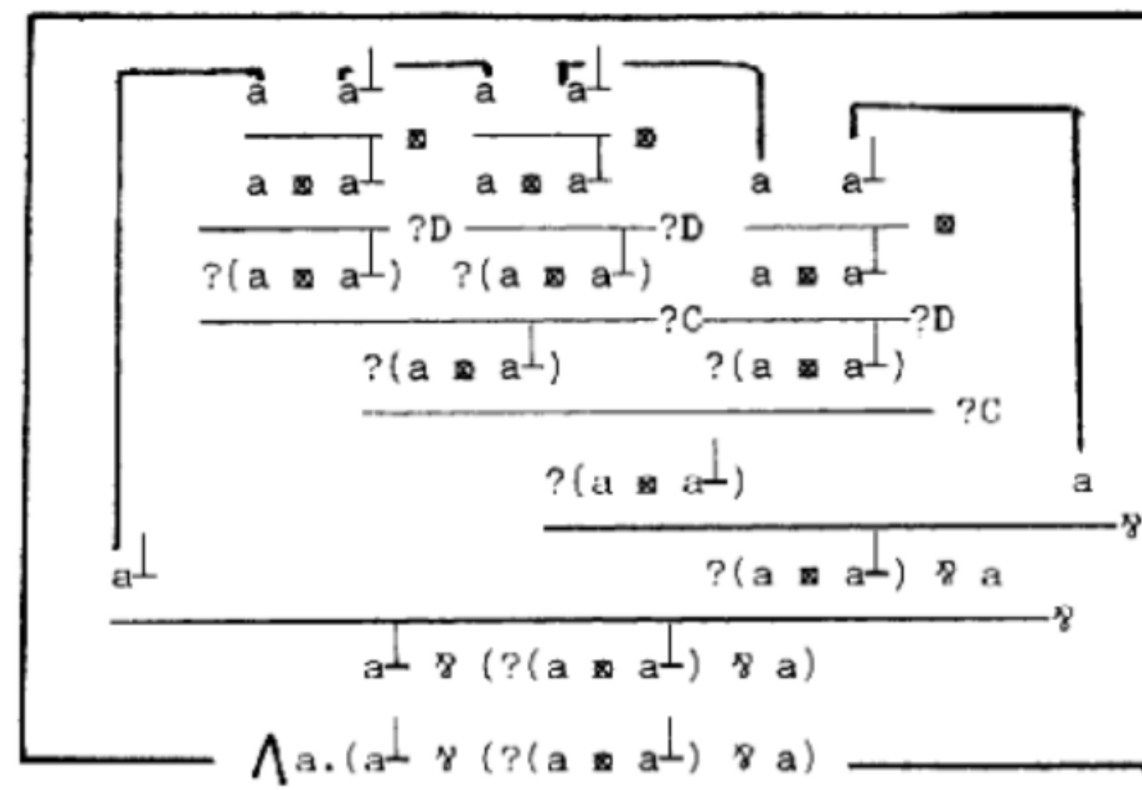


Fig. 57(b). The number "3" represented in PN2.

R. Statman (1974), Structural Complexity of Proofs, PhD Thesis, Stanford University

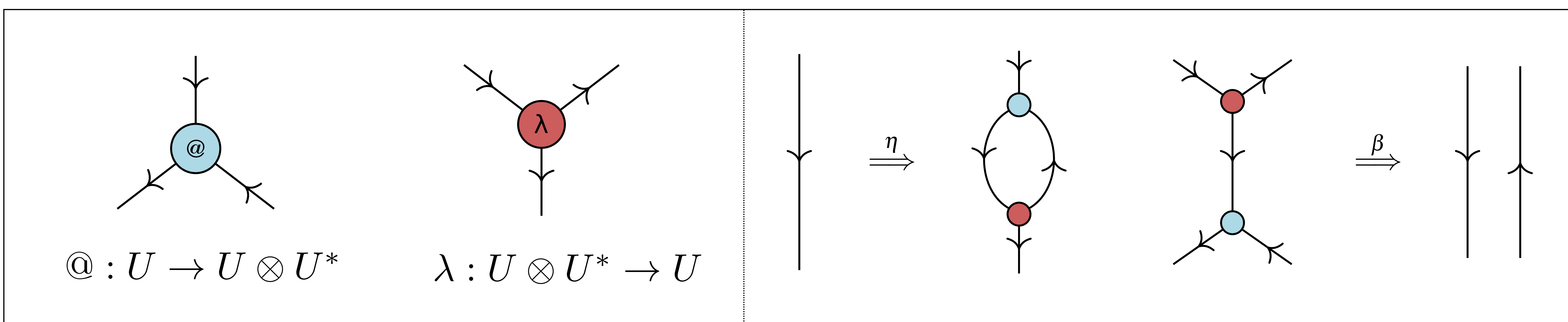
J.-Y. Girard (1987), Linear Logic, Theoretical Computer Science

λ-graphs as string diagrams

Idea (after D. Scott): a linear lambda term may be interpreted as an endomorphism of a reflexive object in a symmetric monoidal closed (bi)category.

$$U \underset{\lambda}{\overset{@}{\underset{\perp}{\rightleftarrows}}} U \multimap U$$

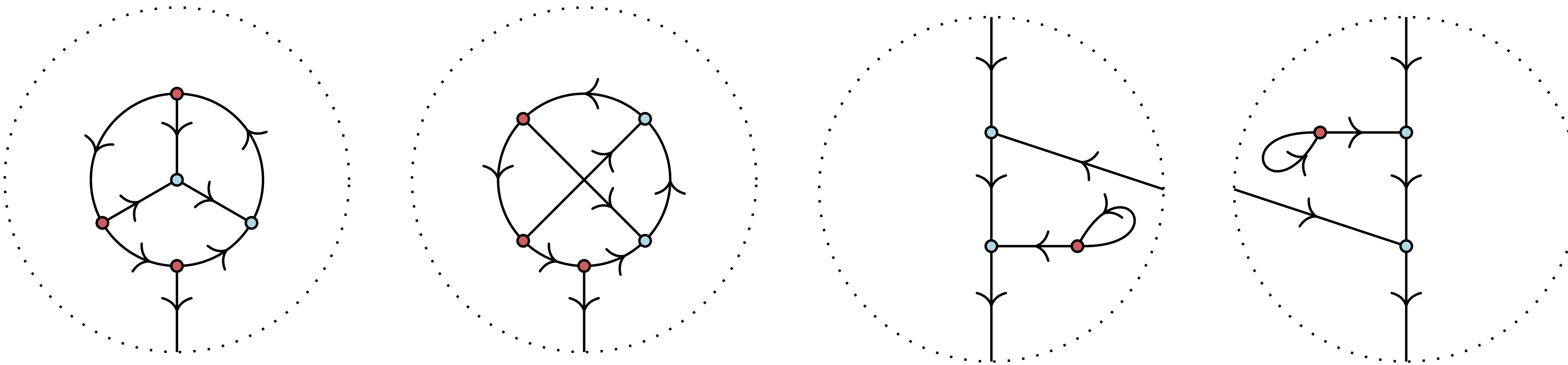
By interpreting this morphism in the graphical language of compact closed (bi)categories, we obtain the traditional diagram associated to the linear lambda term.



From linear terms to rooted 3-valent maps via string diagrams

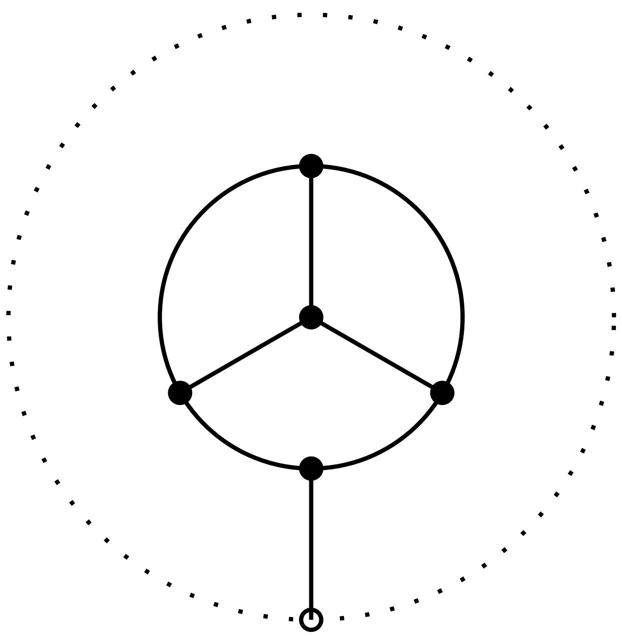
$\lambda x.\lambda y.\lambda z.x(yz)$ $\lambda x.\lambda y.\lambda z.(xz)y$ $x,y \vdash (xy)(\lambda z.z)$ $x,y \vdash x((\lambda z.z)y)$

From linear terms to rooted 3-valent maps via string diagrams

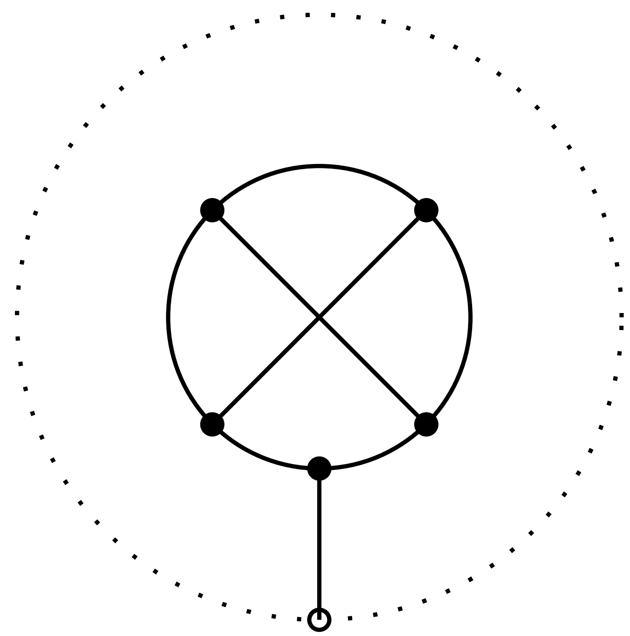


λx.λy.λz.x(yz) λx.λy.λz.(xz)y x,y ⊢ (xy)(λz.z) x,y ⊢ x((λz.z)y)

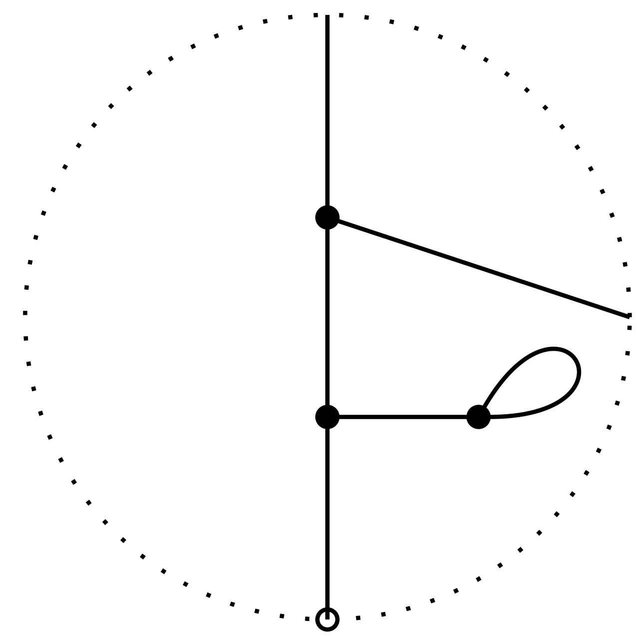
From linear terms to rooted 3-valent maps via string diagrams



