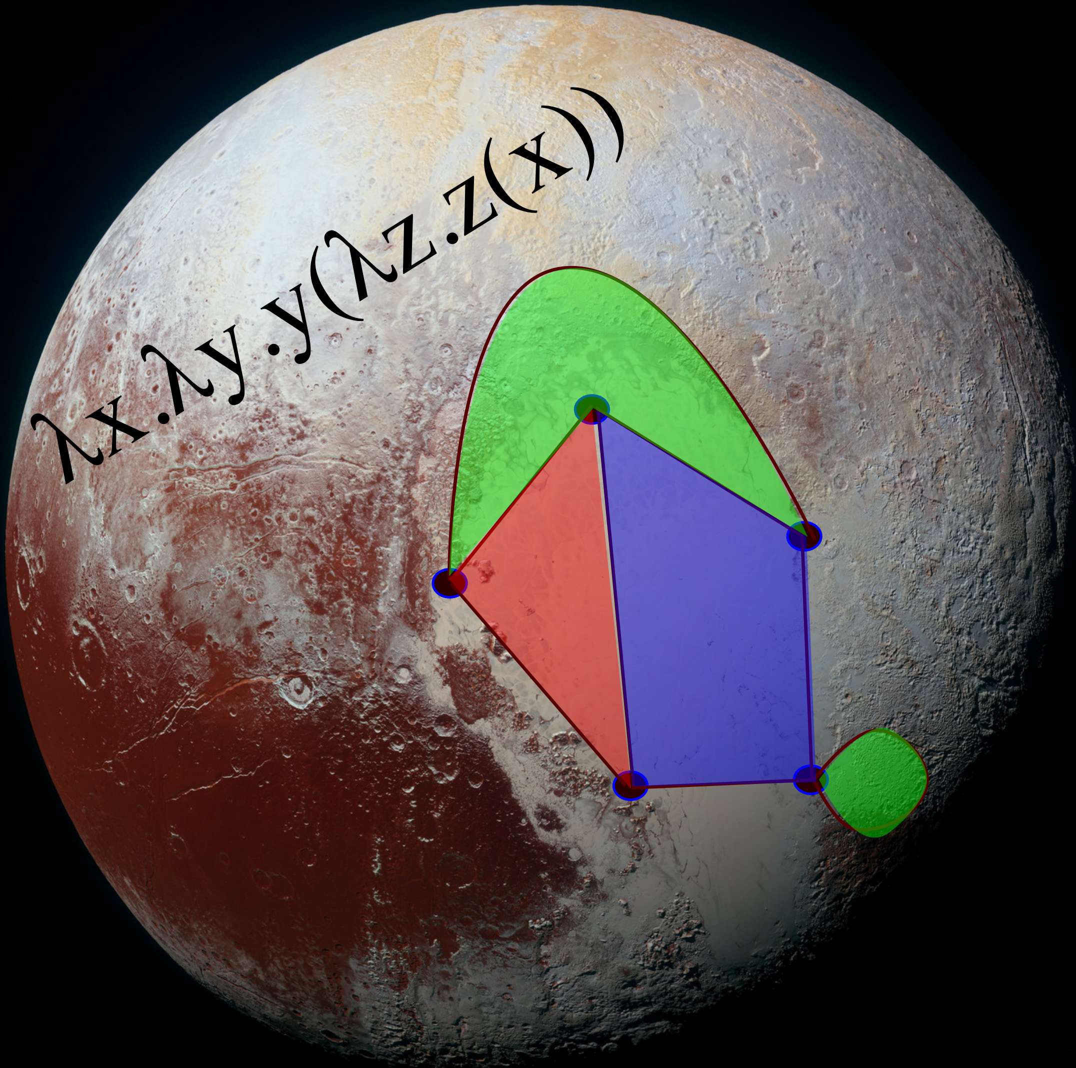


Linear lambda calculus and the combinatorics of embedded graphs

Noam Zeilberger

March 14, 2015

based in part on joint work with Alain Giorgetti

Journées Nationales GEOCAL-LAC-LTP 2015

```
\x -> x
```

1

```
\x -> x (\y -> y)
\x -> \y -> y(x)
```

2

```
\x -> x(\y -> y(\z -> z))
\x -> x(\y -> \z -> z(y))
\x -> x(\y -> y)(\z -> z)
\x -> \y -> y(x(\z -> z))
\x -> \y -> y(\z -> z(x))
\x -> \y -> y(\z -> z)(x)
\x -> \y -> y(x)(\z -> z)
\x -> \y -> \z -> z(y(x))
\x -> \y -> \z -> z(y)(x)
```

9

```
\x -> x(\y -> y(\z -> z(\w -> w)))
\x -> x(\y -> y(\z -> \w -> w(z)))
\x -> x(\y -> y(\z -> z)(\w -> w))
\x -> x(\y -> \z -> z(y(\w -> w)))
\x -> x(\y -> \z -> z(\w -> w(y)))
\x -> x(\y -> \z -> z(\w -> w)(y))
\x -> x(\y -> \z -> z(y)(\w -> w))
\x -> x(\y -> \z -> \w -> w(z(y)))
\x -> x(\y -> \z -> \w -> w(z)(y))
\x -> x(\y -> y)(\z -> z(\w -> w))
\x -> x(\y -> y)(\z -> \w -> w(z))
\x -> x(\y -> y(\z -> z))(\w -> w)
\x -> x(\y -> \z -> z(y))(\w -> w)
\x -> x(\y -> y)(\z -> z)(\w -> w)
\x -> \y -> y(x(\z -> z(\w -> w)))
\x -> \y -> y(x(\z -> \w -> w(z)))
\x -> \y -> y(x(\z -> z)(\w -> w))
\x -> \y -> y(\z -> z(x(\w -> w)))
\x -> \y -> y(\z -> z(\w -> w(x)))
\x -> \y -> y(\z -> z(\w -> w)(x))
\x -> \y -> y(\z -> z(x)(\w -> w))
\x -> \y -> y(\z -> \w -> w(z(x)))
\x -> \y -> y(\z -> \w -> w(z)(x))
\x -> \y -> y(\z -> z)(x(\w -> w))
\x -> \y -> y(\z -> z)(\w -> w(x))
\x -> \y -> y(\z -> z(\w -> w))(x)
\x -> \y -> y(\z -> \w -> w(z))(x)
```

```
\x -> \y -> y(\z -> z)(\w -> w)(x)
\x -> \y -> y(x)(\z -> z(\w -> w))
\x -> \y -> y(x)(\z -> \w -> w(z))
\x -> \y -> y(x(\z -> z))(\w -> w)
\x -> \y -> y(\z -> z(x))(\w -> w)
\x -> \y -> y(\z -> z)(x)(\w -> w)
\x -> \y -> y(x)(\z -> z)(\w -> w)
\x -> \y -> \z -> z(y(x(\w -> w)))
\x -> \y -> \z -> z(y(\w -> w(x)))
\x -> \y -> \z -> z(y(\w -> w)(x))
\x -> \y -> \z -> z(y(x)(\w -> w))
\x -> \y -> \z -> z(\w -> w(y(x)))
\x -> \y -> \z -> z(\w -> w(y)(x))
\x -> \y -> \z -> z(\w -> w)(y(x))
\x -> \y -> \z -> z(y)(x(\w -> w))
\x -> \y -> \z -> z(y)(\w -> w(x))
\x -> \y -> \z -> z(y(\w -> w))(x)
\x -> \y -> \z -> z(\w -> w(y))(x)
\x -> \y -> \z -> z(\w -> w)(y)(x)
\x -> \y -> \z -> z(y)(\w -> w)(x)
\x -> \y -> \z -> z(y(x))(\w -> w)
\x -> \y -> \z -> z(y)(x)(\w -> w)
\x -> \y -> \z -> \w -> w(z(y(x)))
\x -> \y -> \z -> \w -> w(z(y)(x))
\x -> \y -> \z -> \w -> w(z)(y(x))
\x -> \y -> \z -> \w -> w(z(y))(x)
\x -> \y -> \z -> \w -> w(z)(y)(x)
```

54

Invitation: celebrating 50 years of OEIS, 250000 sequences, and Sloane's 75th, there will be a conference at DIMACS, Rutgers, Oct 9-10 2014.

