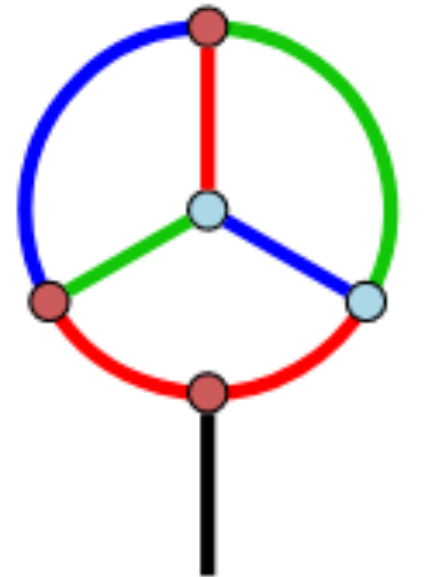


A cartographic quest between lambda-calculus, logic, and combinatorics

$$\begin{array}{c}
 \frac{}{x : \beta \multimap \gamma \vdash x : \beta \multimap \gamma} \quad \frac{}{y : \alpha \multimap \beta \vdash y : \alpha \multimap \beta} \quad \frac{}{z : \alpha \vdash z : \alpha} \\
 \hline
 \frac{}{y : \alpha \multimap \beta, z : \alpha \vdash y(z) : \beta} \\
 \hline
 \frac{x : \beta \multimap \gamma, y : \alpha \multimap \beta, z : \alpha \vdash x(yz) : \gamma}{x : \beta \multimap \gamma, y : \alpha \multimap \beta \vdash \lambda z. x(yz) : \alpha \multimap \gamma} \\
 \hline
 \frac{x : \beta \multimap \gamma \vdash \lambda y. \lambda z. x(yz) : (\alpha \multimap \beta) \multimap (\alpha \multimap \gamma)}{\vdash \lambda x. \lambda y. \lambda z. x(yz) : (\beta \multimap \gamma) \multimap ((\alpha \multimap \beta) \multimap (\alpha \multimap \gamma))}
 \end{array}$$



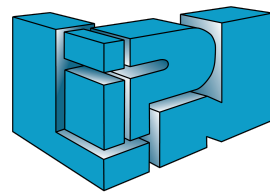
The LambdaComb Project

www.lix.polytechnique.fr/LambdaComb/

a new 4-year ANR project, co-initiated by Olivier Bodini & myself

kick-off meeting: 11 April @ LIX!

five labs involved:



broad goal: develop interdisciplinary connections between
lambda calculus (and related topics) and **comb**inatorics

The LambdaComb Project

www.lix.polytechnique.fr/LambdaComb/

one important motivation: the discovery of a host of links between subsystems of λ -calculus and enumeration of graphs on surfaces, or "maps".

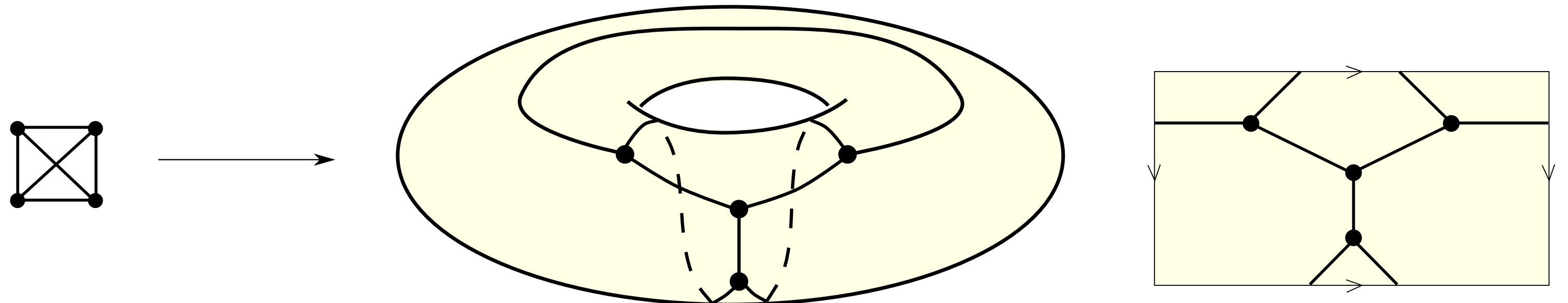
family of lambda terms	family of rooted maps	OEIS
linear	3-valent (genus $g \geq 0$)	A062980
ordered	planar 3-valent	A002005
unitless linear	bridgeless 3-valent ($g \geq 0$)	A267827
unitless ordered	bridgeless planar 3-valent	A000309
normal linear/ \sim	(all maps of genus $g \geq 0$)	A000698
normal ordered	planar	A000168
normal unitless linear/ \sim	bridgeless ($g \geq 0$)	A000699
normal unitless ordered	bridgeless planar	A000260

aim for this talk: explain these links, and give some indications of our motivations for exploring them further.

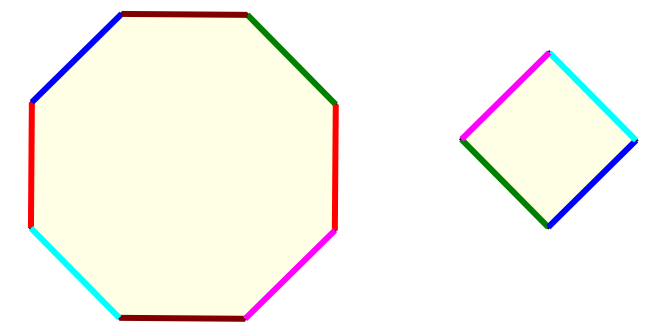
1. What is a map?
(And how many are there?)

Topological definition

map = 2-cell embedding of a graph into a surface^{*}, considered up to deformation of the underlying surface.



Or equivalently, a tiling of a surface by polygons.