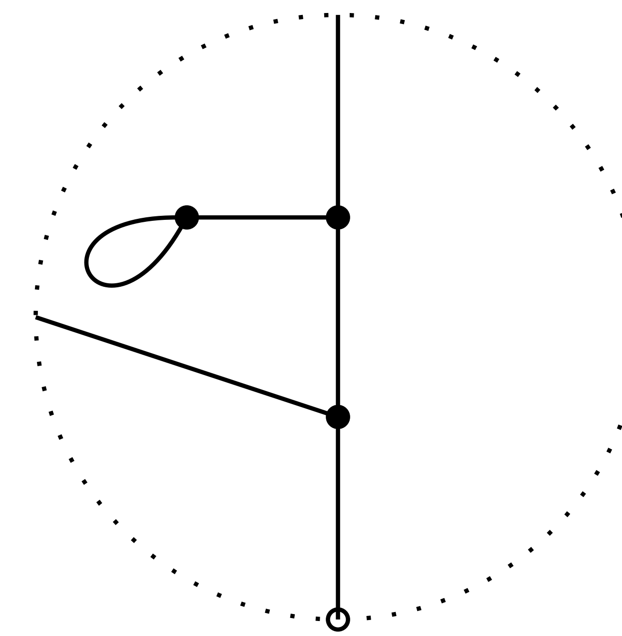
λx.λy.λz.x(yz)



λx.λy.λz.(xz)y



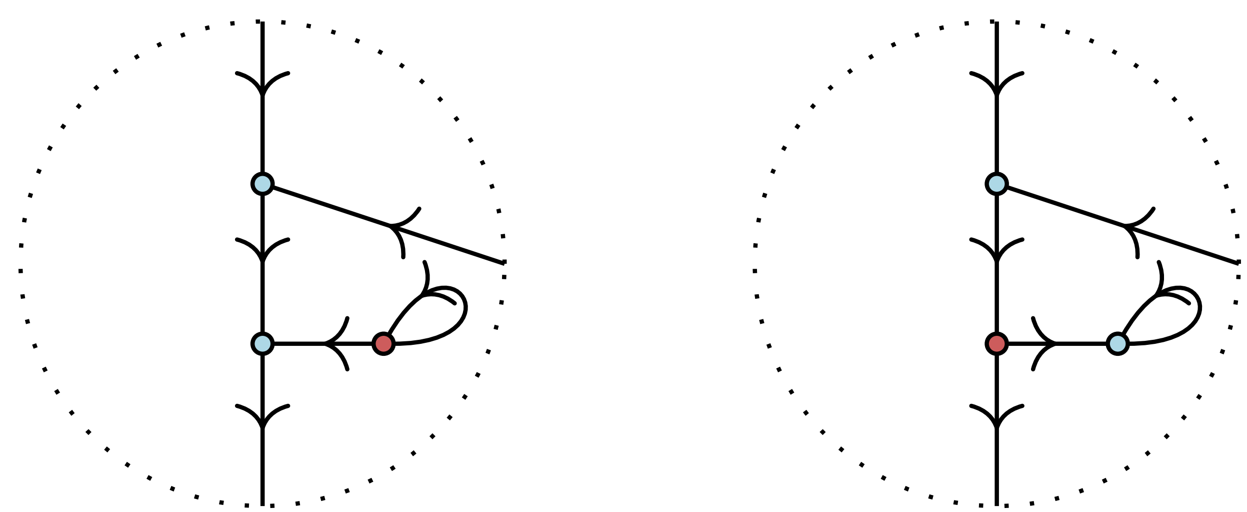
x,y ⊢ (xy)(λz.z)



x,y ⊢ x((λz.z)y)

Diagrams versus Terms

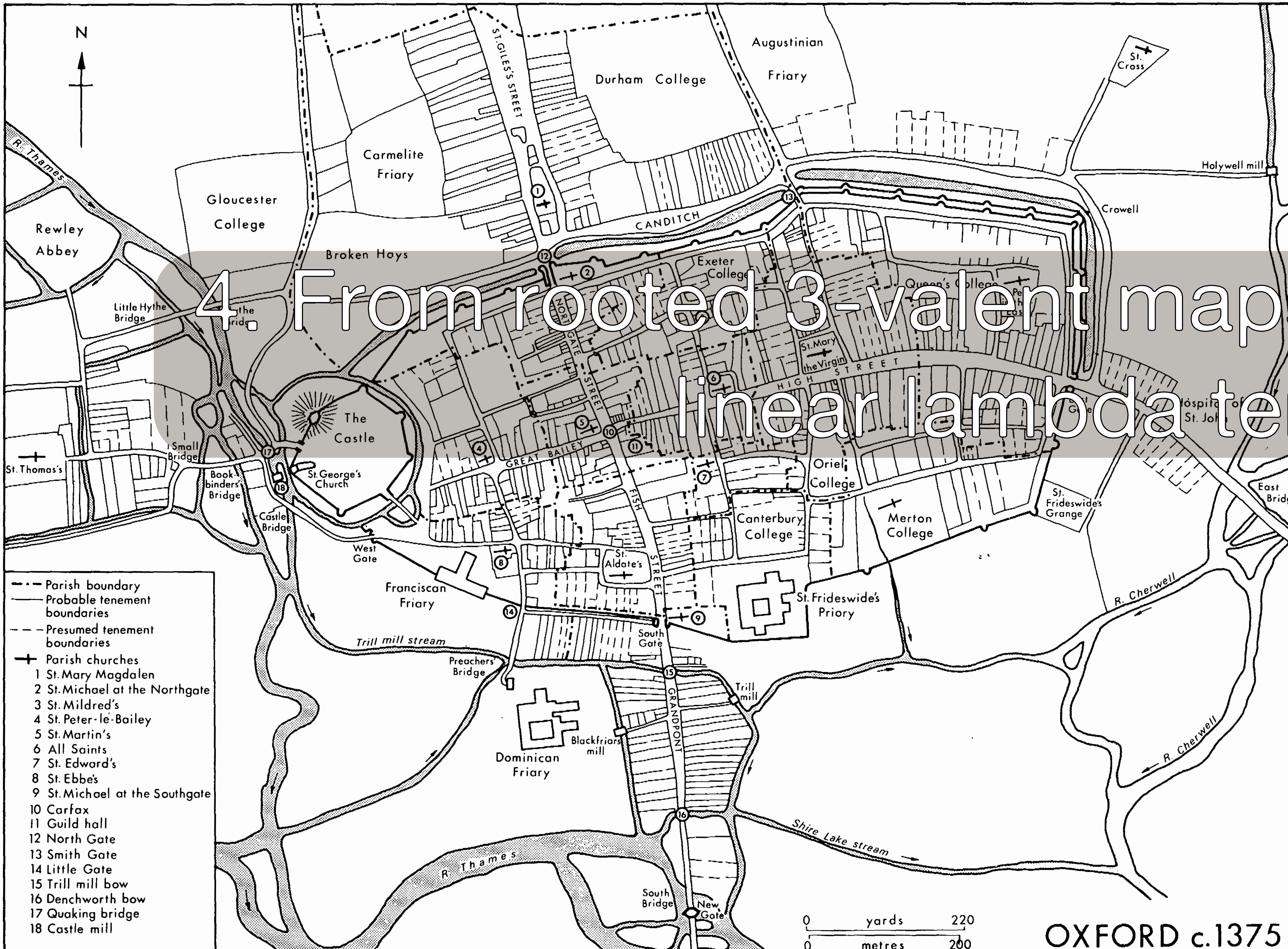
Note: two different diagrams can correspond to the same underlying map.



Indeed, a diagram is just a 3-valent map + a **proper orientation**.

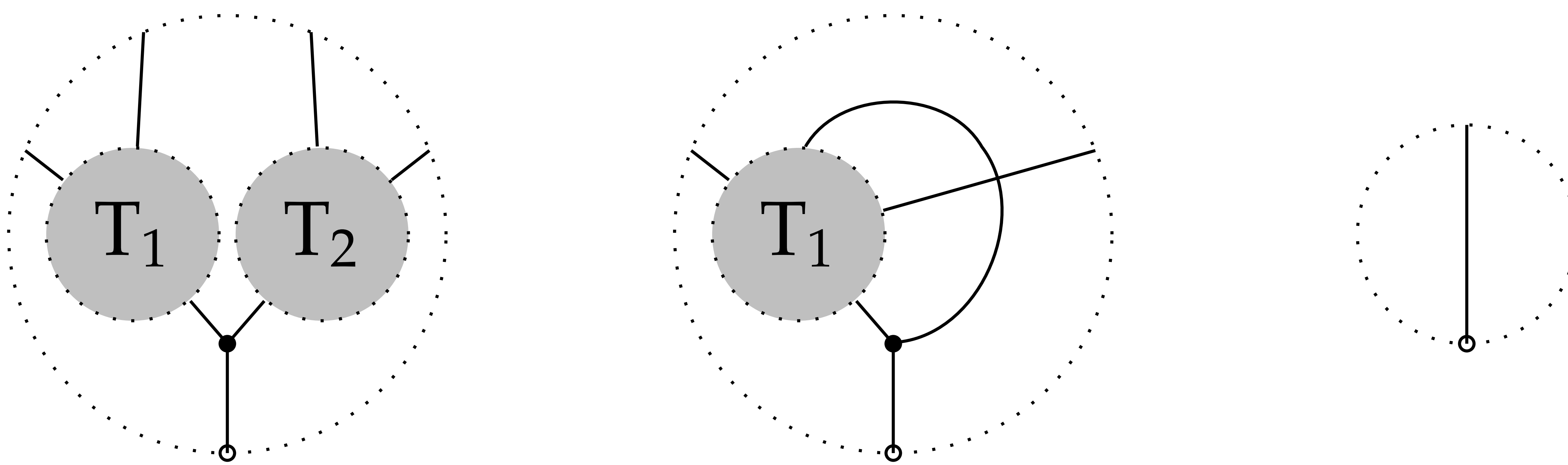
But we will see that every rooted trivalent map has a **unique** orientation corresponding to the diagram of a linear lambda term...

4. From rooted 3-valent maps to linear lambda terms



Rooted 3-valent maps, inductively

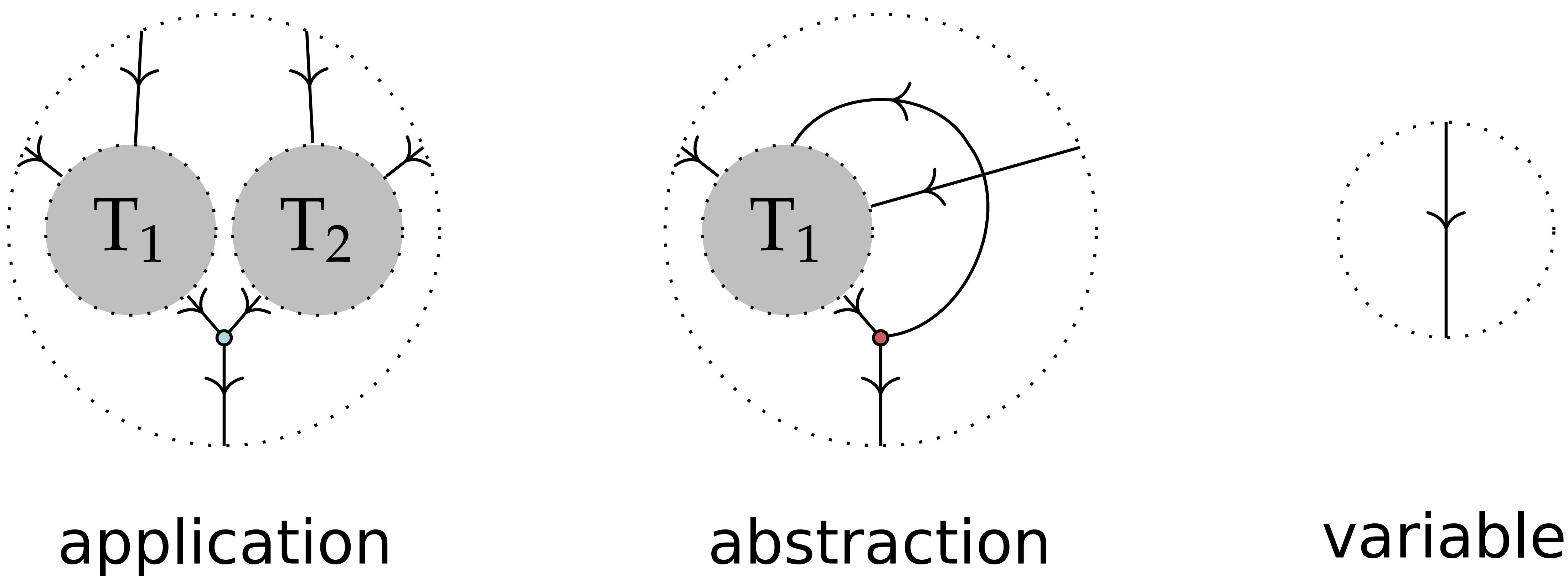
Observation: any rooted 3-valent map must have one of the following forms.