1,2,9,54,378,2916,24057 | Search | Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,2,9,54,378,2916,24057**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A000168 2*3^n*(2*n)!/(n!*(n+2)!). +20 18

(Formerly M1940 N0768)

1, 2, 9, 54, 378, 2916, 24057, 208494, 1876446, 17399772, 165297834, 1602117468, 15792300756, 157923007560, 1598970451545, 16365932856990, 169114639522230, 1762352559231660, 18504701871932430, 195621134074714260, 2080697516976506220, 22254416920705240440, 239234981897581334730, 2583737804493878415084 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

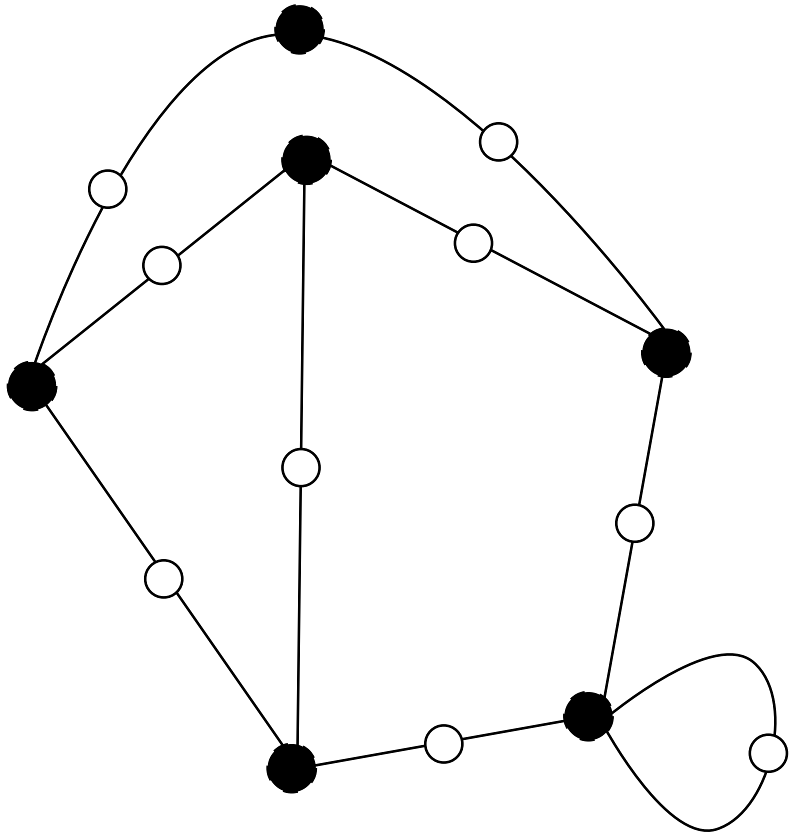
COMMENTS Number of rooted planar maps with n edges. - Don Knuth, Nov 24 2013

Number of rooted 4-regular planar maps with n vertices.

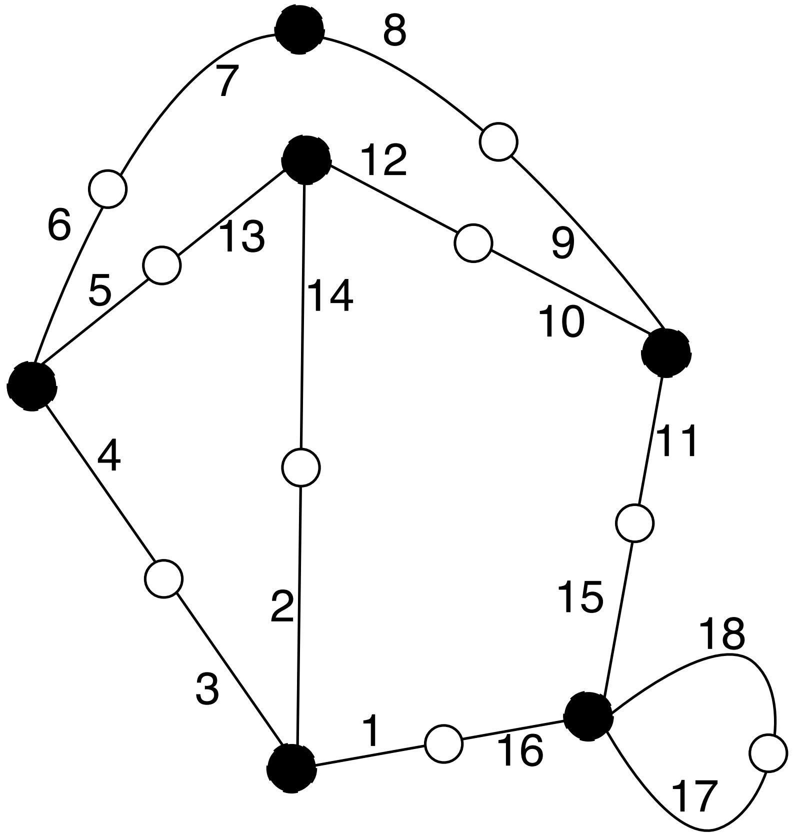
Also, number of doodles with n crossings, irrespective of the number of loops.

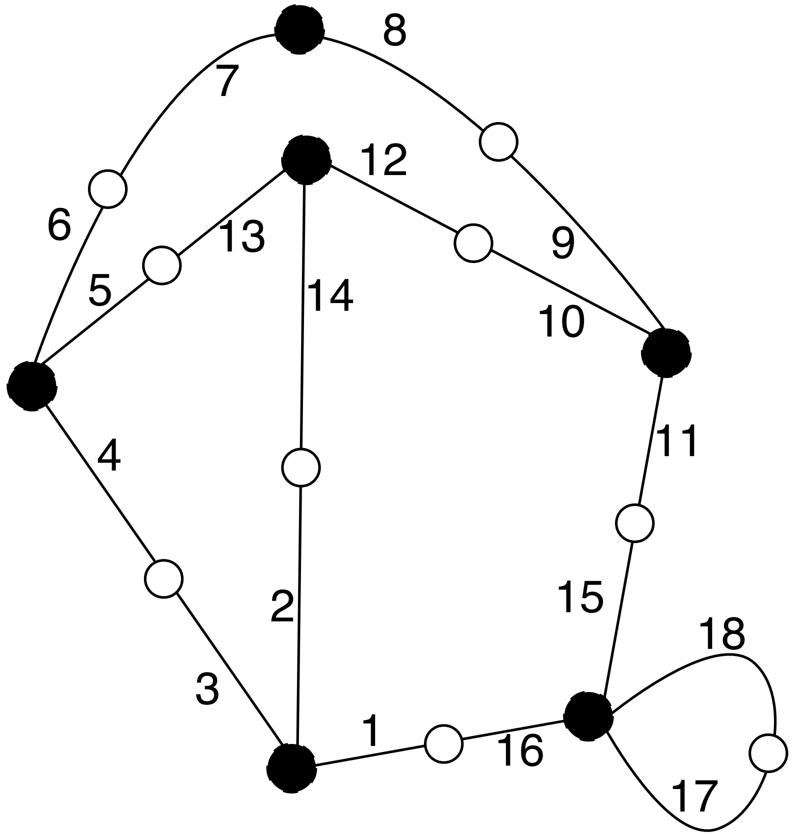
What is a (rooted planar) map?