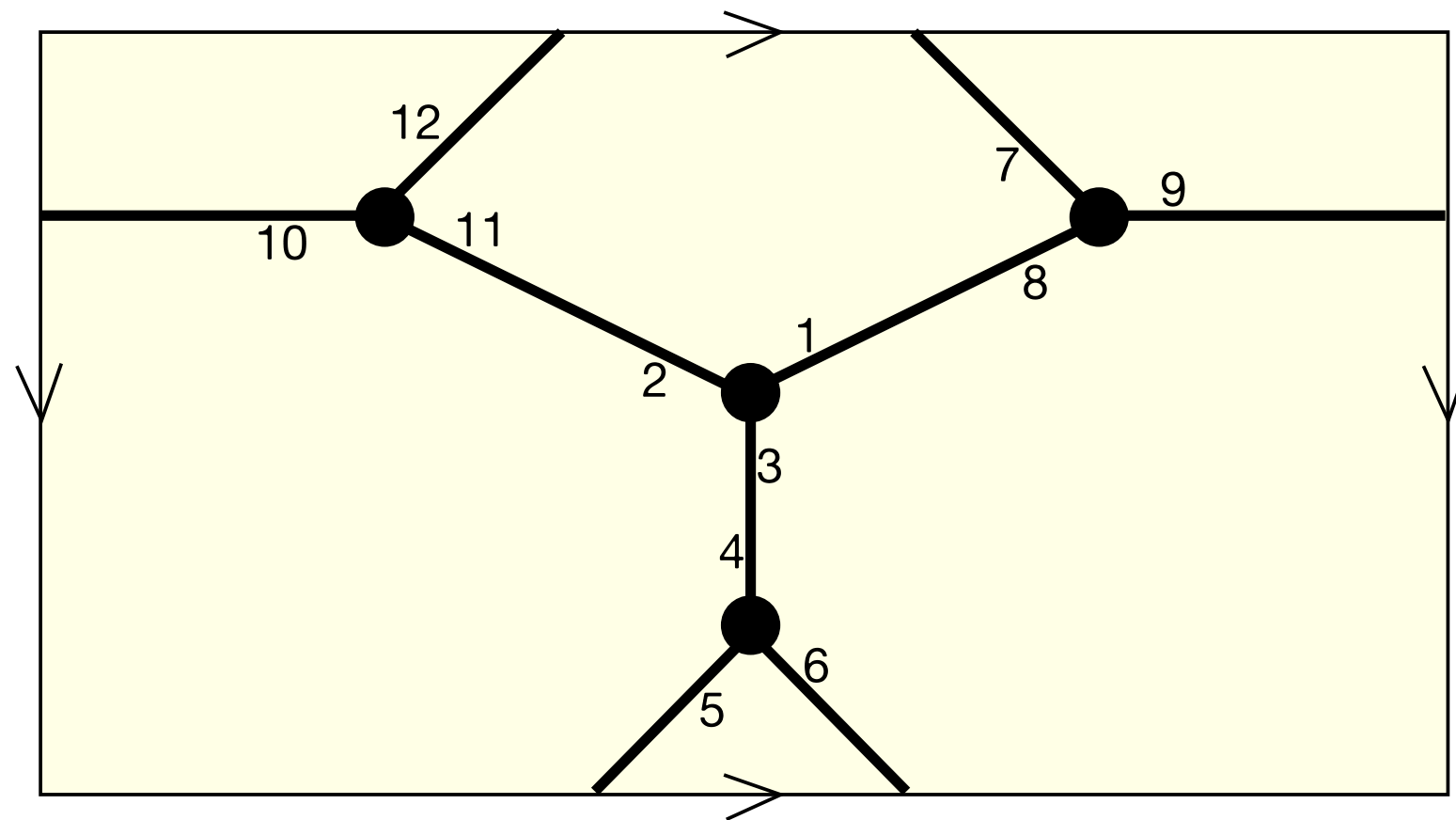


^{*}All surfaces are assumed to be connected and oriented throughout this talk

Algebraic definition

map = transitive permutation representation of the group

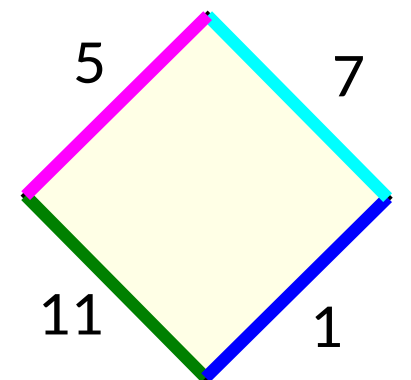
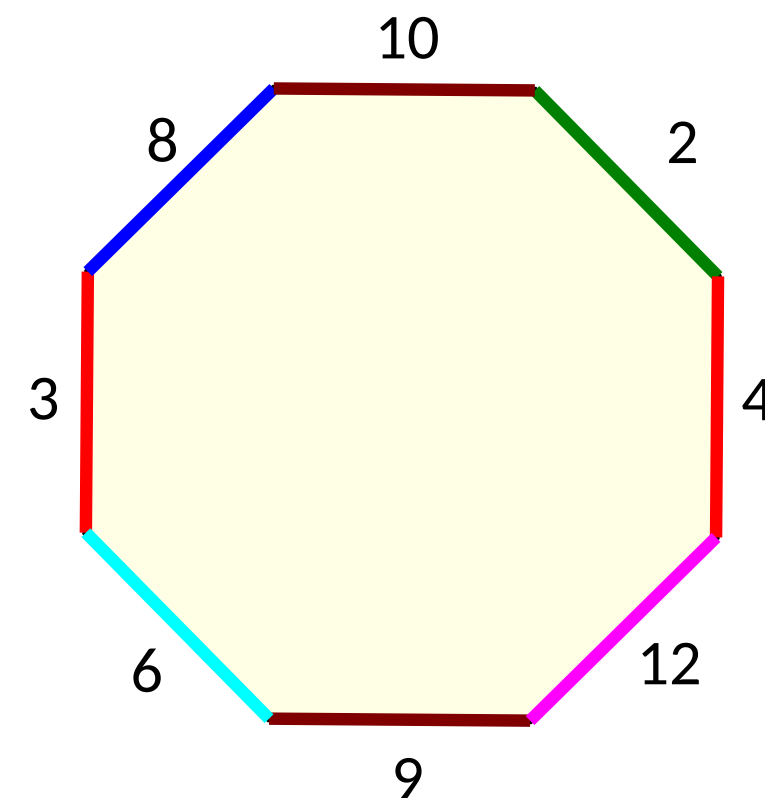
$G = \langle v, e, f \mid e^2 = vef = 1 \rangle$ considered up to G -equivariant relabelling.



$$v = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)$$

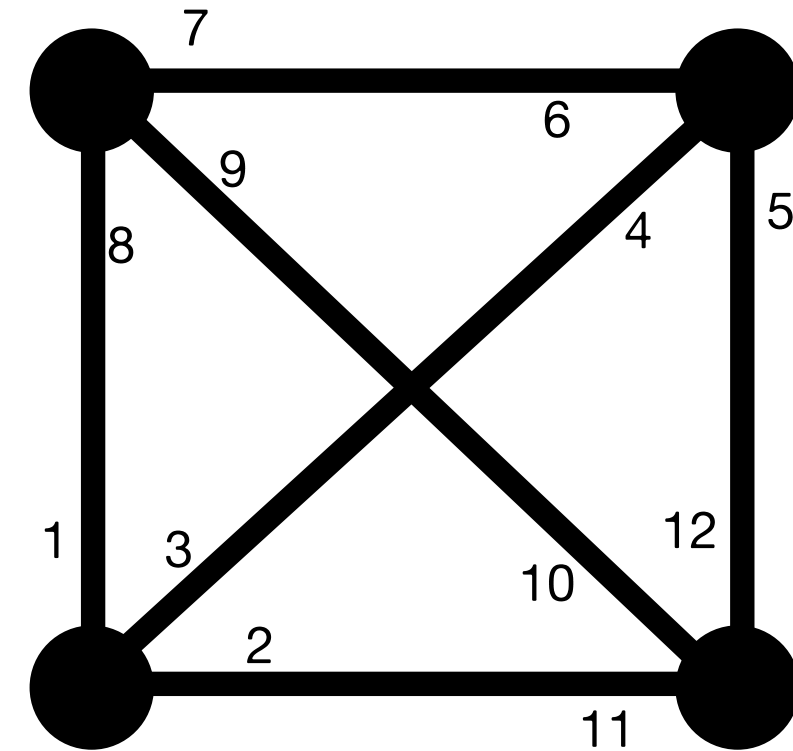
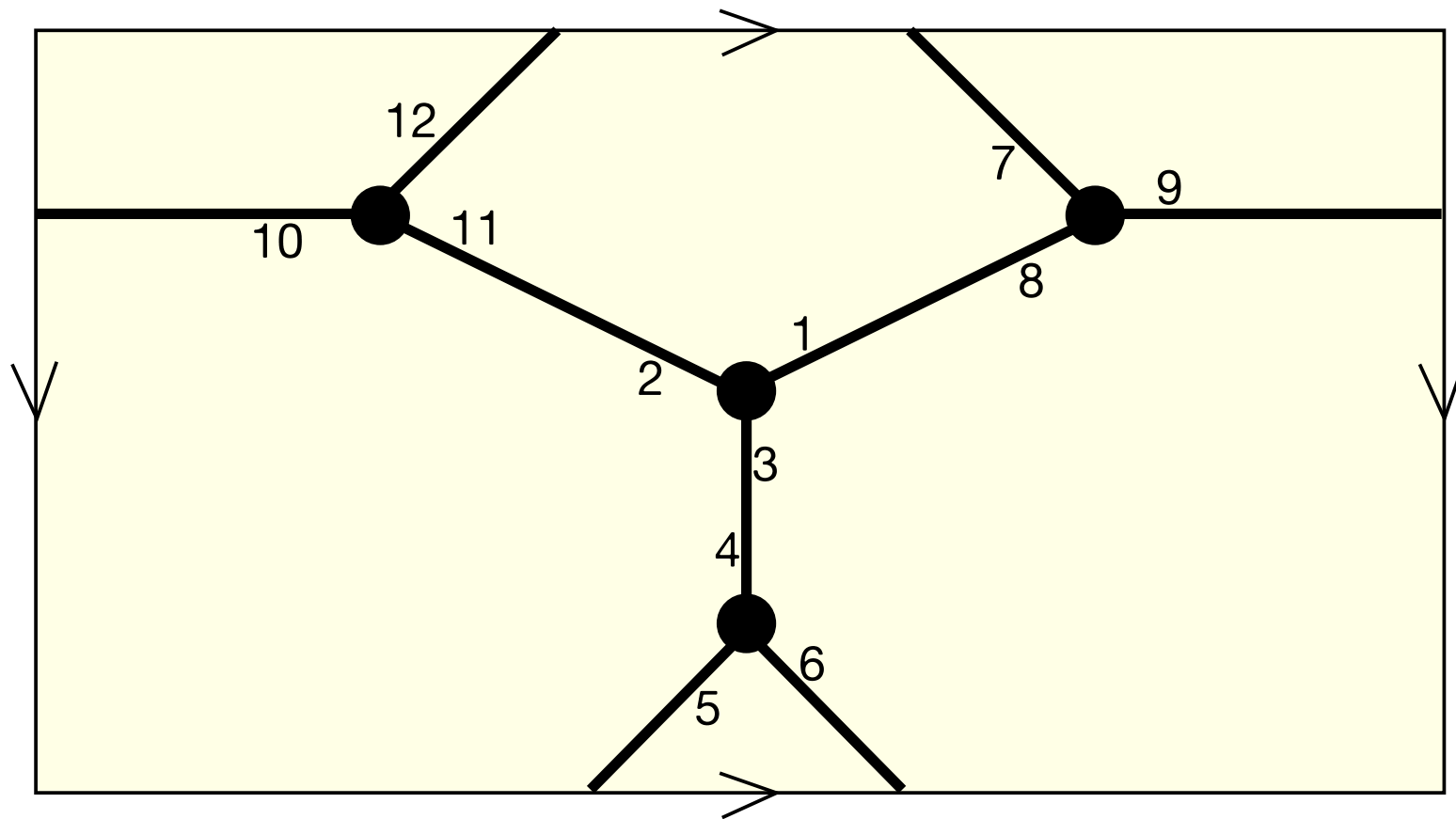
$$e = (1\ 8)(2\ 11)(3\ 4)(5\ 12)(6\ 7)(9\ 10)$$

$$f = (1\ 7\ 5\ 11)(2\ 10\ 8\ 3\ 6\ 9\ 12\ 4)$$



Combinatorial definition

map = connected graph + cyclic ordering of the half-edges around each vertex (e.g, as given by a drawing with "virtual crossings").