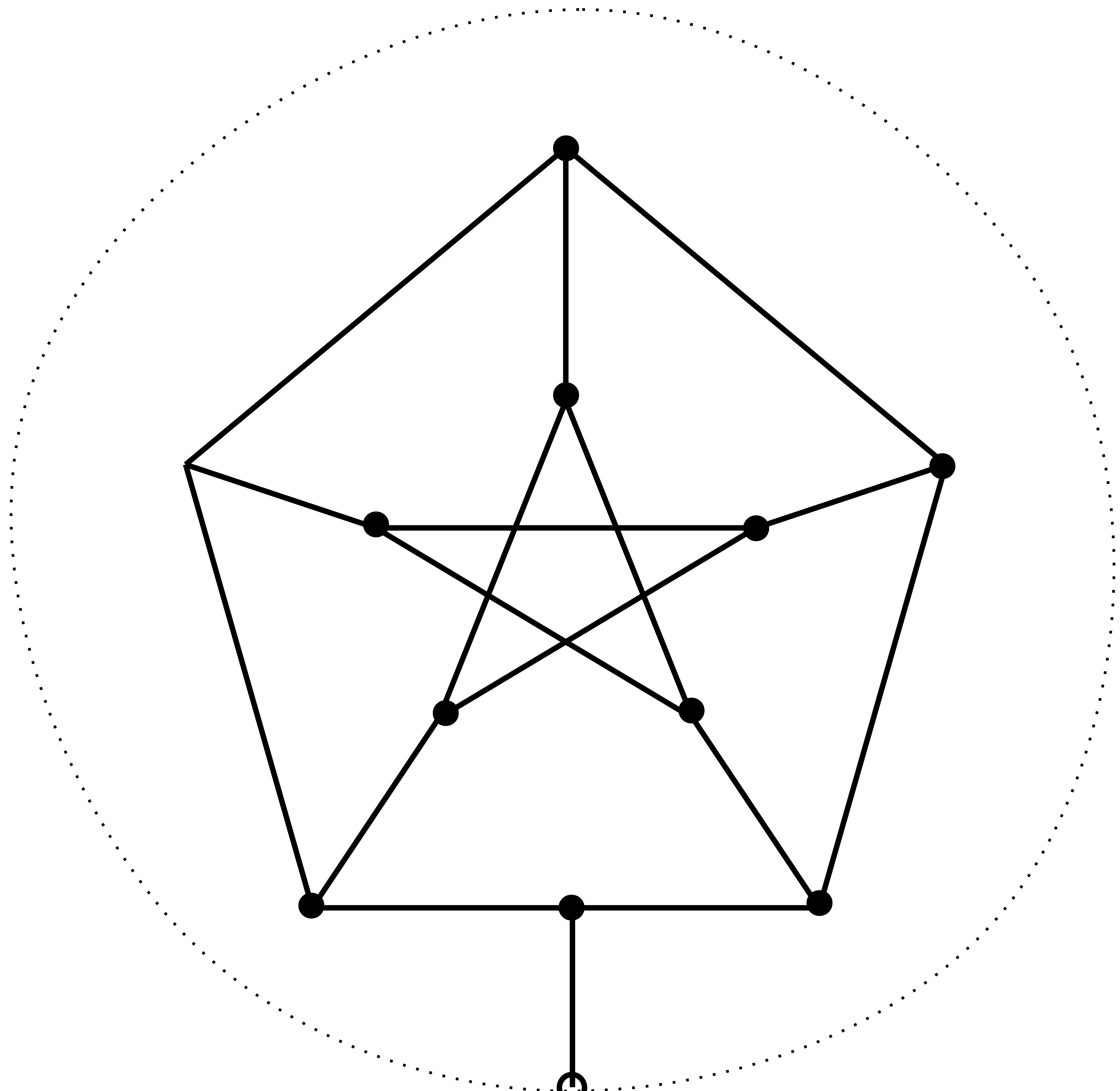
disconnecting root vertex | connecting root vertex | no root vertex

Linear lambda terms, inductively

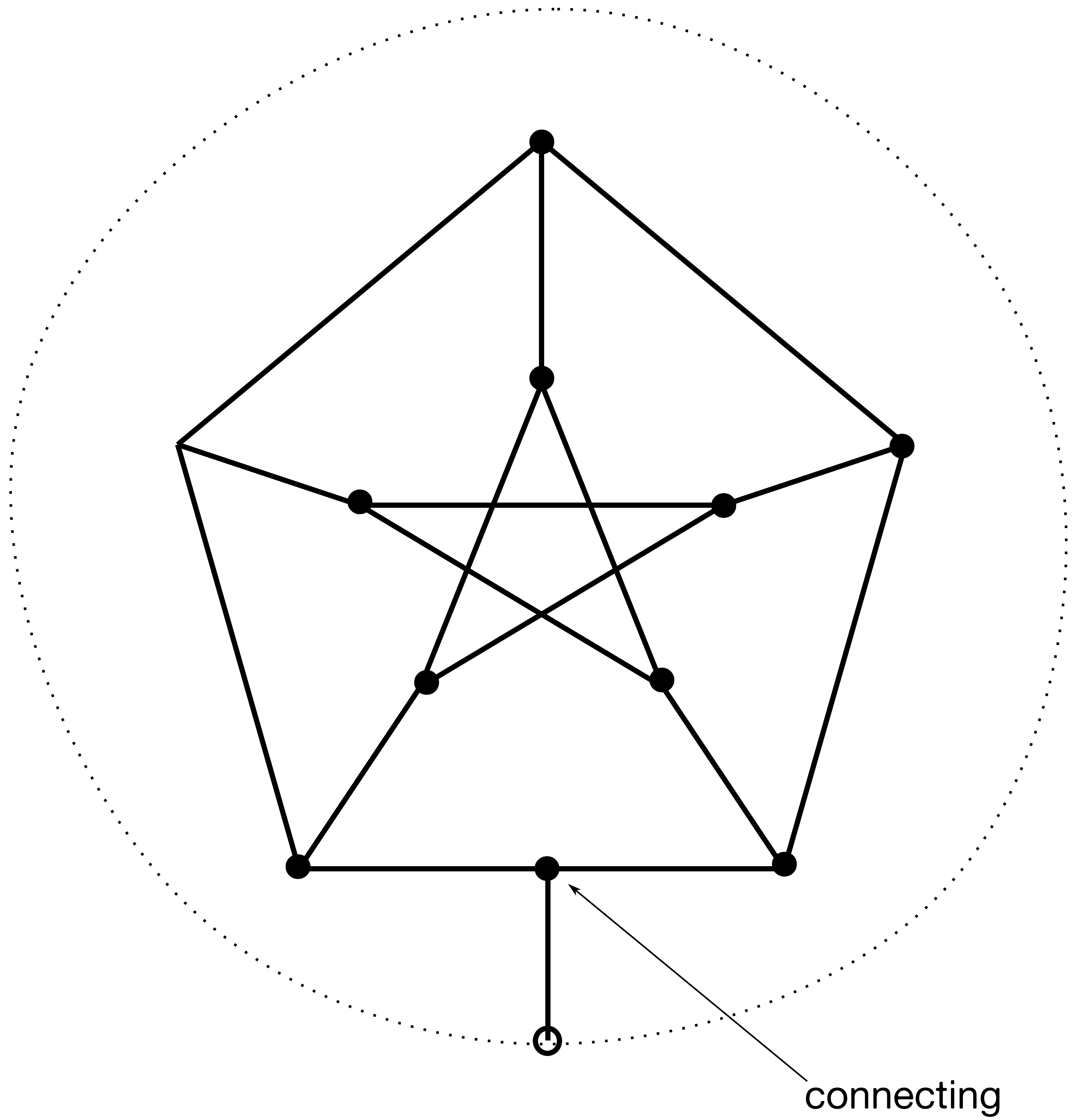
...but this exactly mirrors the inductive structure of linear lambda terms!