1. Pick a surface



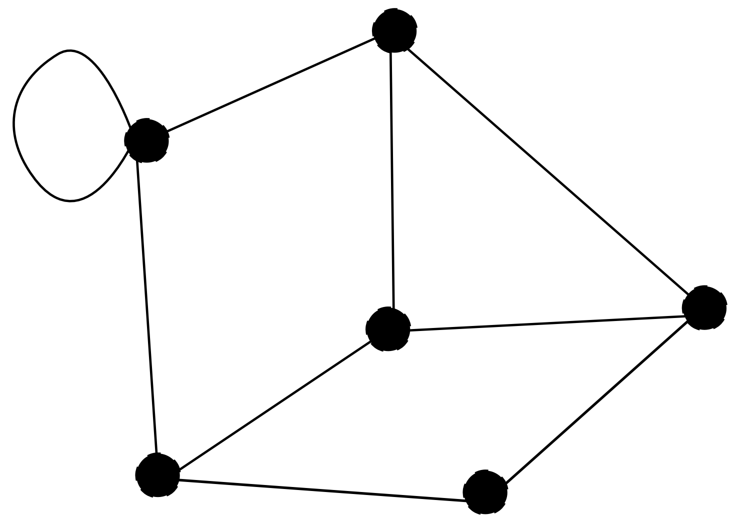
D = { 1...18 }





D = { 1...18 }

V = (1 2 3)(4 5 6)(7 8)(9 10 11)(12 13 14)(15 16 17 18)
E = (1 16)(2 14)(3 4)(5 13)(6 7)(8 9)(10 12)(11 15)(17 18)
F = (1 15 10 14)(2 13 4)(3 6 8 11 18 16)(5 12 9 7)(17)



D = { 1...18 }

V = (1 2 3)(4 5 6)(7 8)(9 10 11)(12 13 14)(15 16 17 18)
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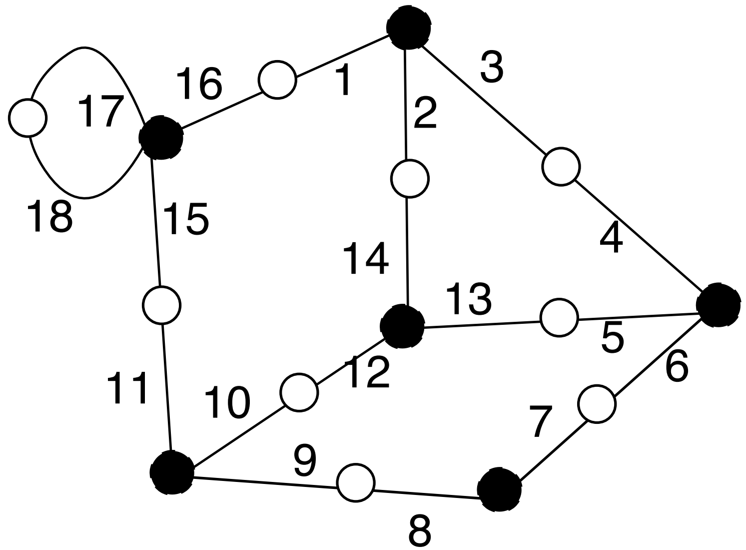
F = (1 15 10 14)(2 13 4)(3 6 8 11 18 16)(5 12 9 7)(17)



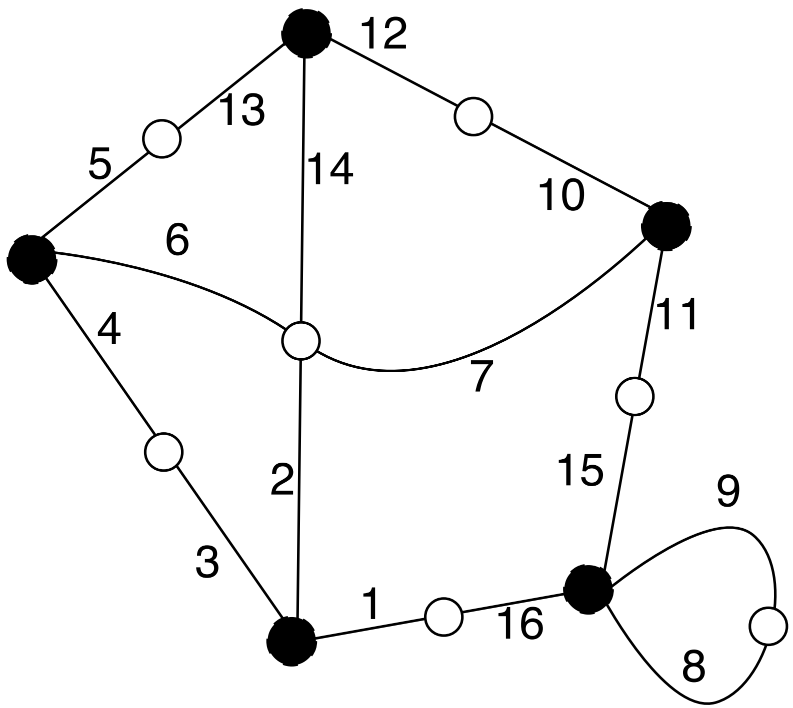
D = { 1...18 }

V = (1 2 3)(4 5 6)(7 8)(9 10 11)(12 13 14)(15 16 17 18)
E = (1 16)(2 14)(3 4)(5 13)(6 7)(8 9)(10 12)(11 15)(17 18)
F = (1 15 10 14)(2 13 4)(3 6 8 11 18 16)(5 12 9 7)(17)

A natural generalization

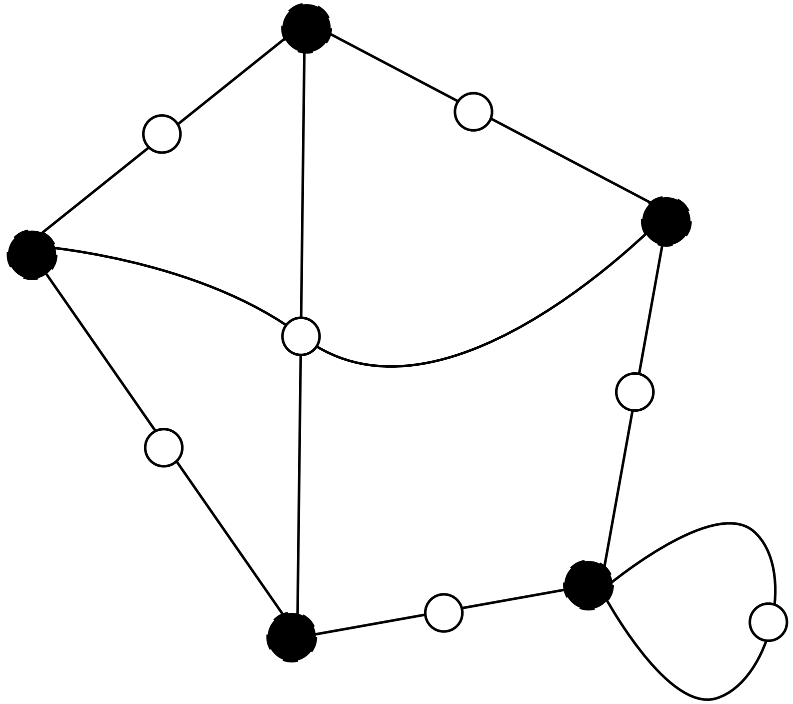
a **combinatorial hypermap** consists of a set D equipped with a triple of permutations (V,E,F) such that:

1. <V,E,F> acts transitively on D
2. ~~E is a fixed-point free involution~~
3. V;E;F = id