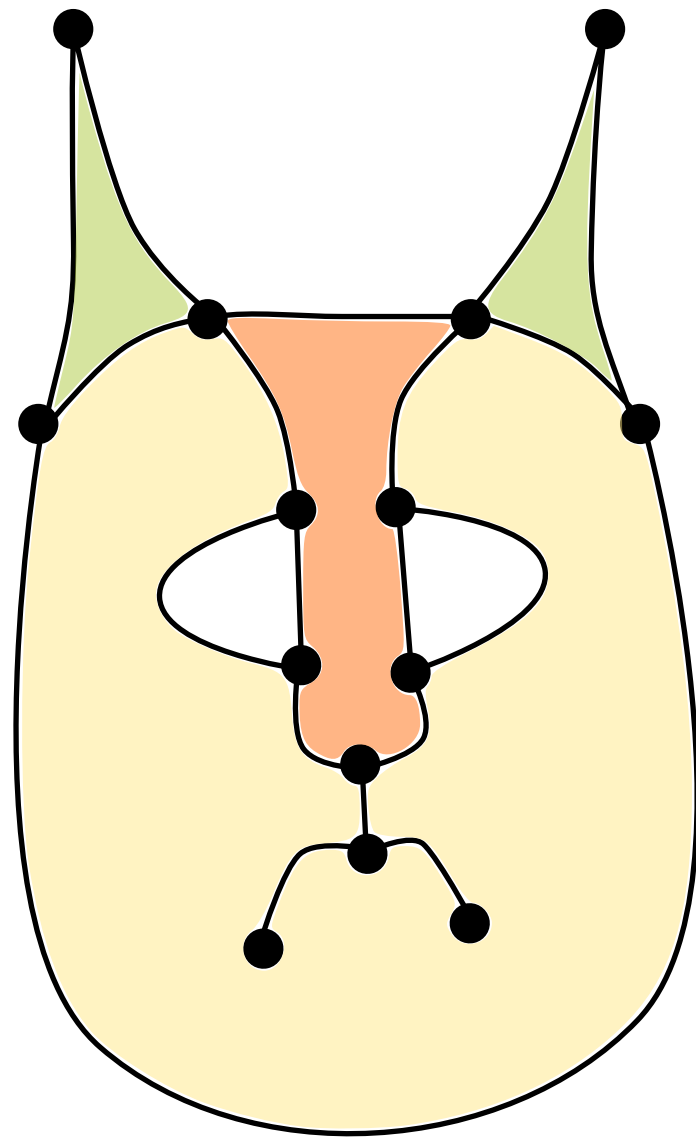


$$v = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)$$

$$e = (1\ 8)(2\ 11)(3\ 4)(5\ 12)(6\ 7)(9\ 10)$$

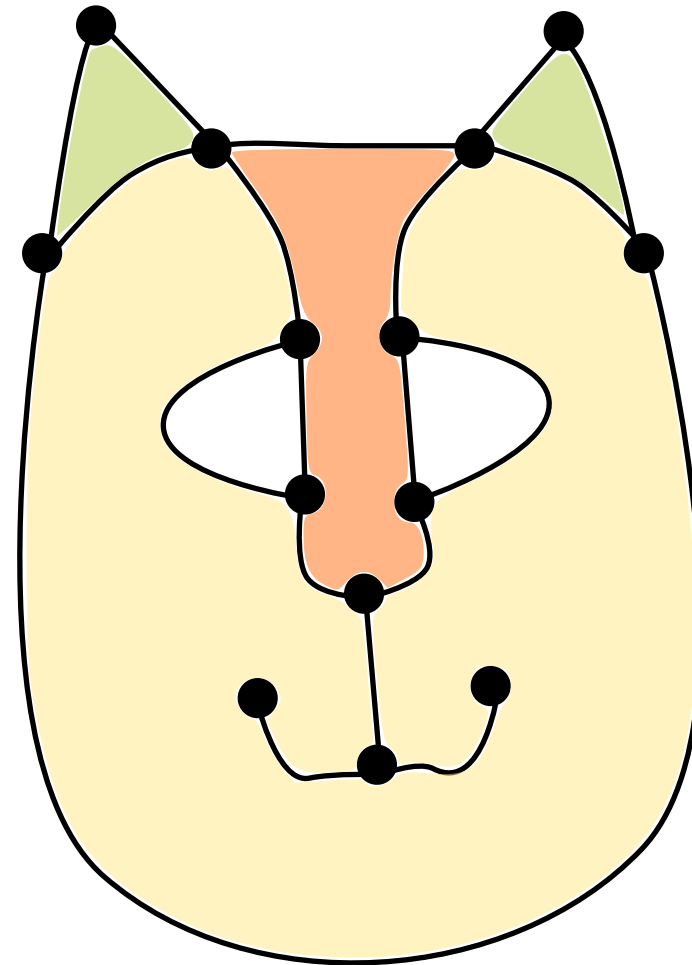
$$f = (ve)^{-1}$$

Graph versus Map



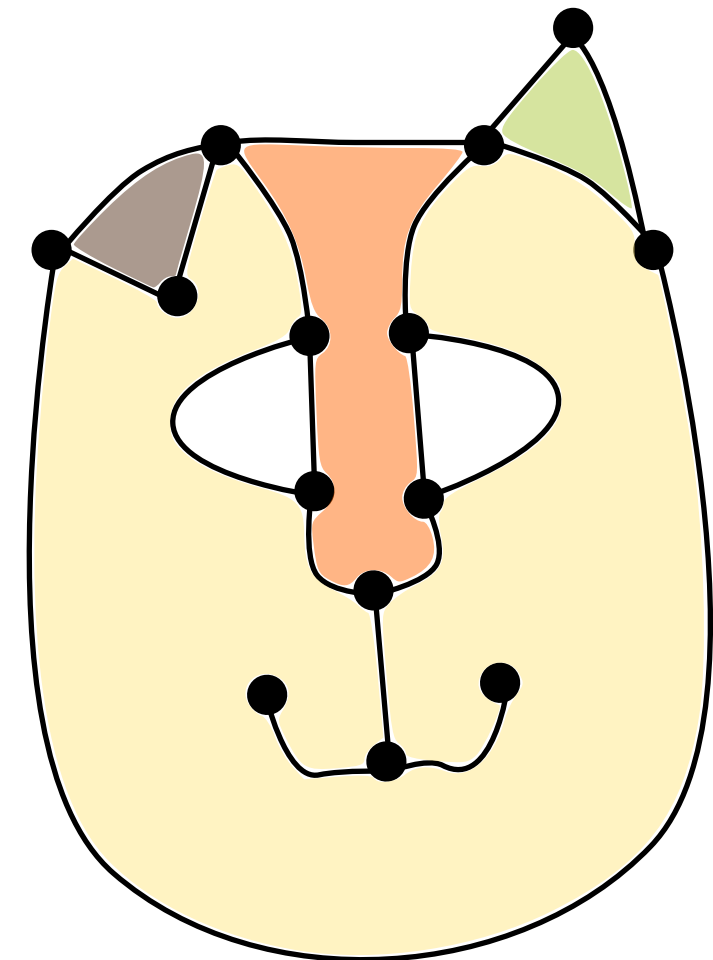
\equiv
map

\equiv
graph

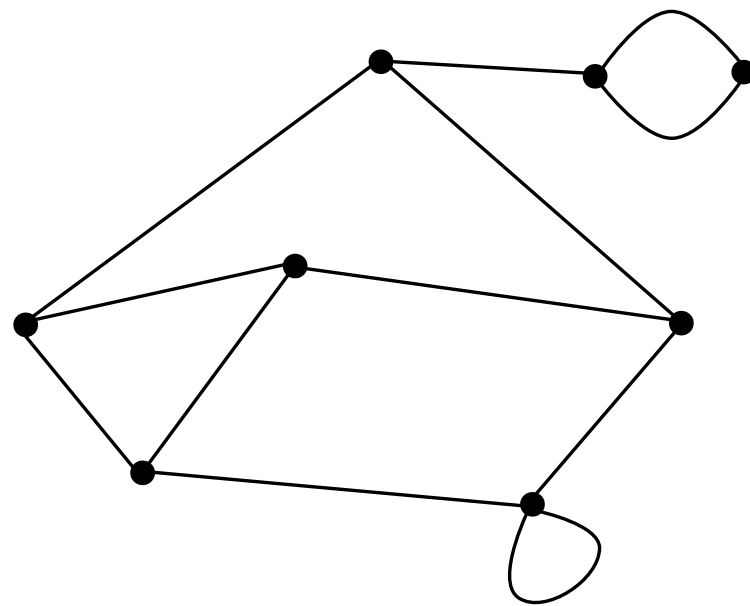