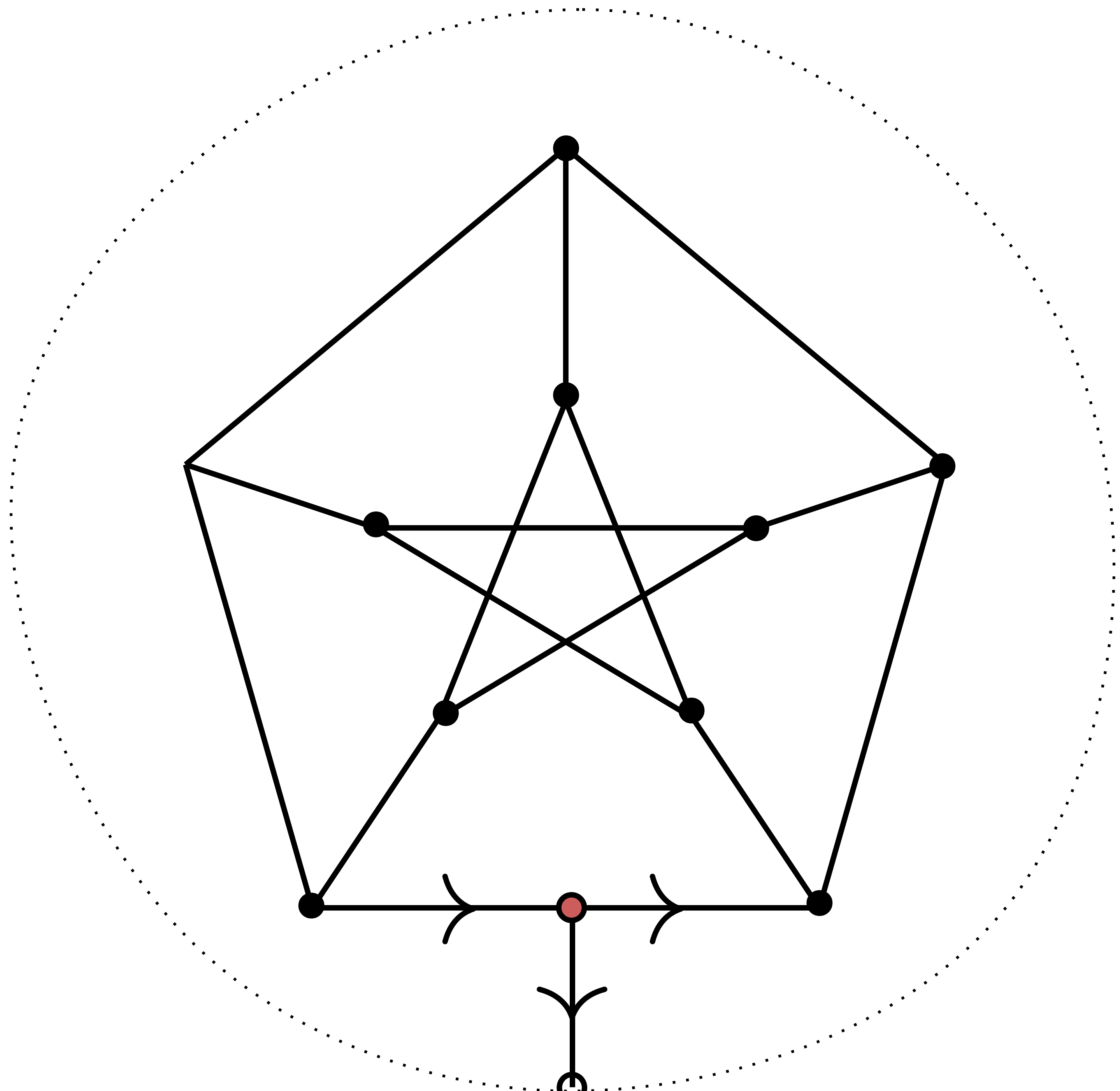
An example



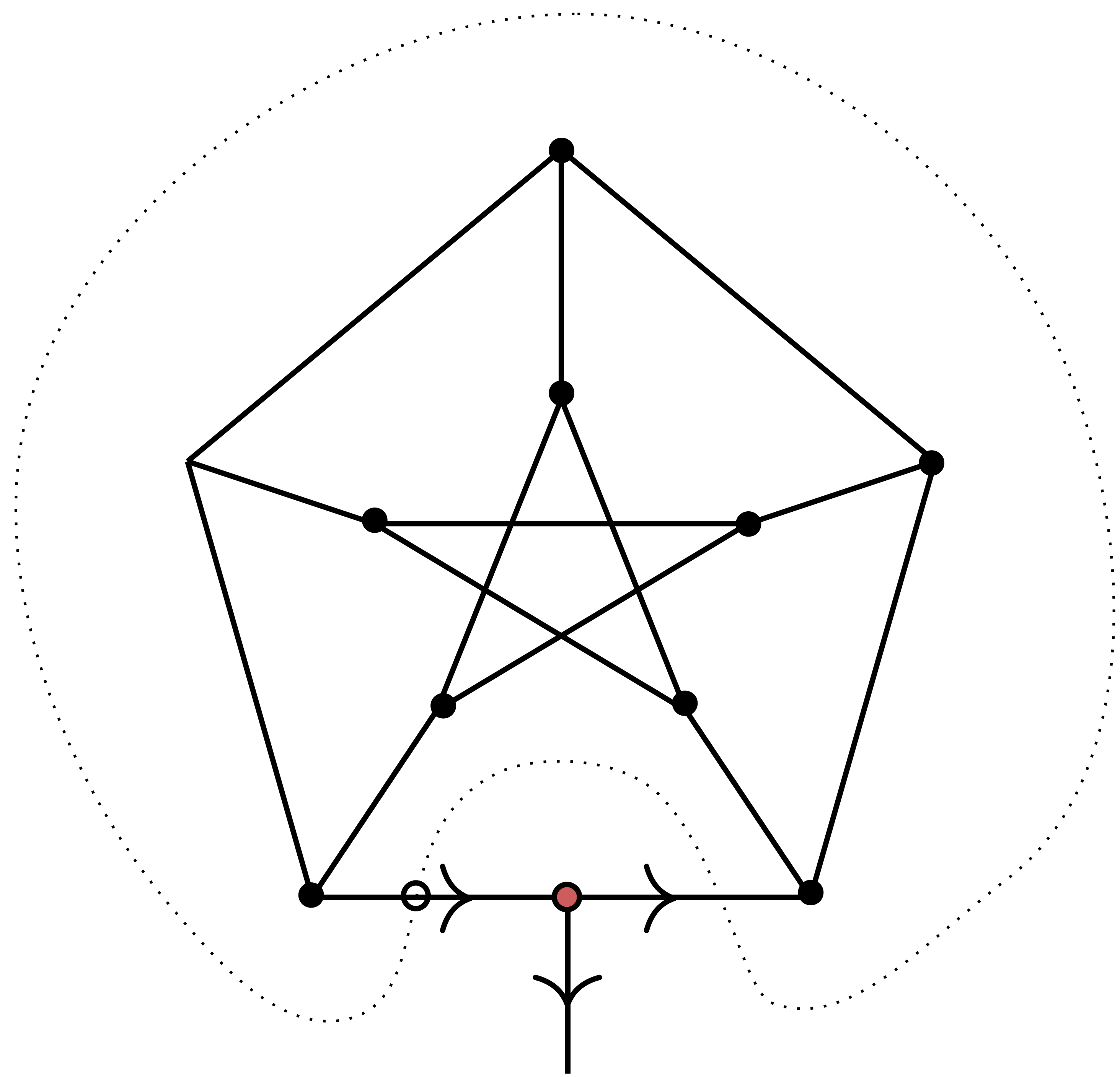
An example



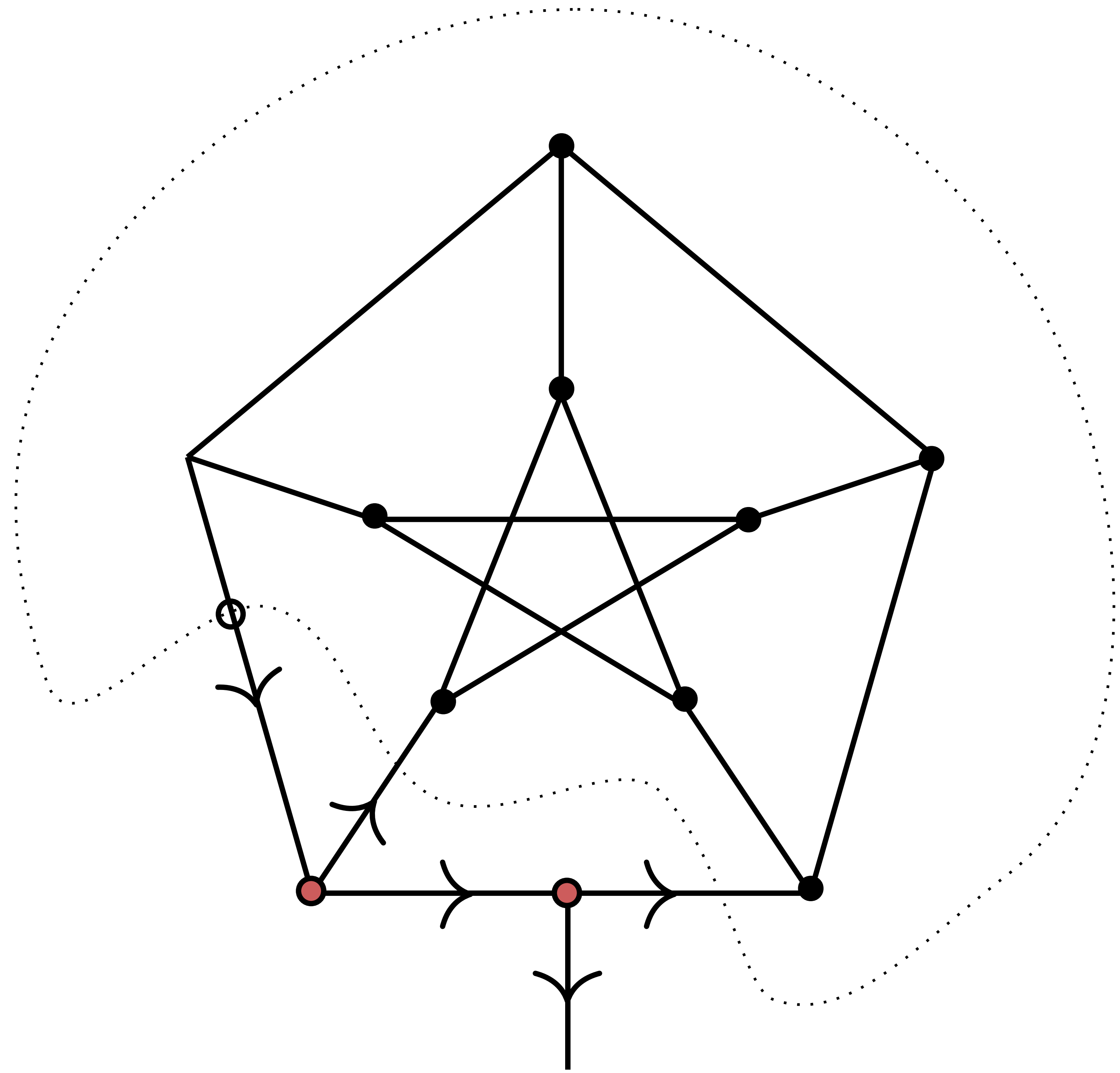
An example



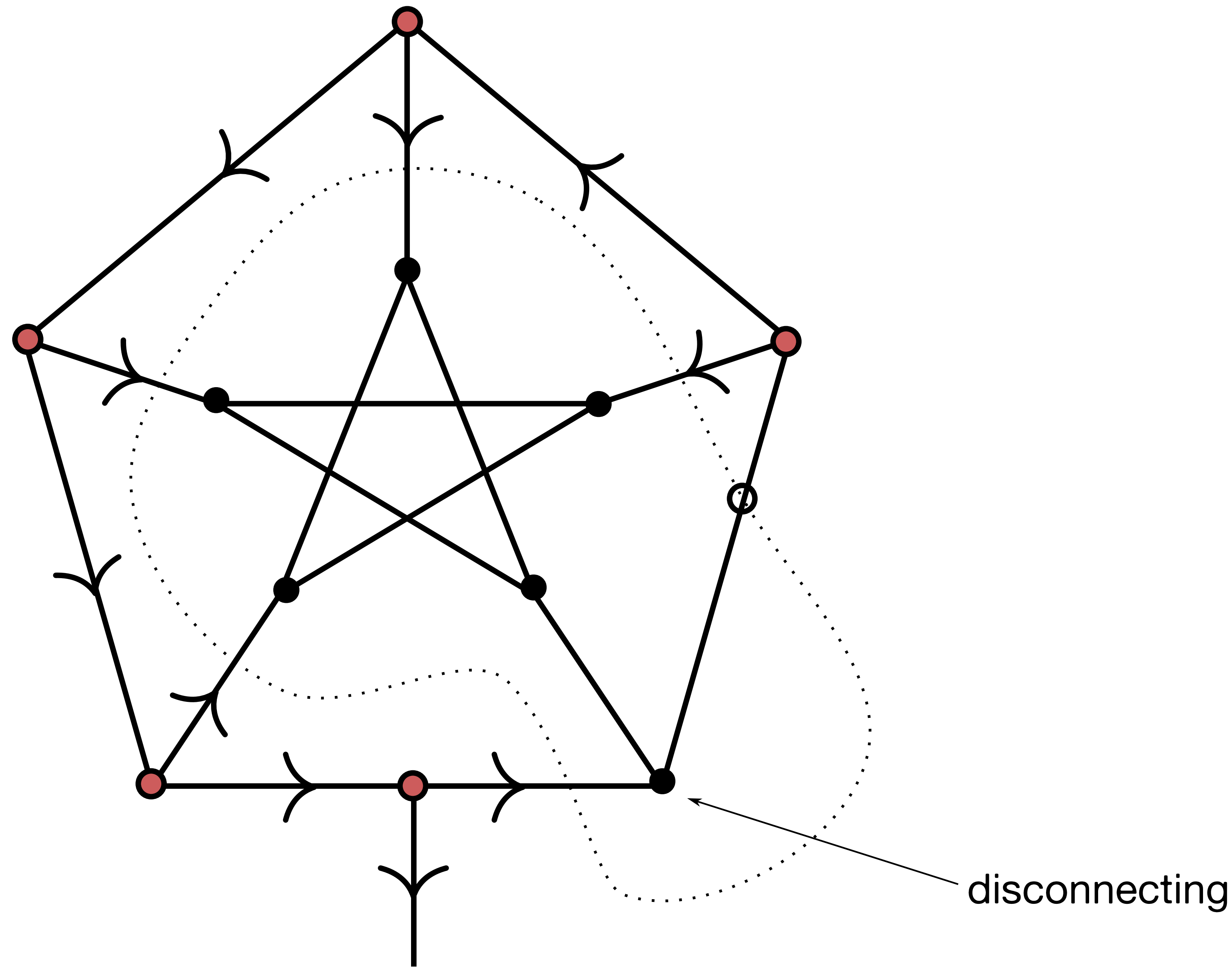
An example



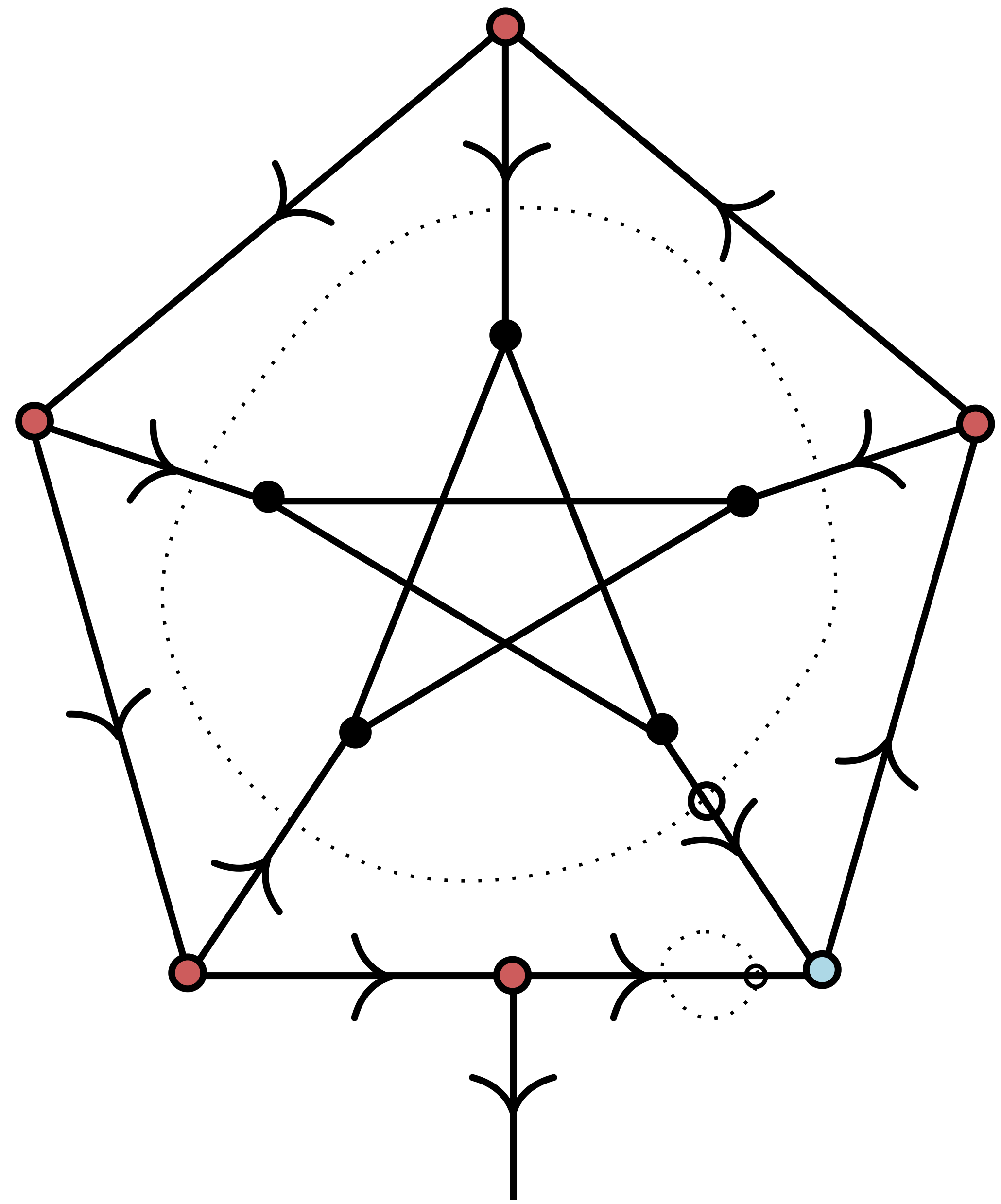
An example



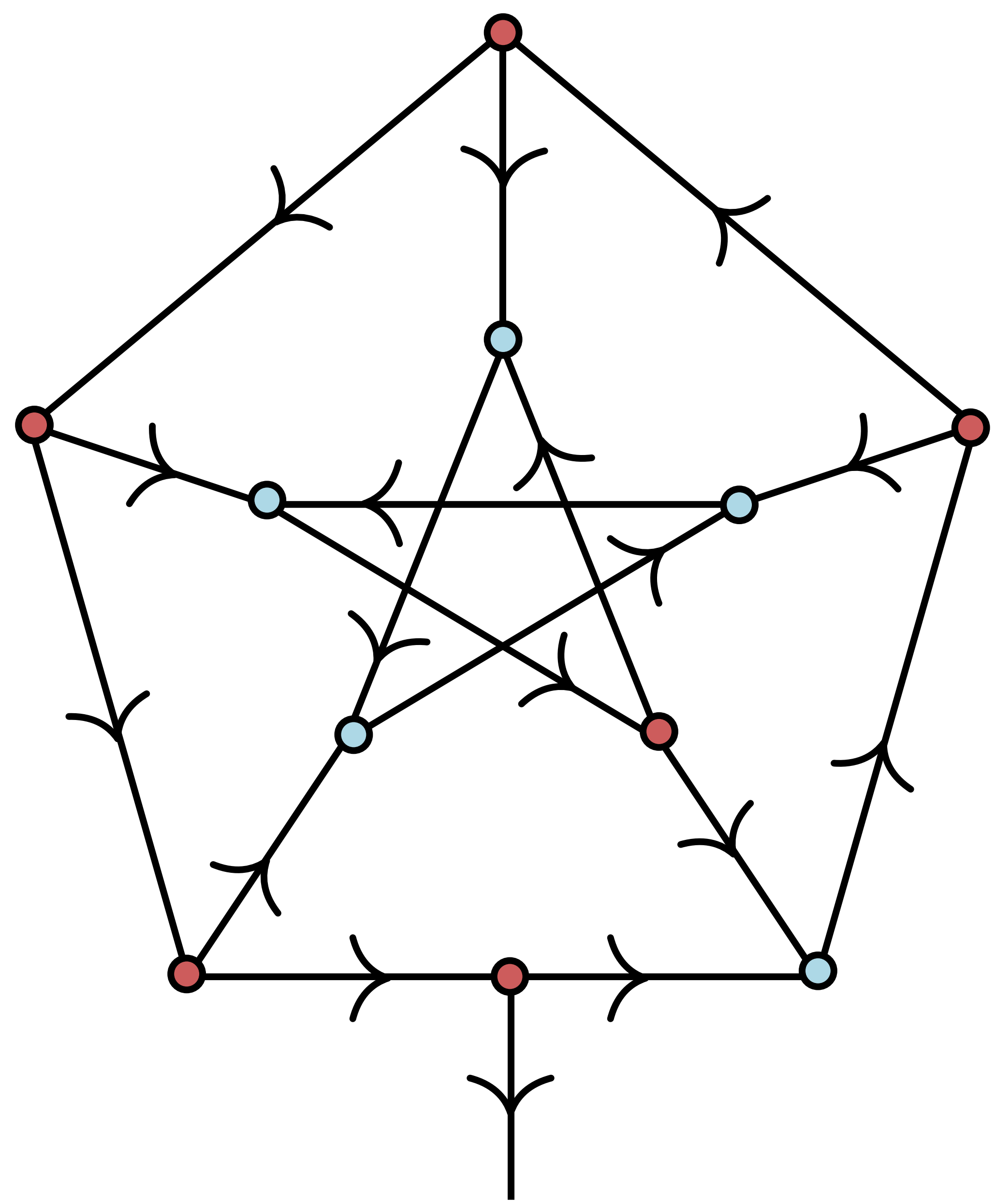
An example



An example



An example



λa.λb.λc.λd.λe.a(λf.c(e(b(df))))

An operadic perspective