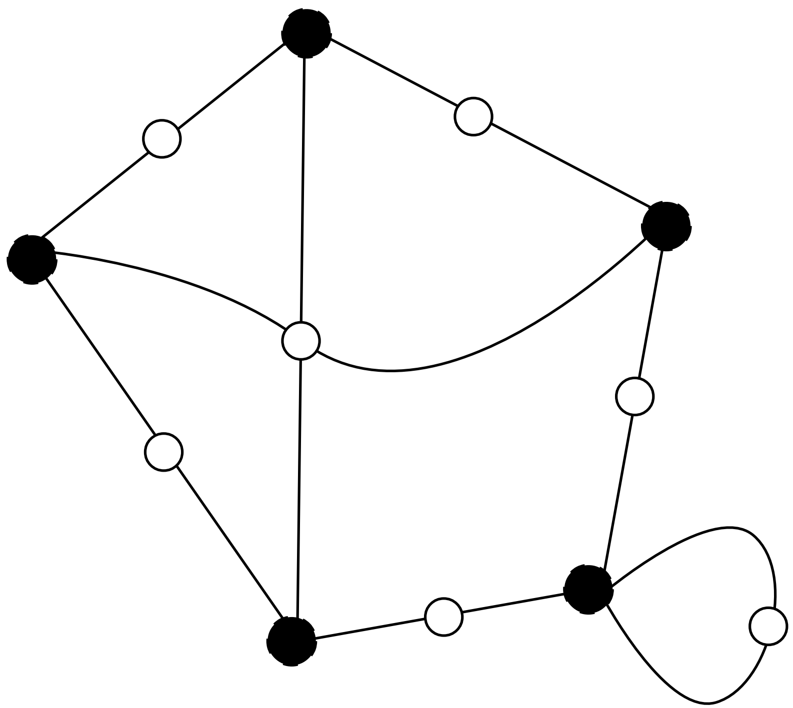


D = { 1...16 }

V = (1 2 3)(4 6 5)(7 11 10)(8 9 15 16)(12 13 14)
E = (1 16)(2 7 14 6)(3 4)(5 13)(8 9)(10 12)(11 15)
F = (1 15 7)(2 4)(3 5 12 11 9 16)(6 13)(8)(10 14)



Alors que dans mes recherches d'avant 1970, mon attention systématiquement était dirigée vers les objets de généralité maximale, afin de dégager un langage d'ensemble adéquat pour le monde de la géométrie algébrique, et que je ne m'attardais sur les courbes algébriques dans la stricte mesure où cela s'avérait indispensable (notamment en cohomologie étale) pour développer des techniques et énoncés "passe-partout" valables en toute dimension et en tous lieux (j'entends, sur tous schémas de base, voire tous topos annelés de base...), me voici donc ramené, par le truchement d'objets si simples qu'un enfant peut les connaître en jouant, aux débuts et origines de la géométrie algébrique, familiers à Riemann et à ses émules!



A. Grothendieck, "Esquisse d'un programme" (1984)

(quoted in Lando & Zvonkin, *Graphs on Surfaces and their Applications*)

August Möbius found a lot of applications for his discoveries.

The **oriented cartographic group**

$$\mathcal{C}_2^+ = \langle a, b, c \mid b^2 = abc = 1 \rangle$$

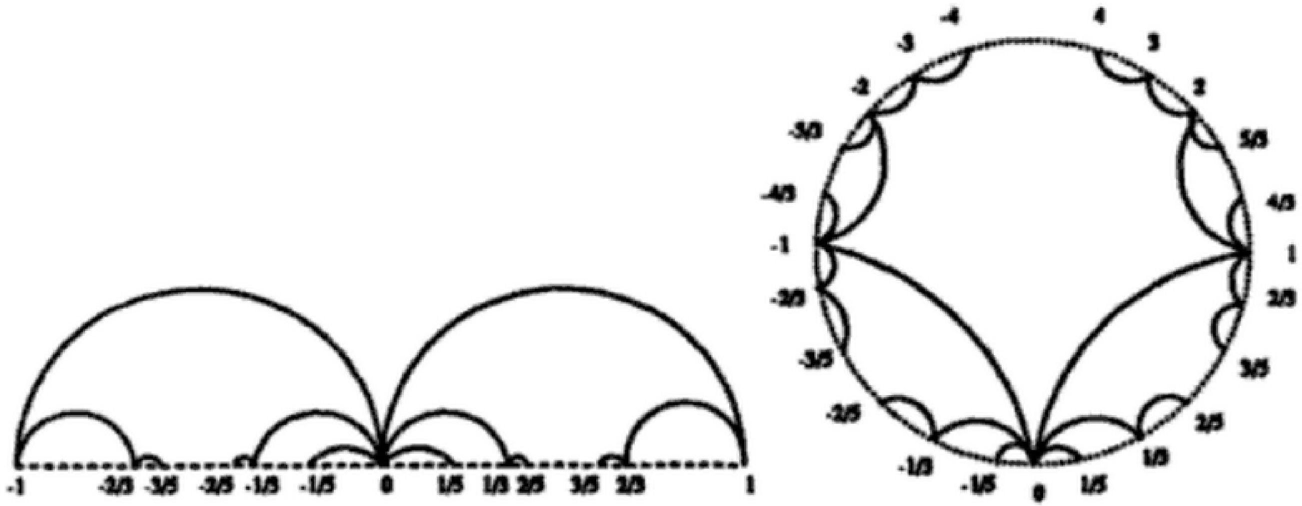
is an index 2 subgroup of the **cartographic group**

$$\mathcal{C}_2 = \langle x, y, z \mid x^2 = y^2 = z^2 = (xz)^2 = 1 \rangle$$

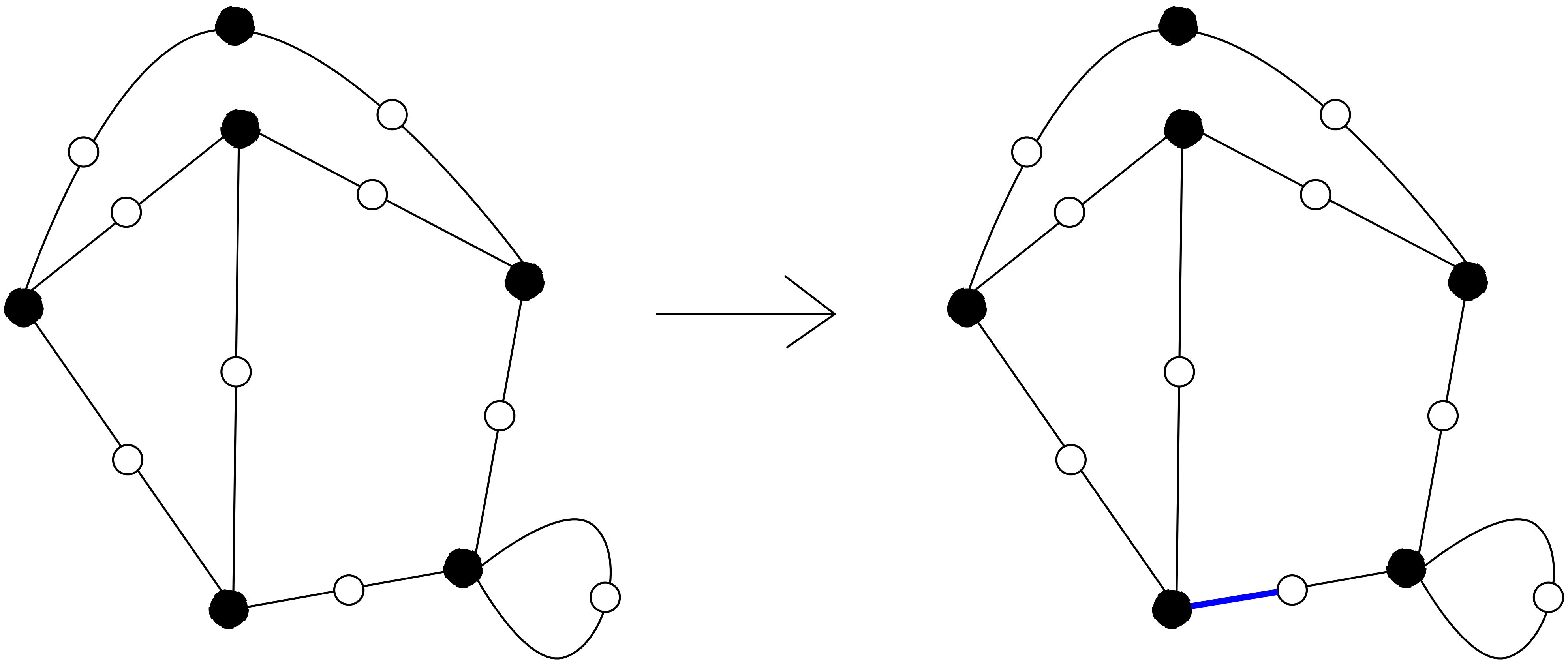
Transitive permutation representations of C_2 give a faithful encoding of maps on unoriented surfaces, possibly with boundary.