$\not\equiv$
map

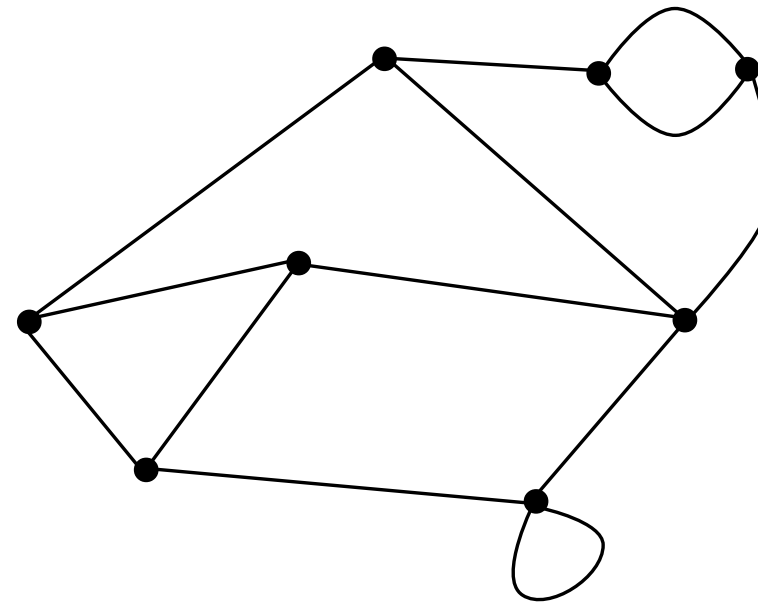
\equiv
graph



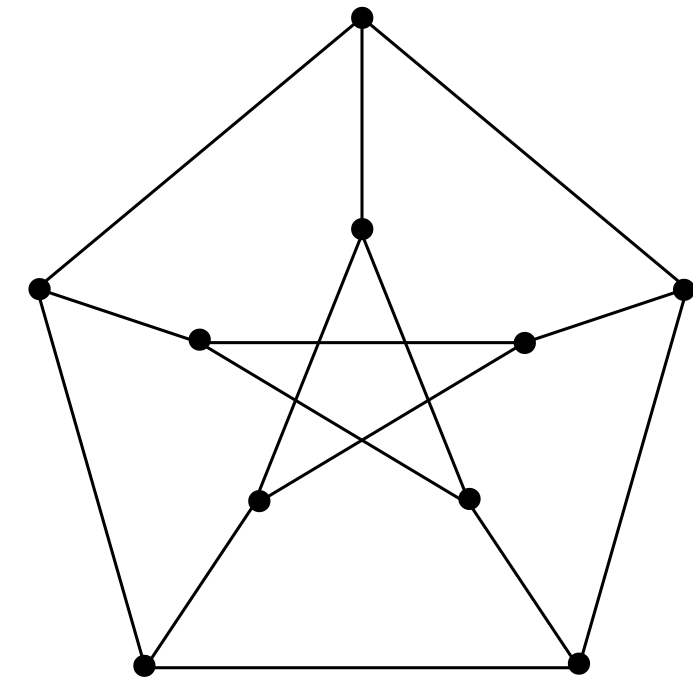
Some special kinds of maps



planar



bridgeless

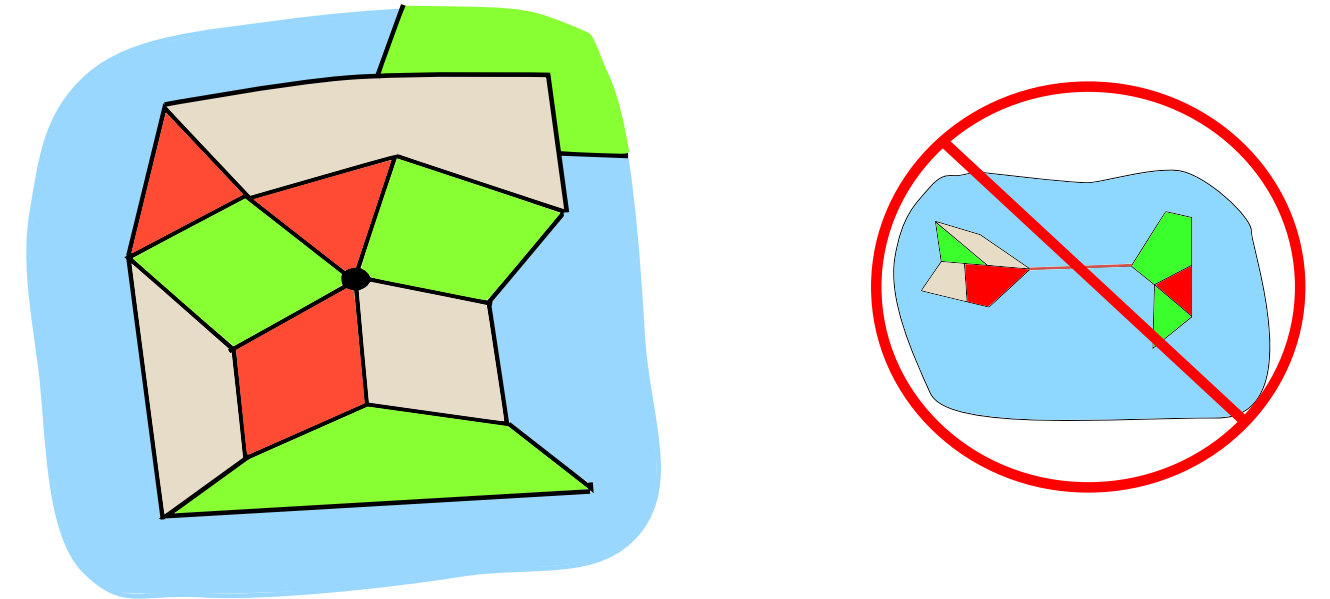


3-valent

Four Color Theorem

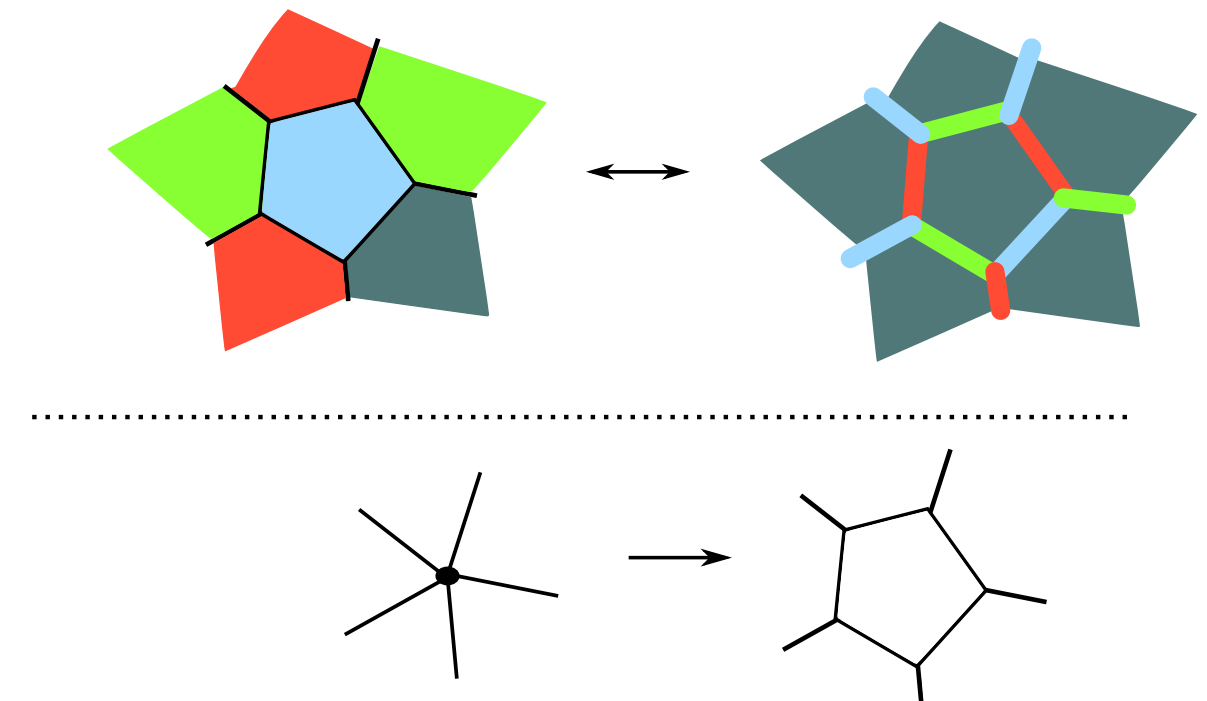
The 4CT is a statement about maps.