Let Θ(n) = set of isomorphism classes of rooted 3-valent maps with n non-root boundary arcs.

Θ defines a **symmetric operad** equipped with operations

$$@ : \Theta(m) \times \Theta(n) \to \Theta(m+n)$$

$$\lambda_i : \Theta(m+1) \to \theta(m) \quad [1 \leq i \leq m+1]$$

naturally isomorphic to the operad of linear lambda terms.

An operadic perspective

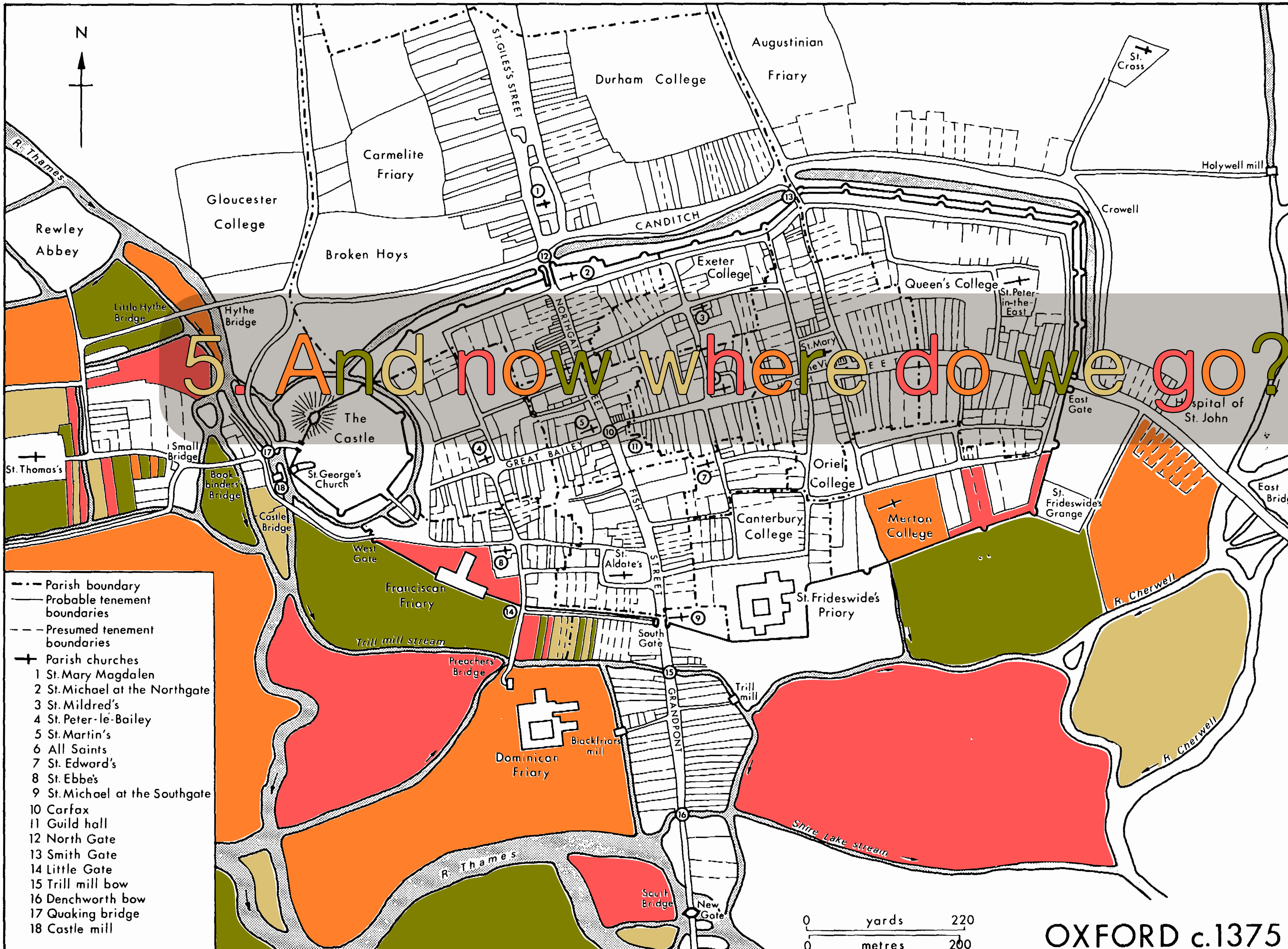
Moreover, Θ has some natural suboperads:

Θ_0 = the *non-symmetric* operad of **planar** 3-valent maps
= **ordered** linear lambda terms (i.e., no exchange rule)

Θ^2 = the *constant-free* operad of **bridgeless** maps
= linear terms **with no closed subterms** ("unitless")

Θ_0^2 = rooted bridgeless planar 3-valent maps
= ordered linear terms with no closed subterms

5. And now where do we go?



family of rooted maps	family of lambda terms	sequence	OEIS
trivalent maps (genus g≥0)	linear terms	1,5,60,1105,27120,...	A062980
planar trivalent maps	planar terms	1,4,32,336,4096,...	A002005
bridgeless trivalent maps	unitless linear terms	1,2,20,352,8624,...	A267827
bridgeless planar trivalent maps	unitless planar terms	1,1,4,24,176,1456,...	A000309
maps (genus g≥0)	normal linear terms (mod ~)	1,2,10,74,706,8162,...	A000698
planar maps	normal planar terms	1,2,9,54,378,2916,...	A000168
bridgeless maps	normal unitless linear terms (mod ~)	1,1,4,27,248,2830,...	A000699
bridgeless planar maps	normal unitless planar terms	1,1,3,13,68,399,...	A000260

1. O. Bodini, D. Gardy, A. Jacquot (2013), Asymptotics and random sampling for BCI and BCK lambda terms, TCS 502: 227-238
2. Z, A. Giorgetti (2015), A correspondence between rooted planar maps and normal planar lambda terms, LMCS 11(3:22): 1-39
3. Z (2015), Counting isomorphism classes of beta-normal linear lambda terms, arXiv:1509.07596
4. Z (2016), Linear lambda terms as invariants of rooted trivalent maps, J. Functional Programming 26(e21)
5. J. Courtiel, K. Yeats, Z (2016), Connected chord diagrams and bridgeless maps, arXiv:1611.04611
6. Z (2017), A sequent calculus for a semi-associative law, FSCD

OEIS = Online Encyclopedia of Integer Sequences (oeis.org)

A000168

We gave a bijective proof of the correspondence based on a simulation of Tutte's techniques in lambda calculus, albeit with an alternative convention for which lambda terms are "planar".

Finding a natural bijection between rooted planar maps and β-normal ordered terms is an open problem.

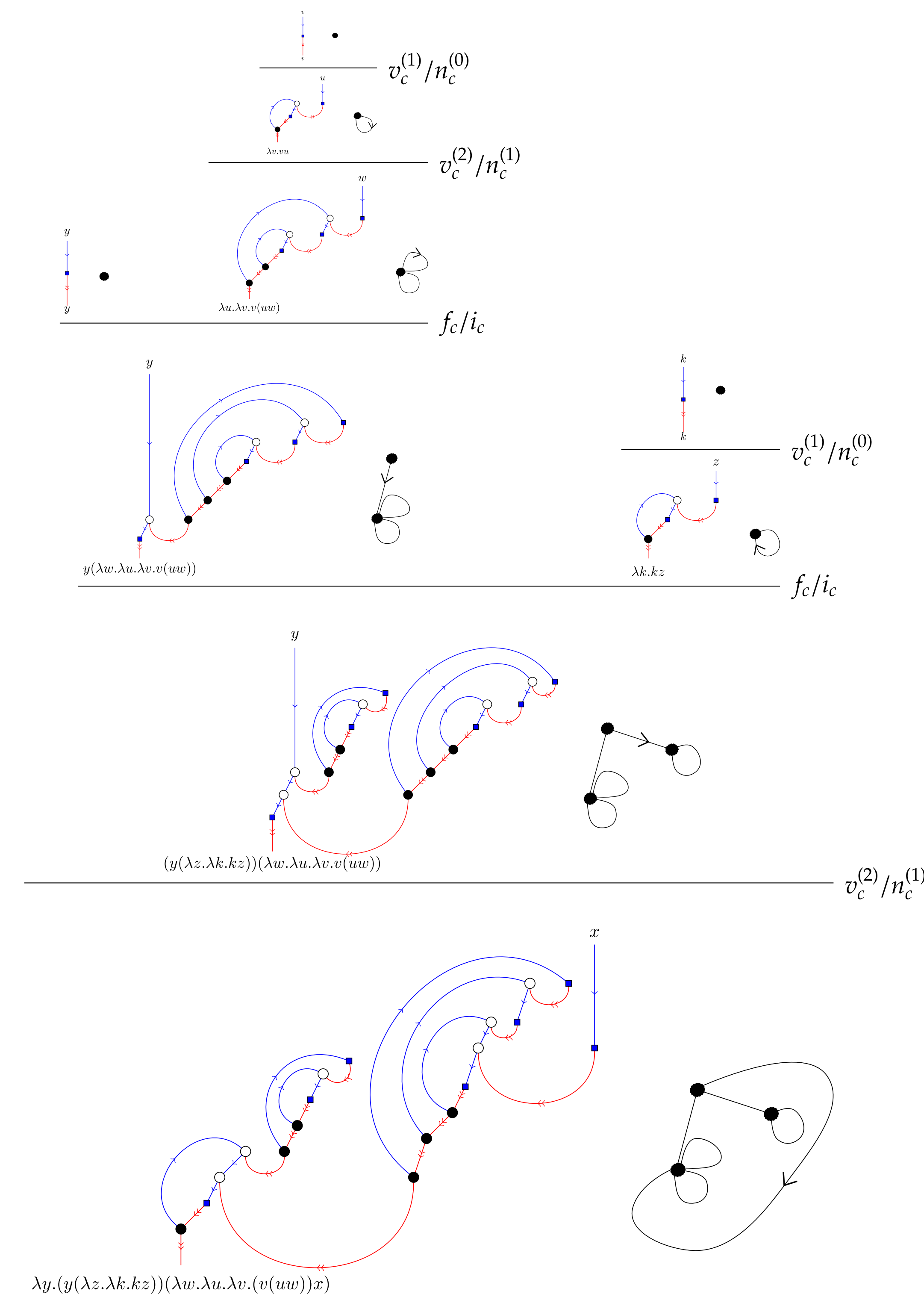


FIGURE 9. Full decomposition of a normal planar lambda term with seven s-nodes and three outer neutral handles, in parallel with the corresponding rooted planar map with six edges and outer face degree two.

A000698

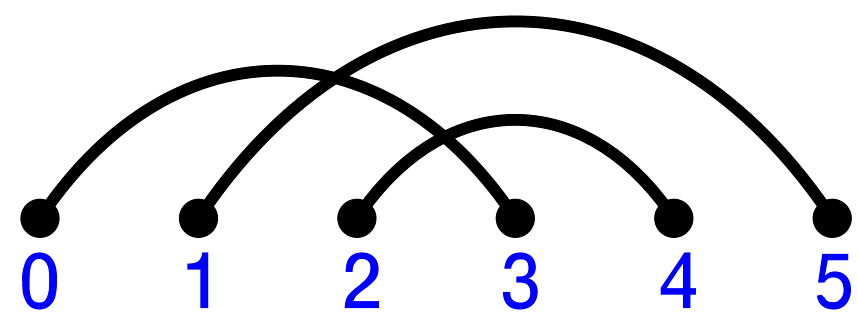
Quotient by the relation $\lambda x.\lambda y.t \sim \lambda y.\lambda x.t$.
(Perhaps more natural to think of this as an isomorphism between their principal types $A \multimap (B \multimap C) \approx B \multimap (A \multimap C)$...)

One can prove that the generating function counting equivalence classes of β-normal linear terms by size and free variables equals the GF counting rooted maps by edges and vertices.

Finding a natural bijection is an open problem.

A000699

A (rooted) **chord diagram** is a perfect matching on a linearly ordered set.

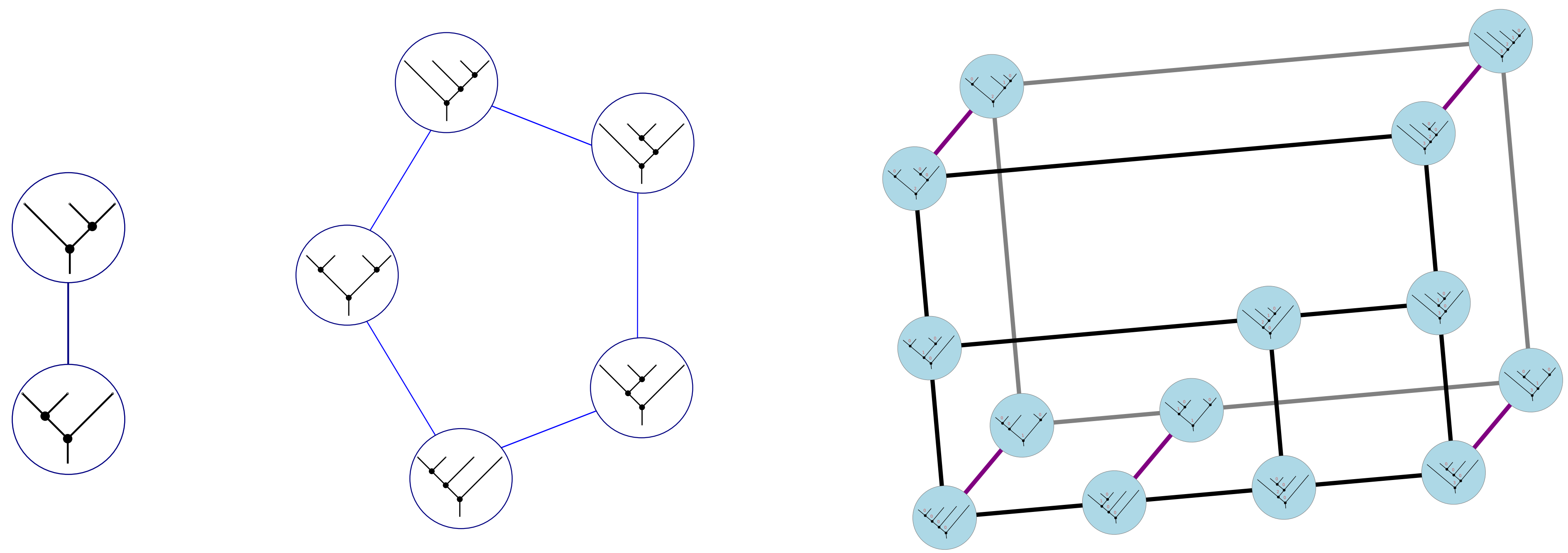


Indecomposable chord diagrams are in bijection with maps.

Connected chord diagrams are in bijection with bridgeless maps.

A000260

The **Tamari lattices** are the posets of binary trees ordered by rotation.