From "Maps, hypermaps, and triangle groups" by Gareth Jones and David Singerman, in *The Grothendieck Theory of Dessins d'Enfants*, Leila Schneps (ed.), CUP, 1994.

so in this case the identification is given by $\rho_i \mapsto U_i$ $(i = 0, 1, 2)$. The corresponding universal map $\hat{\mathcal{M}} = \hat{\mathcal{M}}(\infty, \infty)$ (which first appeared in [Si2]) is illustrated in Figure 5; its vertices are the rationals with odd denominator, and a/b is joined to c/d if and only if $ad - bc = \pm 1$.



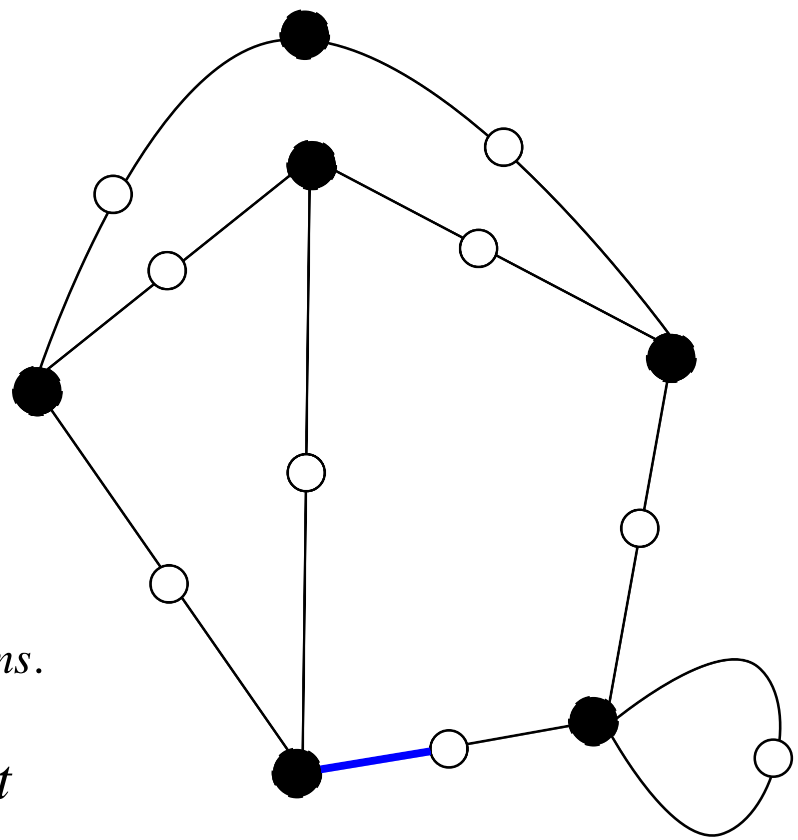
Rooting a map



Having made no progress with the enumeration of these diagrams I bethought myself of Cayley's work on the enumeration of trees. His first successes had been with the rooted trees, in which one vertex is distinguished as the "root". Perhaps I should root the strict triangulations in some way and try to enumerate the rooted ones. Eventually I decided that the rooting should consist of the choice of a face, edge and vertex, mutually incident….

I am not sure what I would have replied at this stage if I had been asked why I preferred these rooted triangulations to the unrooted ones…. Later I realized that the distinguishing feature and the great advantage of rooted maps is that they have no symmetry. Automorphisms seem to complicate enumerative problems.

from W. T. Tutte, *Graph Theory As I Have Known It*



A CENSUS OF PLANAR MAPS

(1963, Canad. J. of Math.)

W. T. TUTTE

The number a_n of rooted maps with n edges is

$$\frac{2(2n)!\,3^n}{n!\,(n+2)!}.$$

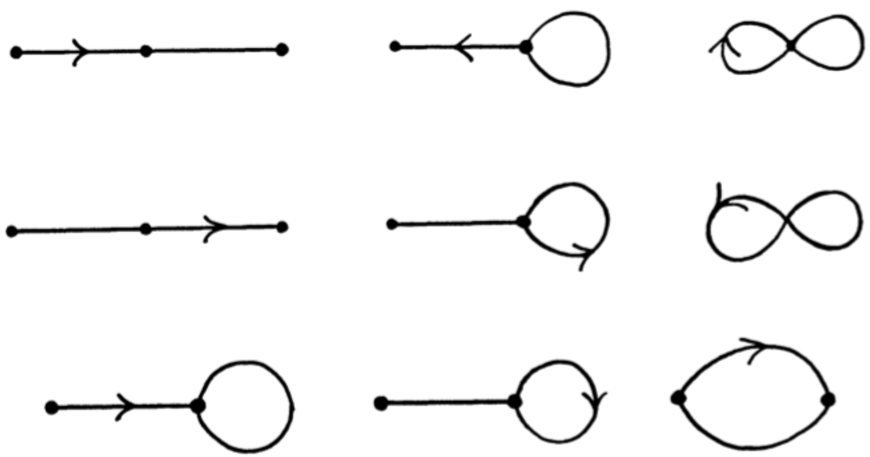
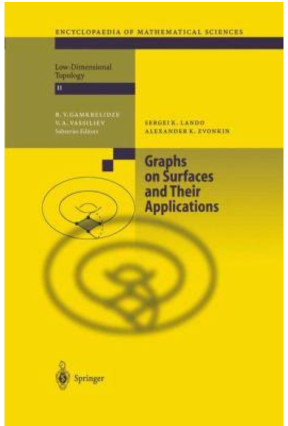


FIGURE 3.



From Ch. 3 ("Matrix Integrals") of Lando & Zvonkin:

This chapter is also a message to enumerative combinatorialists: among a large variety of objects one can enumerate, maps are of special interest because of their revealed relations to physics. If you hesitate what to enumerate, choose maps.

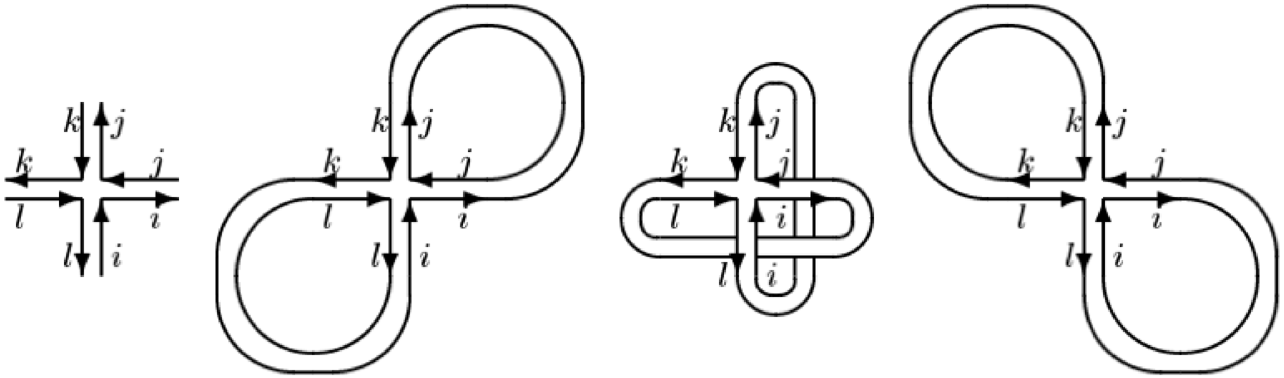
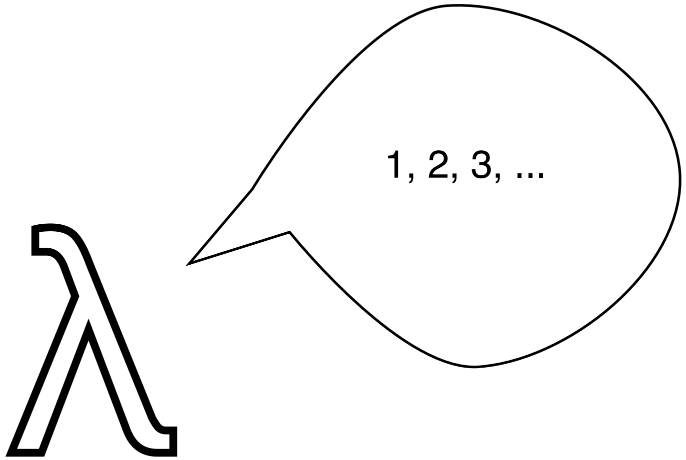