*every bridgeless planar map
has a proper face 4-coloring*



By a well-known reduction (Tait 1880), 4CT is equivalent to a statement about 3-valent maps

*every bridgeless planar 3-valent map
has a proper edge 3-coloring*



Map enumeration

*From time to time in a graph-theoretical career one's thoughts turn to the Four Colour Problem. **It occurred to me once that it might be possible to get results of interest in the theory of map-colourings without actually solving the Problem. For example, it might be possible to find the average number of colourings on vertices, for planar triangulations of a given size.***

One would determine the number of triangulations of $2n$ faces, and then the number of 4-coloured triangulations of $2n$ faces. Then one would divide the second number by the first to get the required average. I gathered that this sort of retreat from a difficult problem to a related average was not unknown in other branches of Mathematics, and that it was particularly common in Number Theory.

W. T. Tutte, Graph Theory as I Have Known It

Map enumeration

Tutte wrote a germinal series of papers (1962-1969)



bust by Gabriella Bollobás

W. T. Tutte (1962), A census of planar triangulations. Canadian Journal of Mathematics 14:21–38

W. T. Tutte (1962), A census of Hamiltonian polygons. Can. J. Math. 14:402–417

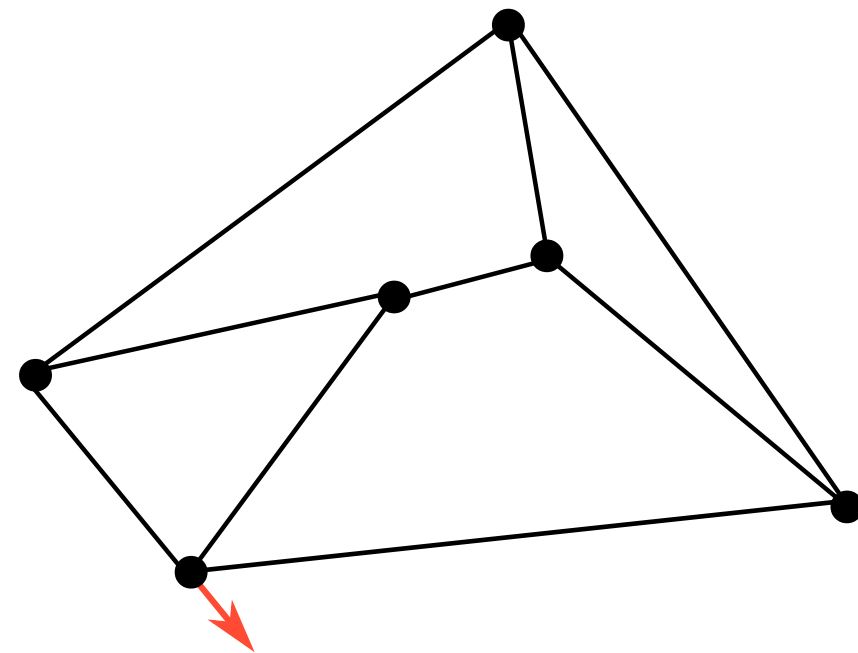
W. T. Tutte (1962), A census of slicings. Can. J. Math. 14:708–722

W. T. Tutte (1963), A census of planar maps. Can. J. Math. 15:249–271

W. T. Tutte (1968), On the enumeration of planar maps. Bulletin of the American Mathematical Society 74:64–74

W. T. Tutte (1969), On the enumeration of four-colored maps. SIAM Journal on Applied Mathematics 17:454–460

One of his insights was to consider **rooted maps**

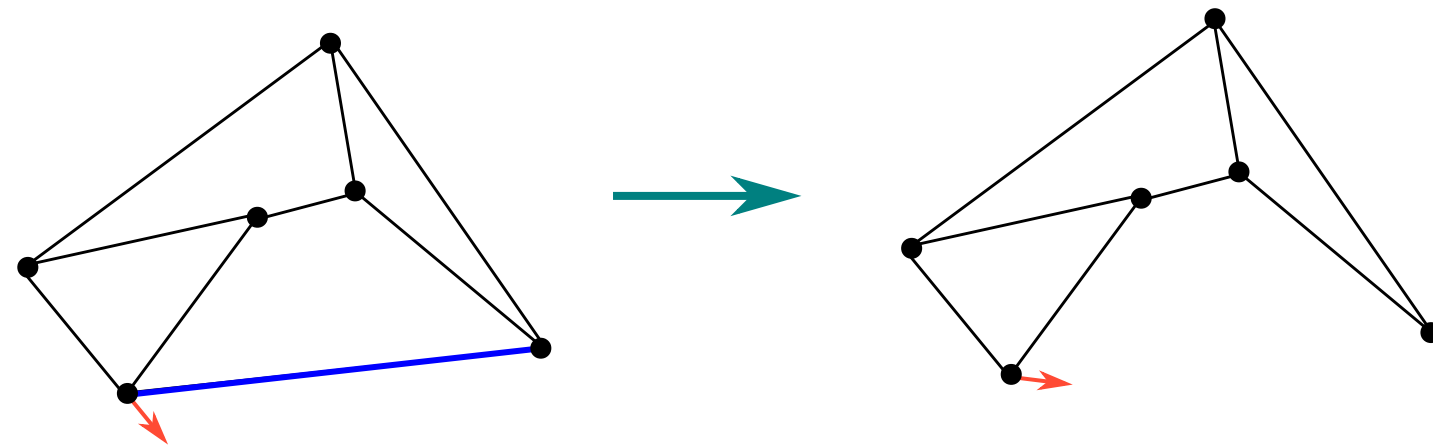


*Key property: rooted maps have
no non-trivial automorphisms*

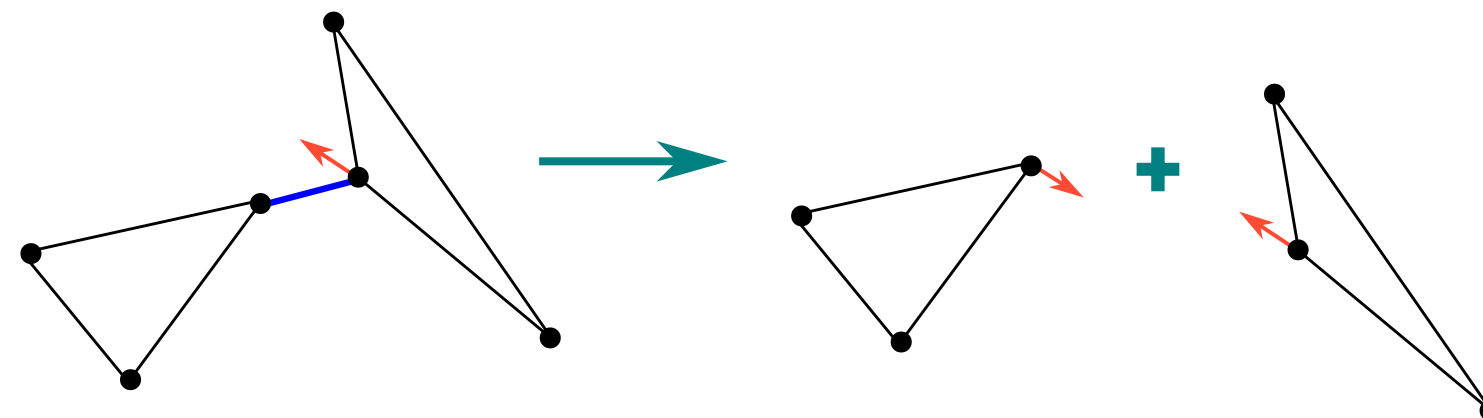
Map enumeration

Idea of 1968 paper: decompose rooted planar maps recursively, by iterated deletion of the root edge.