Intervals of Tamari lattices are in bijection with bridgeless planar maps.

family of rooted maps	family of lambda terms	sequence	OEIS
trivalent maps (genus g≥0)	linear terms	1,5,60,1105,27120,...	A062980
planar trivalent maps	planar terms	1,4,32,336,4096,...	A002005
bridgeless trivalent maps	unitless linear terms	1,2,20,352,8624,...	A267827
bridgeless planar trivalent maps	unitless planar terms	1,1,4,24,176,1456,...	A000309
maps (genus g≥0)	normal linear terms (mod ~)	1,2,10,74,706,8162,...	A000698
planar maps	normal planar terms	1,2,9,54,378,2916,...	A000168
bridgeless maps	normal unitless linear terms (mod ~)	1,1,4,27,248,2830,...	A000699
bridgeless planar maps	normal unitless planar terms	1,1,3,13,68,399,...	A000260

1. O. Bodini, D. Gardy, A. Jacquot (2013), Asymptotics and random sampling for BCI and BCK lambda terms, TCS 502: 227-238
2. Z, A. Giorgetti (2015), A correspondence between rooted planar maps and normal planar lambda terms, LMCS 11(3:22): 1-39
3. Z (2015), Counting isomorphism classes of beta-normal linear lambda terms, arXiv:1509.07596
4. Z (2016), Linear lambda terms as invariants of rooted trivalent maps, J. Functional Programming 26(e21)
5. J. Courtiel, K. Yeats, Z (2016), Connected chord diagrams and bridgeless maps, arXiv:1611.04611
6. Z (2017), A sequent calculus for a semi-associative law, FSCD
7. Z (2018), A theory of linear typings as flows on 3-valent graphs, LICS

OEIS = Online Encyclopedia of Integer Sequences (oeis.org)

Linear typing

$$\frac{\Gamma \vdash t : A \multimap B \qquad \Delta \vdash u : A}{\Gamma, \Delta \vdash t\,u : B}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}$$

$$\frac{}{x : A \vdash x : A}$$

$$\frac{\Gamma, y : B, x : A, \Delta \vdash t : C}{\Gamma, x : A, y : B, \Delta \vdash t : C}$$

Linear typings as flows

$$\frac{\Gamma \vdash t : A \multimap B \qquad \Delta \vdash u : A}{\Gamma, \Delta \vdash t\, u : B} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B} \qquad \frac{}{x : A \vdash x : A}$$

Why not draw types from a *group* G, with $A \multimap B := B \cdot A^{-1}$? A typing is then the same thing as a **G-flow** over the underlying oriented 3-valent map.

$$\frac{\Gamma, y : B, x : A, \Delta \vdash t : C}{\Gamma, x : A, y : B, \Delta \vdash t : C}$$

Proposition: Every unitless ordered linear term has a typing in $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ such that no subterm is assigned the type (0,0).

Linear typings as flows

$$\frac{\Gamma \vdash t : A \multimap B \qquad \Delta \vdash u : A}{\Gamma, \Delta \vdash t\,u : B}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}$$

$$\frac{}{x : A \vdash x : A}$$

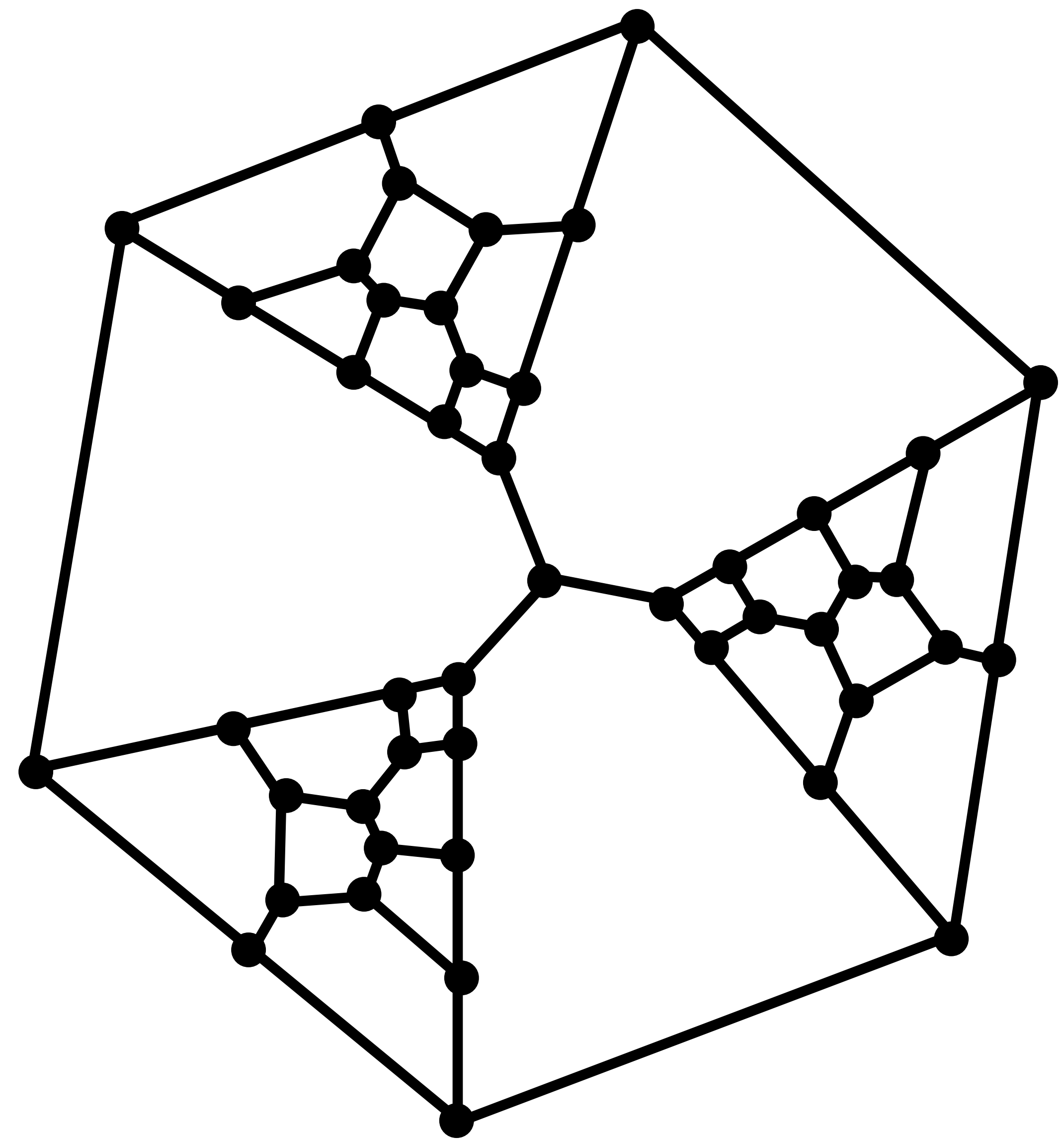
$$\frac{\Gamma, y : B, x : A, \Delta \vdash t : C}{\Gamma, x : A, y : B, \Delta \vdash t : C}$$

Why not draw types from a *group* G, with $A \multimap B := B \cdot A^{-1}$? A typing is then the same thing as a **G-flow** over the underlying oriented 3-valent map.

Proposition: Every unitless ordered linear term has a typing in $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ such that no subterm is assigned the type (0,0).

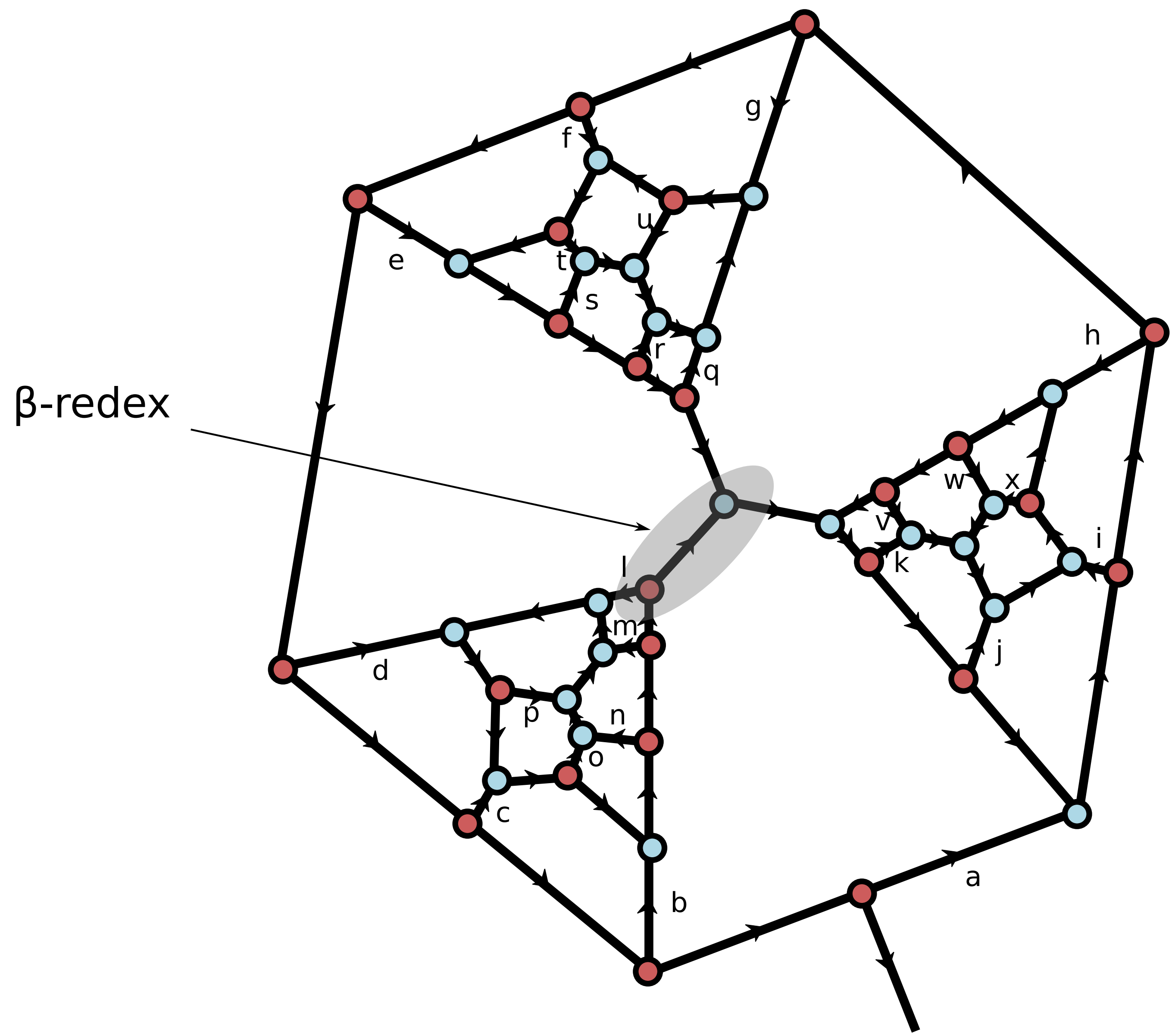
(Proof: This is equivalent to 4CT.)