Figure 3.7: Feynman diagrams for the 4-star

Counting linear lambda terms



Generalities

$$P(z) = \sum_{n=0}^{\infty} a_n z^n$$

ordinary generating function

$$Q(x) = \sum_{k=0}^{\infty} b_k \frac{x^k}{k!}$$

exponential generating function

Let $t_{n,k}$ = # linear lambda terms of size n with k free variables

Define $$L(z,x) = \sum_{n,k} t_{n,k} \frac{z^n x^k}{k!}$$

$$L(z,x) = zx + L(z,x)^2 + \frac{\partial}{\partial x} L(z,x)$$

$$L(z,0) = z + 5z^2 + 60z^3 + 1105z^4 + 27120z^5 + \dots$$

```
\x -> x
```

```
(\x -> x)(\y -> y)
\x -> x(\y -> y)
\x -> (\y -> y)(x)
\x -> \y -> x(y)
\x -> \y -> y(x)
```

```
(\x -> x)((\y -> y)(\z -> z))
(\x -> x)(\y -> y(\z -> z))
(\x -> x)(\y -> (\z -> z)(y))
(\x -> x)(\y -> \z -> y(z))
(\x -> x)(\y -> \z -> z(y))
(\x -> x)(\y -> y)(\z -> z)
(\x -> x(\y -> y))(\z -> z)
(\x -> (\y -> y)(x))(\z -> z)
(\x -> \y -> x(y))(\z -> z)
(\x -> \y -> y(x))(\z -> z)
\x -> x((\y -> y)(\z -> z))
\x -> x(\y -> y(\z -> z))
\x -> x(\y -> (\z -> z)(y))
\x -> x(\y -> \z -> y(z))
\x -> x(\y -> \z -> z(y))
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\x -> (\y -> y)(x)(\z -> z)
\x -> (\y -> x(y))(\z -> z)
\x -> (\y -> y(x))(\z -> z)
\x -> (\y -> y)(x(\z -> z))
\x -> (\y -> y)((\z -> z)(x))
\x -> (\y -> y)(\z -> x(z))
\x -> (\y -> y)(\z -> z(x))
\x -> (\y -> y)(\z -> z)(x)
\x -> (\y -> y(\z -> z))(x)
\x -> (\y -> (\z -> z)(y))(x)
\x -> (\y -> \z -> y(z))(x)
\x -> (\y -> \z -> z(y))(x)
\x -> \y -> x(y)(\z -> z)
\x -> \y -> y(x)(\z -> z)
```