case 1 (non-bridge):



case 2 (bridge):



case 3 (no edge):



To count # maps, only need to keep track of # edges + *degree of outer face*

Map enumeration

Ultimately, Tutte obtained some remarkably simple formulas for counting different families of rooted planar maps by size n , e.g.:

$$\# \text{ maps} = 2(2n)!3^n / n!(n+2)! \quad \text{A000168}$$

$$\# \text{ bridgeless maps} = 2(4n+1)! / (n+1)!(3n+2)! \quad \text{A000260}$$

$$\# \text{ bridgeless 3-valent maps} = 2^n(3n)! / (n+1)!(2n+1)! \quad \text{A000309}$$

For more on map-counting see:

Mireille Bousquet-Mélou, [Enumerative Combinatorics of Maps](#) (recorded lecture series)

Gilles Schaeffer, "Planar maps", in Handbook of Enumerative Combinatorics (ed. Bóna)

Bertrand Eynard, *Counting Surfaces*, Birkhäuser, 2016

2. A crash course in λ -calculus (and its linear subsystems)

Lambda calculus: a very brief history*

Invented by Alonzo Church in late 20s, published in 1932

Original goal: foundation for logic without free variables

Minor defect: inconsistent!

Resolution: separate into an **untyped calculus** for computation, and a **typed calculus** for logic.

(Both have since found many uses.)



*Source: Cardone & Hindley's "History of Lambda-calculus and Combinatory Logic"

Untyped lambda calculus: syntax and computation

Minimalistic syntax of terms:

$$t, u ::= x \quad | \quad t(u) \quad | \quad \lambda x. t$$

variable application abstraction

may freely
rename variables

Computation through **the rule of β -reduction**:

$$(\lambda x. t)(u) \rightarrow^{\beta} t[u/x]$$

can apply to any matching subterm,
but confluence \Rightarrow unique normal form

Sometimes paired with **the rule of η -expansion**:

$$t \rightarrow^{\eta} \lambda x. t(x)$$

Example

$$\begin{aligned} & (\lambda x. \lambda y. \lambda z. x(yz))(\lambda a. a)(t) \\ & \rightarrow^{\beta} (\lambda y. \lambda z. (\lambda a. a)(yz))(t) \\ & \rightarrow^{\beta} (\lambda y. \lambda z. yz)(t) \\ & \rightarrow^{\beta} \lambda z. t(z) \quad \eta \leftarrow t \end{aligned}$$

Simply-typed lambda calculus and the Curry-Howard-Lambek correspondence

In Church's simple typing discipline, every subterm is annotated by a **type** subject to the following constraints:



A term t_A can be interpreted as a **constructive proof**^{*†} of A

^{*}: in purely implicative intuitionistic logic

[†]: possibly under assumptions, corresponding to free variables

STLC is closely related to the theory of cartesian closed categories...

Fixpoints and non-linearity



Turing published first **fixed-point combinator** (1937)

(key to Turing-completeness of untyped λ -calculus)

$$Y = (\lambda x. \lambda y. y(xxy))(\lambda x. \lambda y. y(xxy))$$

$$f(Yf) =^{\beta} Yf$$

Observe doubled uses of variables x and y .

By restricting to terms where every variable is used exactly once, one gets a well-behaved **linear** subsystem of lambda calculus.

(no longer Turing-complete...actually P-complete)

An algebraic view

cf. Hyland's "Classical lambda calculus in modern dress"

Different subsystems of untyped λ -calculus may be naturally organized into **operads**, by defining the n-ary operations as terms $\underbrace{x_1, \dots, x_n}_{\Gamma} \vdash t$

"t is a term in context of free variables Γ "

term-in-context formation: $\frac{}{x \vdash x} \quad \frac{\Gamma \vdash t \quad \Delta \vdash u}{\Gamma, \Delta \vdash t(u)} \quad \frac{\Gamma, x \vdash t}{\Gamma \vdash \lambda x.t}$

operadic composition: $\frac{\Gamma, x, \Delta \vdash t \quad \Omega \vdash u}{\Gamma, \Omega, \Delta \vdash t[u/x]}$

structural rules: $\frac{\Gamma, x, y, \Delta \vdash t}{\Gamma, y, x, \Delta \vdash t}$ "*exchange*" $\frac{\Gamma, \Delta \vdash t}{\Gamma, x, \Delta \vdash t}$ "*weakening*" $\frac{\Gamma, x, y, \Delta \vdash t}{\Gamma, x, \Delta \vdash t[x/y]}$ "*contraction*"

all structural rules \Rightarrow **cartesian operad** of general terms

only exchange \Rightarrow **symmetric operad** of linear terms

no structural rules \Rightarrow **(plain) operad** of ordered linear terms

Free closed multicategories

Typed terms may be similarly organized into **multicategories** (= "colored" operads)

The typing rules for application and abstraction...

$$\frac{\Gamma \vdash t:A \multimap B \quad \Delta \vdash u:A}{\Gamma, \Delta \vdash t(u):B} \qquad \frac{\Gamma, x:A \vdash t:B}{\Gamma \vdash \lambda x.t:A \multimap B}$$

...together with the $\beta + \eta$ equations ensure that we have a **closed** multicategory.

Indeed, simply-typed (general/linear/ordered) terms give a presentation of the **free** closed (cartesian/symmetric/arbitrary) multicategory!

For example, any simply-typed term can be interpreted (in Set) as a higher-order function between sets. A linear term can moreover be interpreted (in Vect) as a multilinear mapping between higher-order vector spaces.

**3. What do parts 1 & 2
have to do with each other?**

An innocent idea

In May 2014, I thought it could be fun* to count untyped *closed β -normal ordered linear terms* by size ($\# \lambda s$)...

*for reasons related to certain categorical models of typing, cf. Melliès & Zeilberger POPL 2015

$\lambda x.x$

$\lambda x. x(\lambda y. y)$

$\lambda x. \lambda y. x(y)$

$\lambda x.x(\lambda y.y(\lambda z.z))$

$\lambda x.x(\lambda y.\lambda z.y(z))$

$\lambda x.x(\lambda y.y)(\lambda z.z)$

$\lambda x.\lambda y.x(y(\lambda z.z))$

$\lambda x.\lambda y.x(\lambda z.y(z))$

$\lambda x.\lambda y.x(\lambda z.z)(y)$

$\lambda x.\lambda y.x(y)(\lambda z.z)$

$\lambda x.\lambda y.\lambda z.x(y(z))$

$\lambda x.\lambda y.\lambda z.x(y)(z)$

$\lambda x.x(\lambda y.y(\lambda z.z(\lambda w.w)))$
 $\lambda x.x(\lambda y.y(\lambda z.\lambda w.z(w)))$
 $\lambda x.x(\lambda y.y(\lambda z.z)(\lambda w.w))$
 $\lambda x.x(\lambda y.\lambda z.y(z(\lambda w.w)))$
 $\lambda x.x(\lambda y.\lambda z.y(\lambda w.z(w)))$
 $\lambda x.x(\lambda y.\lambda z.y(\lambda w.w)(z))$
 $\lambda x.x(\lambda y.\lambda z.y(z)(\lambda w.w))$
 $\lambda x.x(\lambda y.\lambda z.\lambda w.y(z(w)))$
 $\lambda x.x(\lambda y.\lambda z.\lambda w.y(z)(w))$

$\lambda x.x(\lambda y.y)(\lambda z.z(\lambda w.w))$
 $\lambda x.x(\lambda y.y)(\lambda z.\lambda w.z(w))$
 $\lambda x.x(\lambda y.y(\lambda z.z))(\lambda w.w)$
 $\lambda x.x(\lambda y.\lambda z.y(z))(\lambda w.w)$
 $\lambda x.x(\lambda y.y)(\lambda z.z)(\lambda w.w)$
 $\lambda x.\lambda y.x(y(\lambda z.z(\lambda w.w)))$
 $\lambda x.\lambda y.x(y(\lambda z.\lambda w.z(w)))$
 $\lambda x.\lambda y.x(y(\lambda z.z)(\lambda w.w))$
 $\lambda x.\lambda y.x(\lambda z.y(z(\lambda w.w)))$