Example ("the Tutte graph")



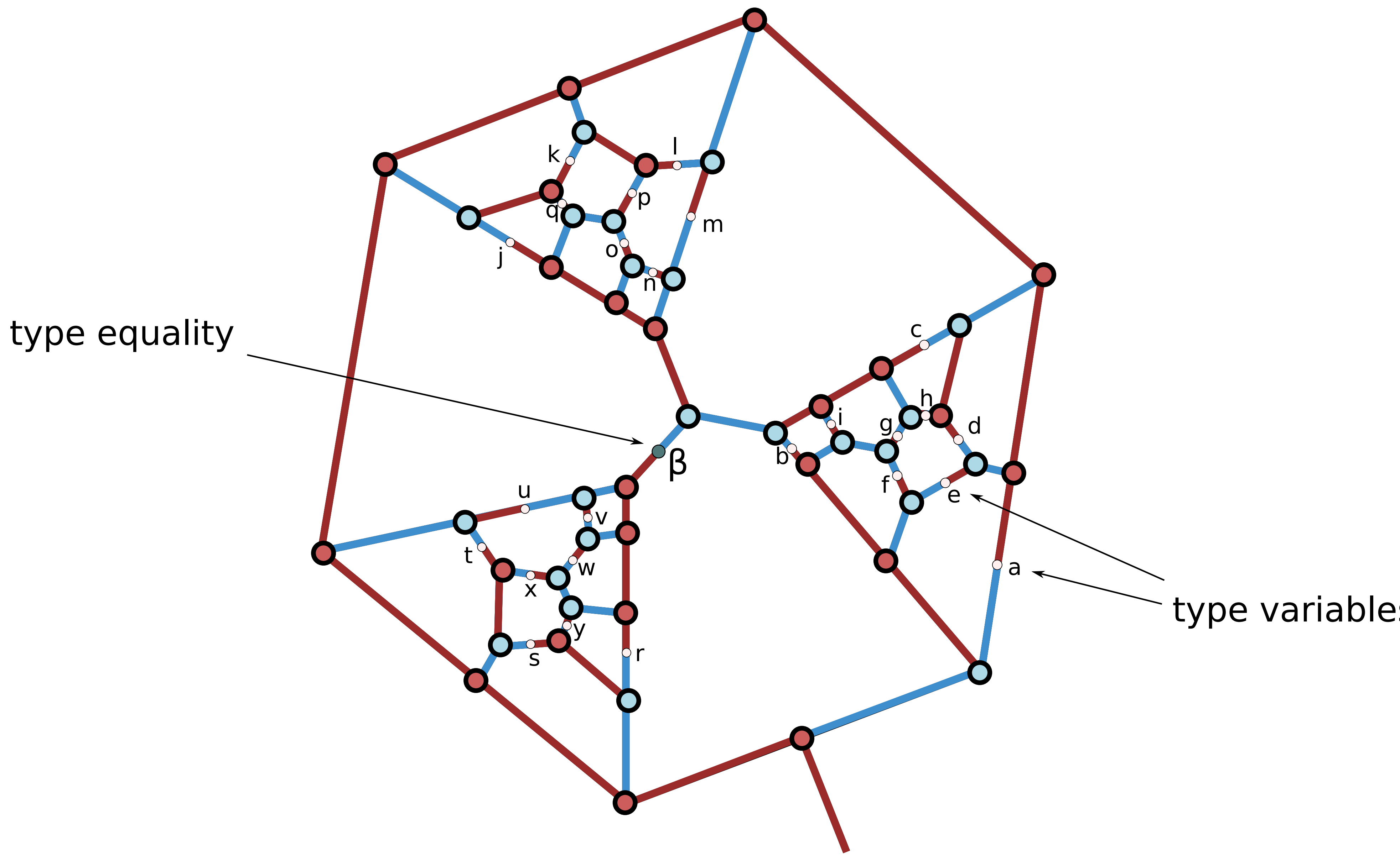
(From W. T. Tutte, “On Hamiltonian Circuits”, *Journal of the London Mathematical Society* 21 (1946), 98–101.)

The associated lambda term

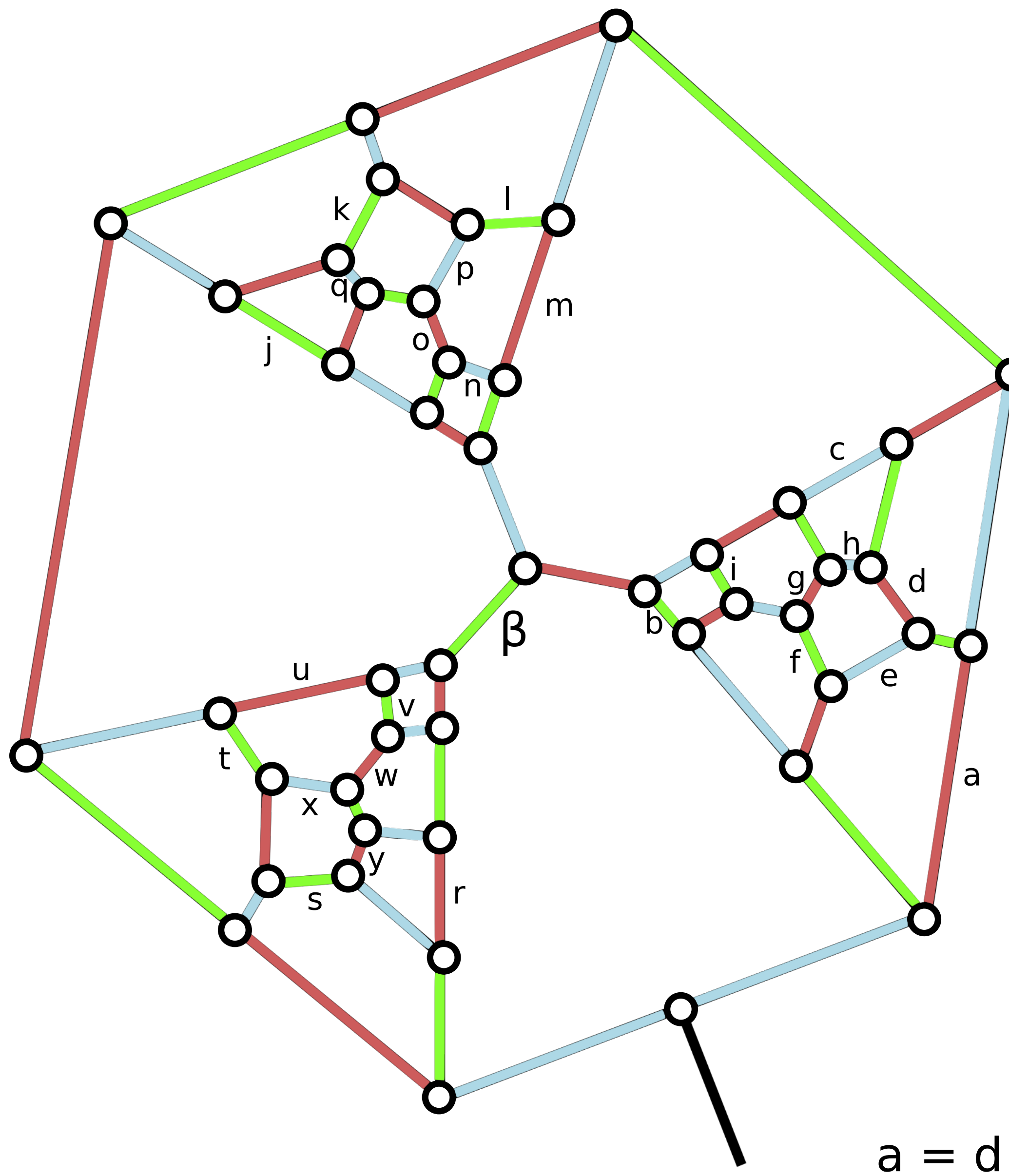


$$\lambda a \lambda b \lambda c \lambda d \lambda e \lambda f \lambda g \lambda h \lambda i.a(\lambda j \lambda k.\underline{((\lambda l \lambda m \lambda n.b(\lambda o.c(\lambda p.d(l(m((no)p))))))(\lambda q \lambda r \lambda s.e(\lambda t.f(\lambda u.g(q(r((st)u)))))))}(\lambda v \lambda w.h(\lambda x.i(j((kv)(wx))))))$$

The principal typing



A $\mathbb{Z}_2 \times \mathbb{Z}_2$-typing



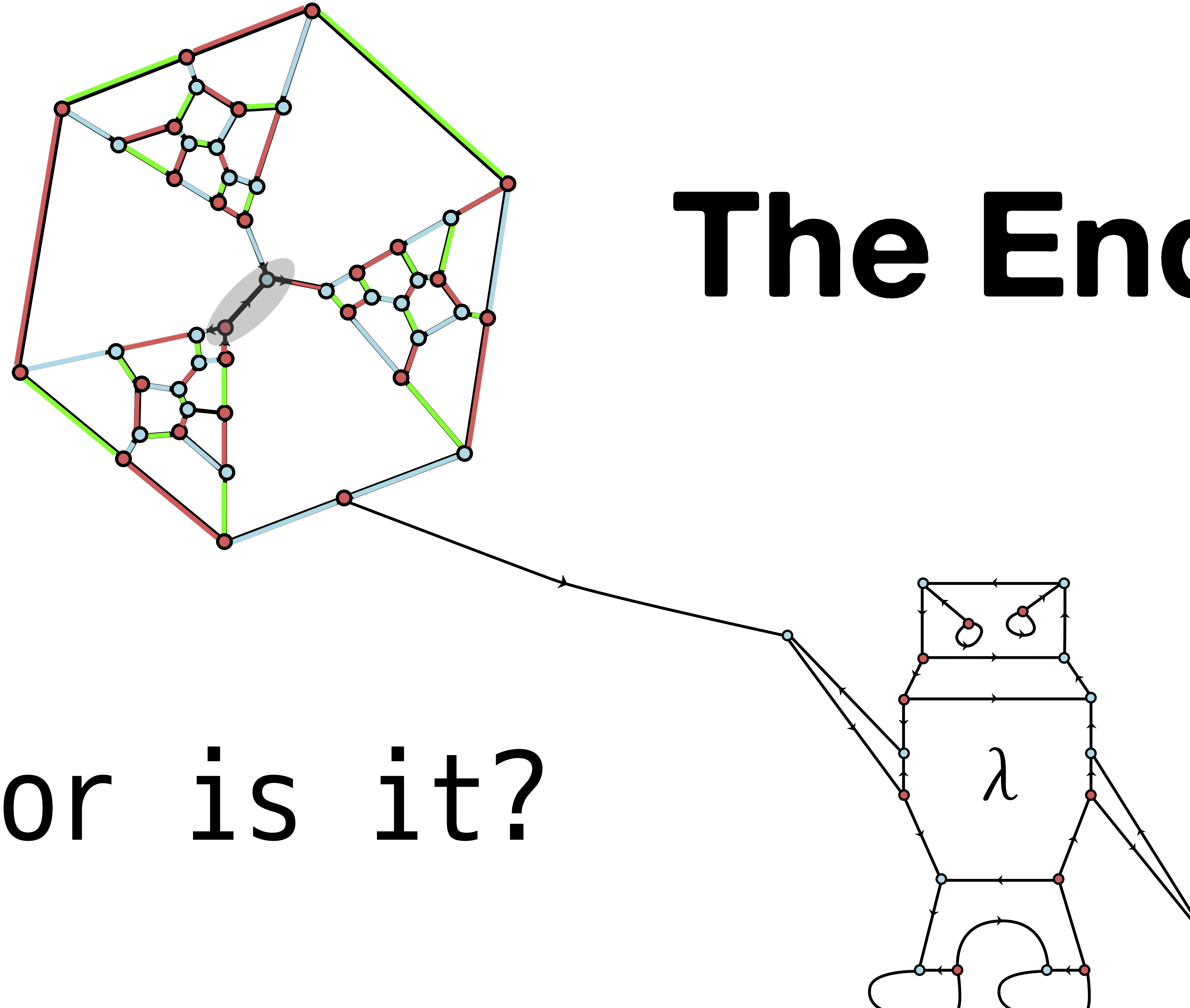
$a = d = g = m = o = r = u = w = y = R$

$b = f = i = j = k = l = s = t = v = G$

$c = e = h = n = p = q = x = B$

$\beta : G = G$

The End



...or is it?