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\x -> \y -> x(y(\z -> z))
\x -> \y -> x((\z -> z)(y))
\x -> \y -> x(\z -> y(z))
\x -> \y -> x(\z -> z(y))
\x -> \y -> x(\z -> z)(y)
\x -> \y -> (\z -> z)(x)(y)
\x -> \y -> (\z -> x(z))(y)
\x -> \y -> (\z -> z(x))(y)
\x -> \y -> y(x(\z -> z))
\x -> \y -> y((\z -> z)(x))
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\x -> \y -> (\z -> z)(x(y))
\x -> \y -> (\z -> z)(y(x))
\x -> \y -> \z -> x(y)(z)
\x -> \y -> \z -> y(x)(z)
\x -> \y -> \z -> x(z)(y)
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\x -> \y -> \z -> x(y(z))
\x -> \y -> \z -> x(z(y))
\x -> \y -> \z -> y(z)(x)
\x -> \y -> \z -> z(y)(x)
\x -> \y -> \z -> y(x(z))
\x -> \y -> \z -> y(z(x))
\x -> \y -> \z -> z(x(y))
\x -> \y -> \z -> z(y(x))
```

1,5,60,1105,27120,828250 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,5,60,1105,27120,828250**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A062980 a(0) = 1, a(1) = 5; for n>1, a(n) = 6n*a(n-1) + Sum_{k=1..n-2} a(k)*a(n-k-1). +20 4

1, 5, 60, 1105, 27120, 828250, 30220800, 1282031525, 61999046400, 3366961243750, 202903221120000, 13437880555850250, 970217083619328000, 75849500508999712500, 6383483988812390400000, 575440151532675686278125 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS Number of rooted unlabeled connected triangular maps on a compact closed oriented surface with 2n faces (and thus 3n edges). [Vidal]

Equivalently, the number of pair of permutations (sigma,tau) up to simultaneous conjugacy on a pointed set of size 6*n with sigma^3=tau^2=1, acting transitively and with no fixed point. [Vidal]

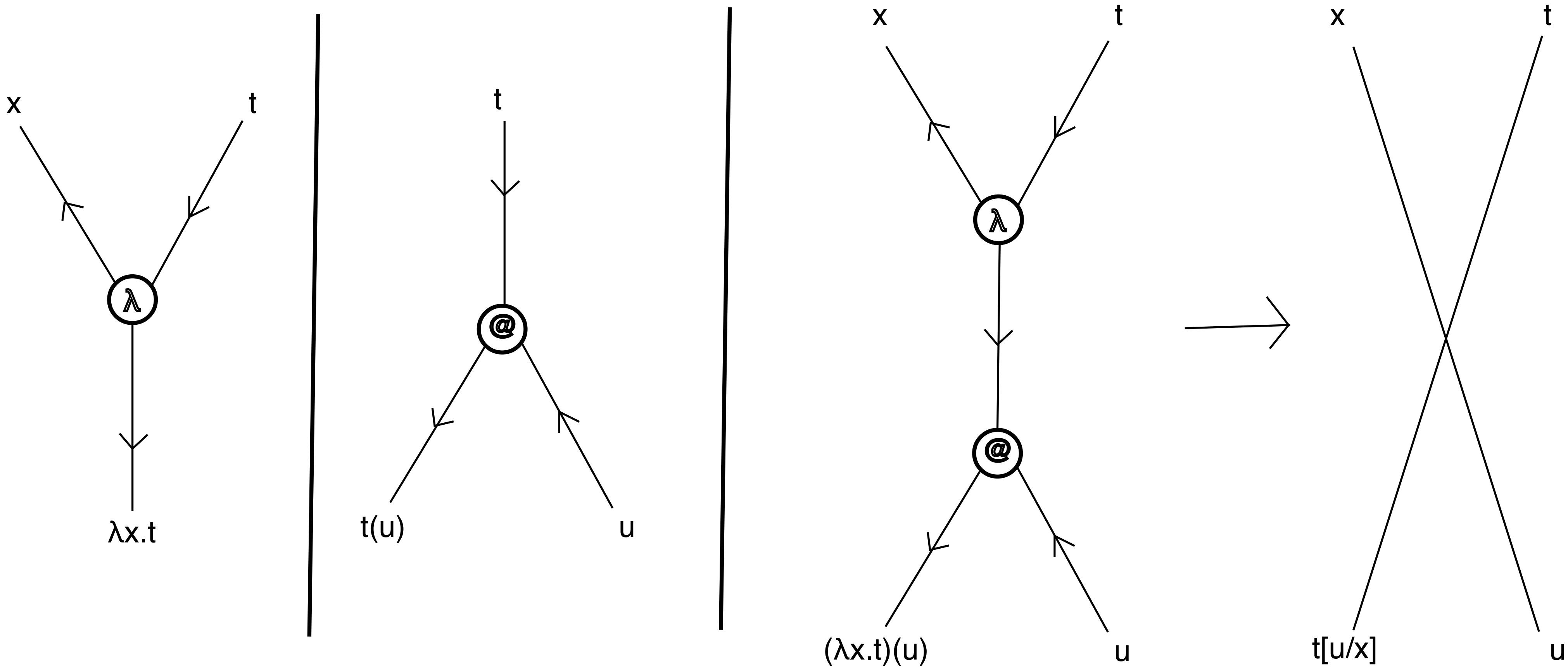
Combinatorics of trivalent/triangular maps (= transitive representations of PSL(Z,2)) studied in:

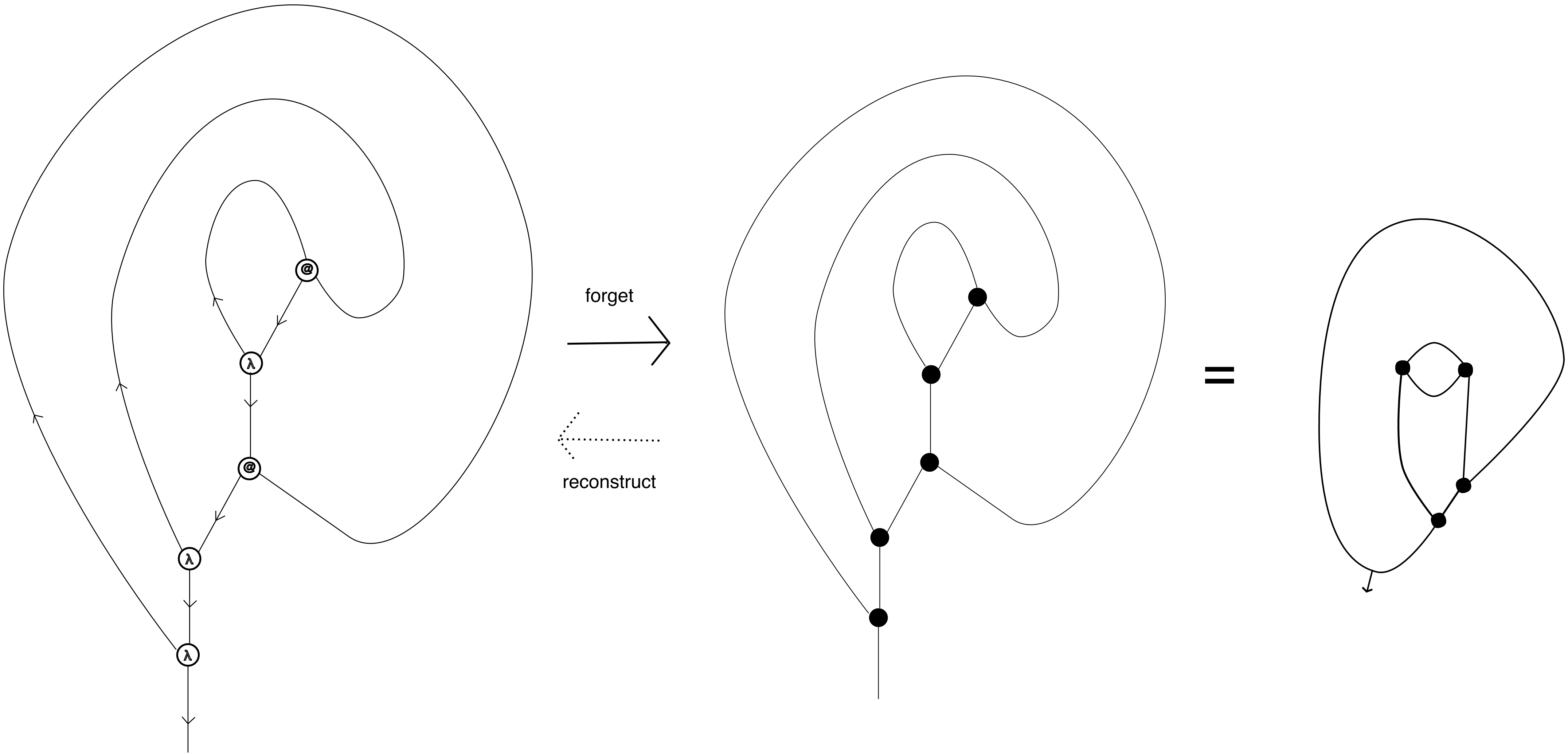
Samuel Vidal. *Groupe Modulaire et Cartes Combinatoires: Génération et Comptage.* PhD thesis, Université Lille I, France, July 2010.

A bijection [closed linear lambda terms <-> rooted trivalent maps] is given in:

Olivier Bodini, Danièle Gardy, and Alice Jacquot. Asymptotics and random sampling for BCI and BCK lambda terms. *Theoretical Computer Science*, 502:227–238, 2013.

String diagrams for linear lambda terms





λx.λy.(λz.zy)(x)

Let $b_{n,k}$ = # neutral linear lambda terms of size n with k free variables
Let $r_{n,k}$ = # normal linear lambda terms of size n with k free variables

$$L_B(z,x) = x + L_B(z,x)L_R(z,x)$$
$$L_R(z,x) = zL_B(z,x) + \frac{\partial}{\partial x}L_R(z,x)$$

$$L_R(z,0) = z + 3z^2 + 26z^3 + 367z^4 + 7142z^5 + \ldots$$

Let $b_{n,k}$ = # neutral **planar** lambda terms of size n with k free variables
Let $r_{n,k}$ = # normal **planar** lambda terms of size n with k free variables

$$P_B(z,x) = x + P_B(z,x)P_R(z,x)$$
$$P_R(z,x) = zP_B(z,x) + \frac{1}{x}(P_R(z,x) - P_R(z,0))$$

$$P_R(z,0) = z + 2z^2 + 9z^3 + 54z^4 + 378z^5 + \ldots$$

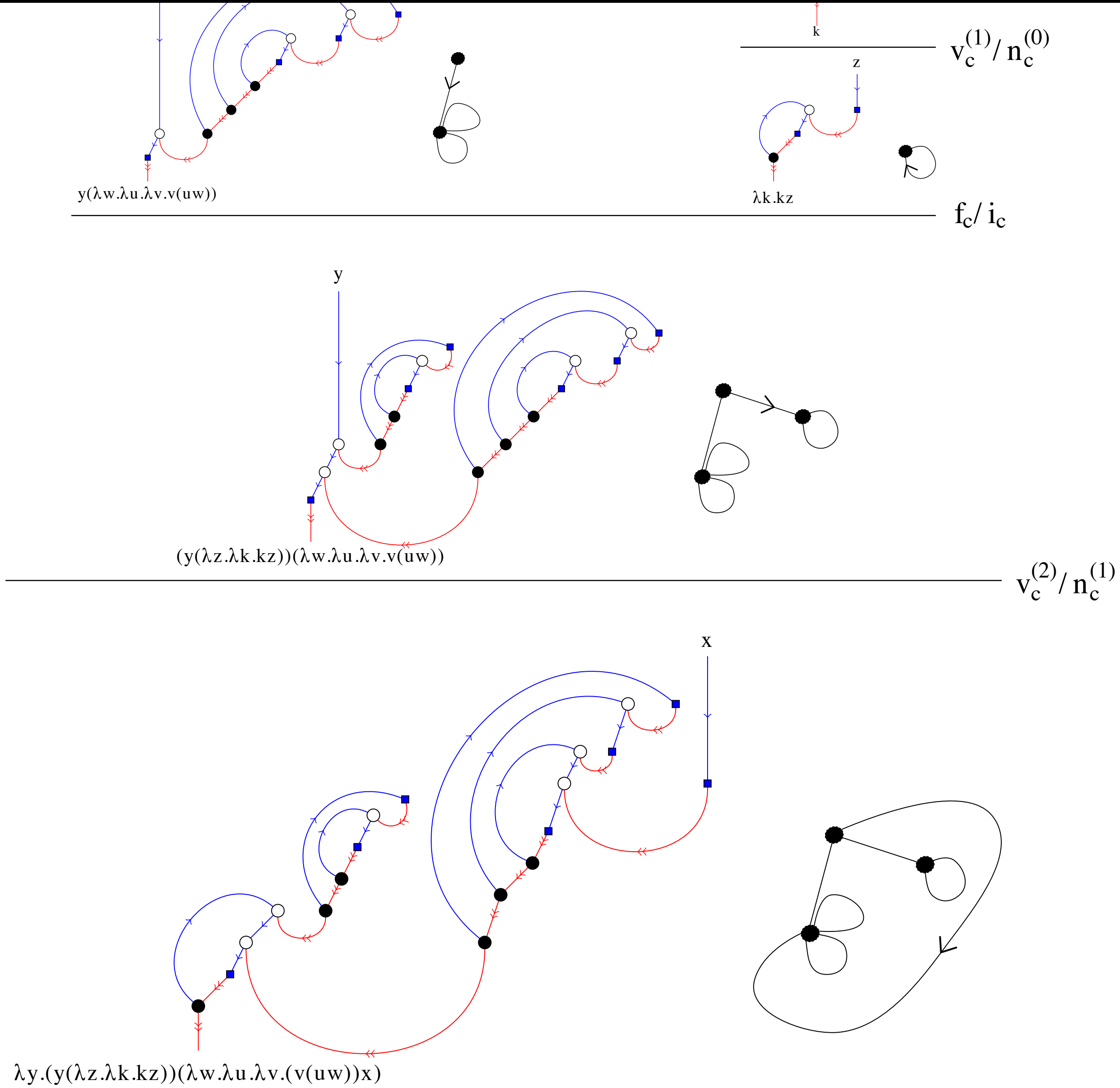
Original work on combinatorics of (rooted) planar maps:

W. T. Tutte. A census of planar maps. *Canadian Journal of Math.*, 15:249–271, 1963.

W. T. Tutte. On the enumeration of planar maps. *Bulletin of the AMS*, 74:64–74,1968.

A bijection [closed normal planar lambda terms <-> rooted planar maps] is given in:

Noam Zeilberger and Alain Giorgetti. A correspondence between rooted planar maps and normal planar lambda terms. *LMCS*, 11(3:22):1-39, 2015.


k
z
$v_c^{(1)} / n_c^{(0)}$
y(λw.λu.λv.v(uw))
λk.kz
f_c / i_c
y
(y(λz.λk.kz))(λw.λu.λv.v(uw))
$v_c^{(2)} / n_c^{(1)}$
x
λy.(y(λz.λk.kz))(λw.λu.λv.(v(uw))x)

Correspondences

planar lambda terms	rooted trivalent maps on the sphere
linear lambda terms	rooted trivalent maps on oriented surfaces
β-normal planar lambda terms	rooted maps on the sphere
β-normal linear lambda terms	???

New draft:

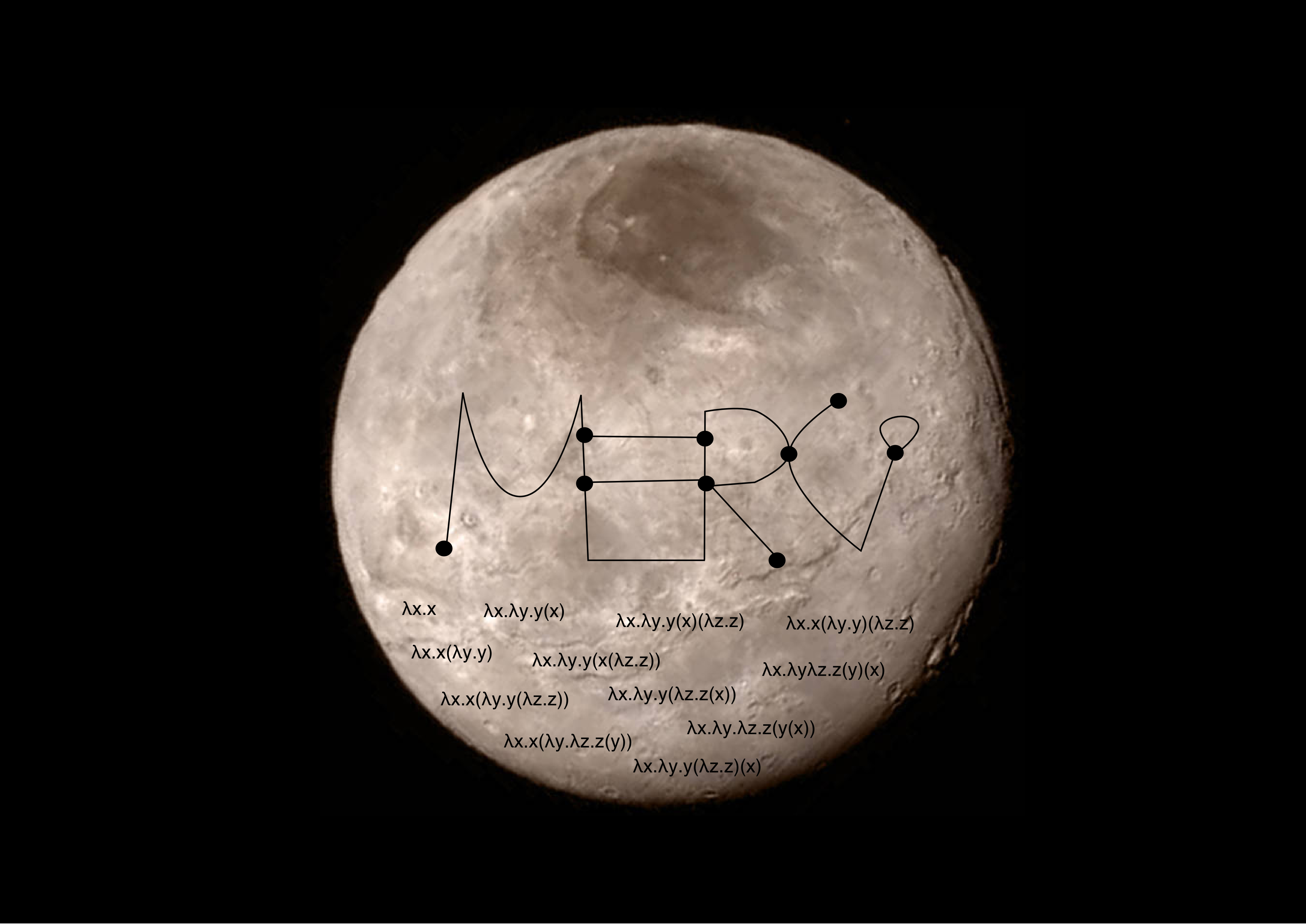
Counting isomorphism classes of β-normal linear lambda terms. arXiv:1509.07596

Idea: $\lambda x.\lambda y.u \cong \lambda y.\lambda x.u$

$$\tilde{L}_B(z,x) = x + \tilde{L}_B(z,x)\tilde{L}_R(z,x)$$
$$\tilde{L}_R(z,x) = z\sum_{i=0}^{\infty}\frac{1}{i!}\cdot\frac{\partial^i}{\partial x^i}\tilde{L}_B(z,x)$$

$$\tilde{L}_R(z,x) = z\tilde{L}_B(z,x+1)$$
$$\tilde{L}_R(z,0) = z + 2z^2 + 10z^3 + 74z^4 + 706z^5 + \dots$$

recover A000698, counting rooted maps on oriented surfaces!


λx.x
λx.λy.y(x)
λx.λy.y(x)(λz.z)
λx.x(λy.y)(λz.z)
λx.x(λy.y)
λx.λy.y(x(λz.z))
λx.λyλz.z(y)(x)
λx.x(λy.y(λz.z))
λx.λy.y(λz.z(x))
λx.λy.λz.z(y(x))
λx.x(λy.λz.z(y))
λx.λy.y(λz.z)(x)