$\lambda x.\lambda y.x(\lambda z.y(\lambda w.z(w)))$
 $\lambda x.\lambda y.x(\lambda z.y(\lambda w.w)(z))$
 $\lambda x.\lambda y.x(\lambda z.y(z)(\lambda w.w))$
 $\lambda x.\lambda y.x(\lambda z.\lambda w.y(z(w)))$
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 $\lambda x.\lambda y.x(\lambda z.z)(\lambda w.y(w))$
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$\lambda x.\lambda y.x(\lambda z.z)(\lambda w.w)(y)$
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 $\lambda x.\lambda y.x(\lambda z.z)(y)(\lambda w.w)$
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$\lambda x.\lambda y.\lambda z.x(\lambda w.w)(y)(z)$
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1,2,9,54,378,2916,24057

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Search: **seq:1,2,9,54,378,2916,24057**

Displaying 1-1 of 1 result found.

page 1

Sort: relevance | [references](#) | [number](#) | [modified](#) | [created](#) Format: long | [short](#) | [data](#)

[A000168](#)

$2 \cdot 3^n \cdot (2 \cdot n)! / (n! \cdot (n+2)!)$.
(Formerly M1940 N0768)

+20
18

1, 2, 9, 54, 378, 2916, 24057, 208494, 1876446, 17399772, 165297834, 1602117468,
15792300756, 157923007560, 1598970451545, 16365932856990, 169114639522230,
1762352559231660, 18504701871932430, 195621134074714260, 2080697516976506220,
22254416920705240440, 239234981897581334730, 2583737804493878415084 ([list](#); [graph](#); [refs](#); [listen](#); [history](#);
[text](#); [internal format](#))

OFFSET 0, 2

COMMENTS Number of rooted planar maps with n edges. - [Don Knuth](#), Nov 24 2013
Number of rooted 4-regular planar maps with n vertices.
Also, number of doodles with n crossings, irrespective of the number of
loops.

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page 1

Sort: relevance | [references](#) | [number](#) | [modified](#) | [created](#)

Format: long | [short](#) | [data](#)

[A000168](#)

$2 \cdot 3^n \cdot (2n)! / (n! \cdot (n+2)!)$.

+20

18

(Formerly M1940 N0768)

1, 2, 9, 54, 378, 2916,

15792300756, 157923007560,

1762352559231660, 18504701871932430,

22254416920705240440, 239234981897581334730,

[text](#); [internal format](#))

OFFSET

0, 2

COMMENTS

Number of rooted planar maps with n edges. - [Don Knuth](#), NOV 24 2013

Number of rooted 4-regular planar maps with n vertices.

Also, number of doodles with n crossings, irrespective of the number of loops.

The number a_n of rooted maps with n edges is

$$\frac{2(2n)! 3^n}{n! (n+2)!}.$$

One piece of a larger puzzle

family of rooted maps	family of lambda terms	sequence	OEIS
planar maps	normal ordered terms	1,2,9,54,378,2916,...	A000168

Z, A. Giorgetti (2015), A correspondence between rooted planar maps and normal planar lambda terms, LMCS 11(3:22): 1-393.

bijection by replaying Tutte's 1968 analysis on lambda terms, but not completely satisfying...

One piece of a larger puzzle

family of rooted maps	family of lambda terms	sequence	OEIS
trivalent maps (genus $g \geq 0$)	linear terms	1,5,60,1105,27120,...	A062980
planar maps	normal ordered terms	1,2,9,54,378,2916,...	A000168

compute a term from a map by depth-first search, but can also be explained more conceptually...

O. Bodini, D. Gardy, A. Jacquot (2013), Asymptotics and random sampling for BCI and BCK lambda terms, TCS 502: 227-238.

Z, A. Giorgetti (2015), A correspondence between rooted planar maps and normal planar lambda terms, LMCS 11(3:22): 1-393.

One piece of a larger puzzle

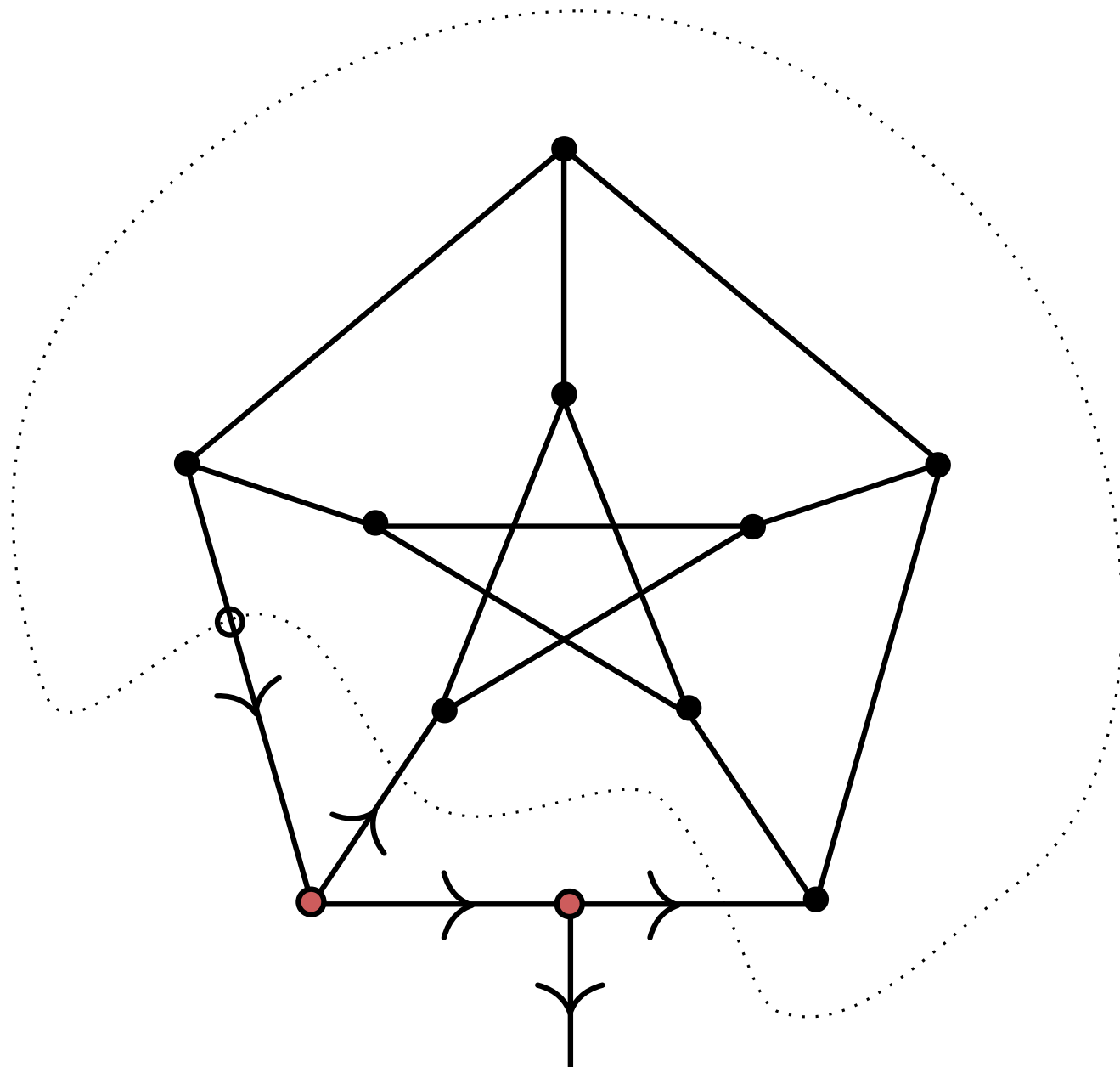
family of rooted maps	family of lambda terms	sequence	OEIS
trivalent maps (genus $g \geq 0$)	linear terms	1,5,60,1105,27120,...	A062980
planar trivalent maps	ordered terms	1,4,32,336,4096,...	A002005
bridgeless trivalent maps	unitless linear terms	1,2,20,352,8624,...	A267827
bridgeless planar trivalent maps	unitless ordered terms	1,1,4,24,176,1456,...	A000309
maps (genus $g \geq 0$)	normal linear terms (mod \sim)	1,2,10,74,706,8162,...	A000698
planar maps	normal ordered terms	1,2,9,54,378,2916,...	A000168
bridgeless maps	normal unitless linear terms (mod \sim)	1,1,4,27,248,2830,...	A000699
bridgeless planar maps	normal unitless ordered terms	1,1,3,13,68,399,...	A000260

O. Bodini, D. Gardy, A. Jacquot (2013), Asymptotics and random sampling for BCI and BCK lambda terms, TCS 502: 227-238.
Z, A. Giorgetti (2015), A correspondence between rooted planar maps and normal planar lambda terms, LMCS 11(3:22): 1-393.
: (see refs at <https://www.lix.polytechnique.fr/LambdaComb/docs/lambdacomb-scientific.pdf>)

Some comments:

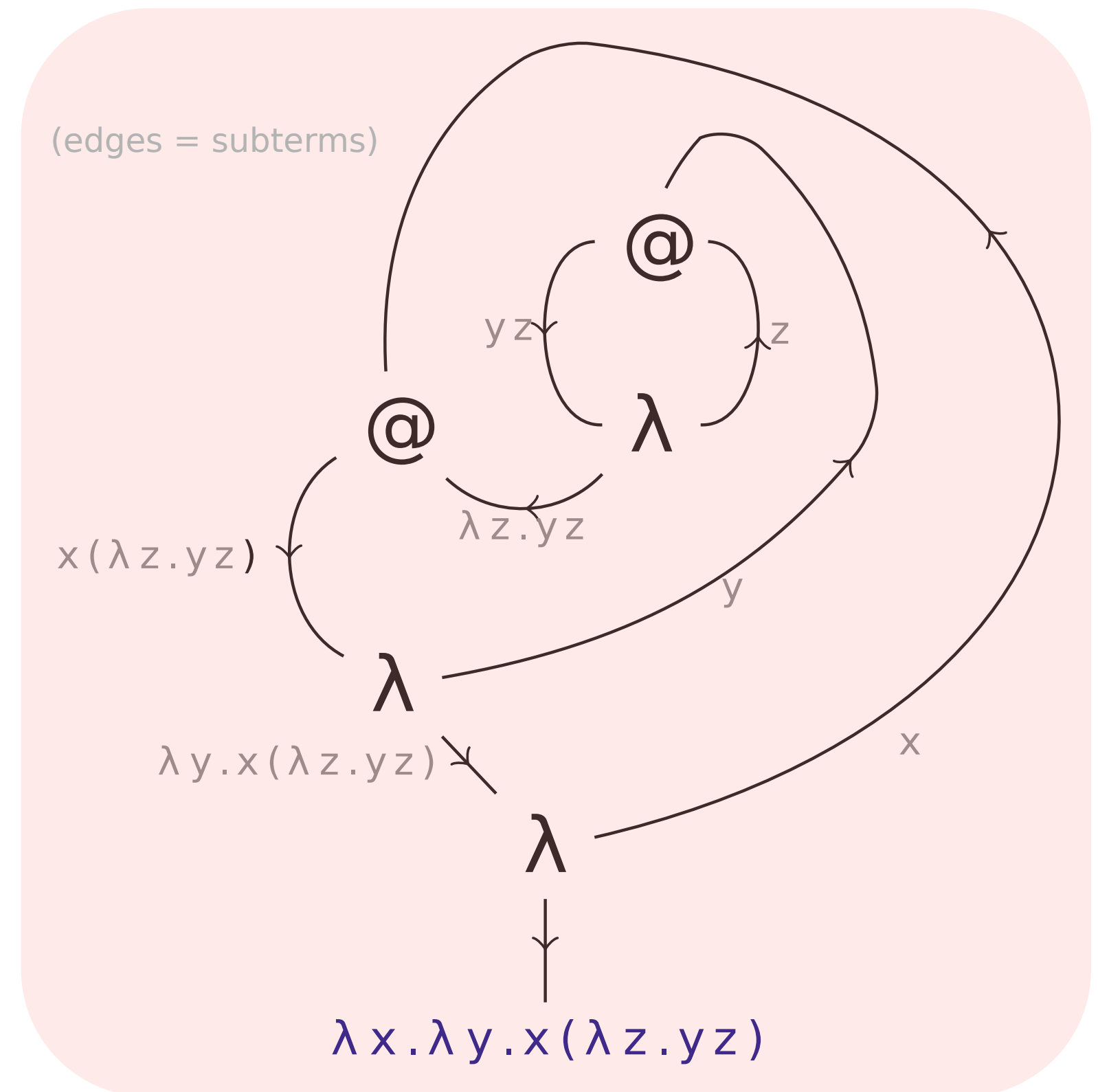
- "unitless" = no closed subterms (such terms can be organized into **non-unitary operads**)
- upper half of table can be explained by a single natural bijection (coming up...)
- "mod \sim " = modulo exchange of adjacent lambdas $\lambda x. \lambda y. t \sim \lambda y. \lambda x. t$
- lower half not yet well-understood...but see **Wenjie Fang's recent draft!** (arXiv:2202.03542)

4. Between linear λ -terms and rooted 3-valent maps



Idea (folklore*): representing λ -terms as graphs

Can represent a term as tree w/two kinds of nodes ($@/\lambda$), with "pointers" from λ -nodes to bound variables. This idea is especially natural for linear terms.



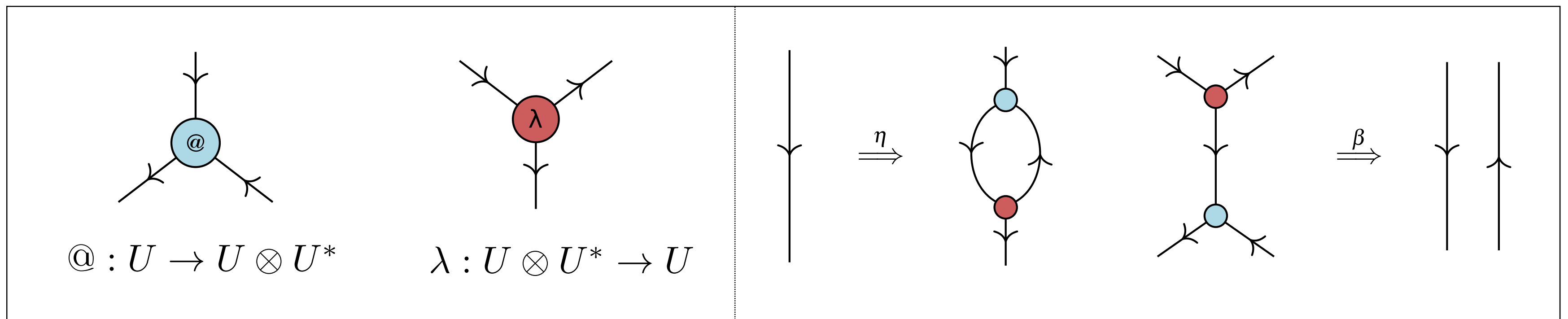
*The idea itself is natural and should probably be called folklore. The earliest explicit description I know of (currently) is in Knuth's "Examples of Formal Semantics" (1970), but it was developed more deeply and independently from different perspectives in the PhD theses of C. P. Wadsworth (1971) and R. Statman (1974).

λ -graphs as string diagrams

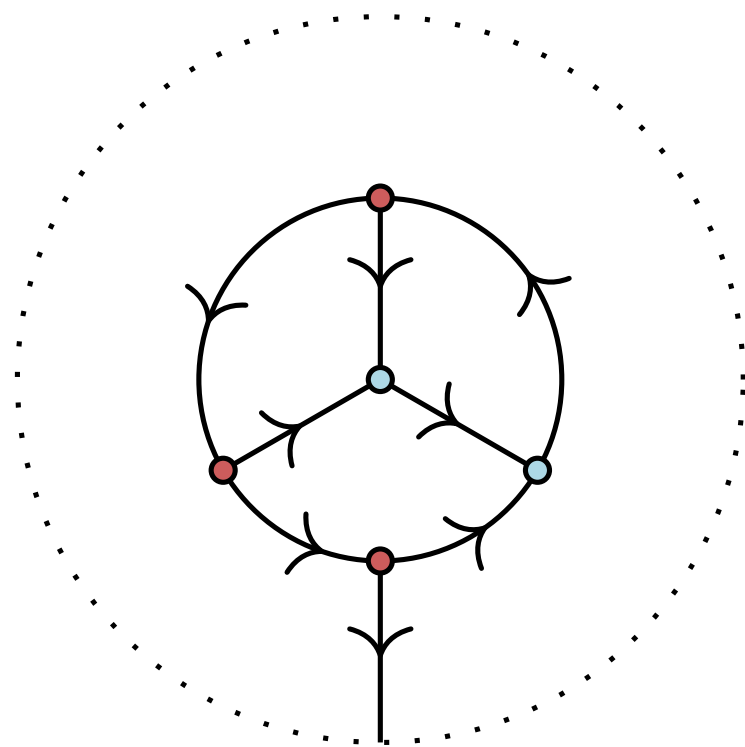
I proposed a preliminary analysis (JFP, 2016) of this graphical syntax within the categorical framework of *string diagrams* (Joyal & Street 1991), by interpreting untyped linear λ -terms as endomorphisms of a *reflexive object* (D. Scott 1980)

$$U \begin{array}{c} \xrightarrow{@} \\ \xleftarrow[\lambda]{\perp} \end{array} U \multimap U$$

in a symmetric monoidal (compact) closed bicategory.

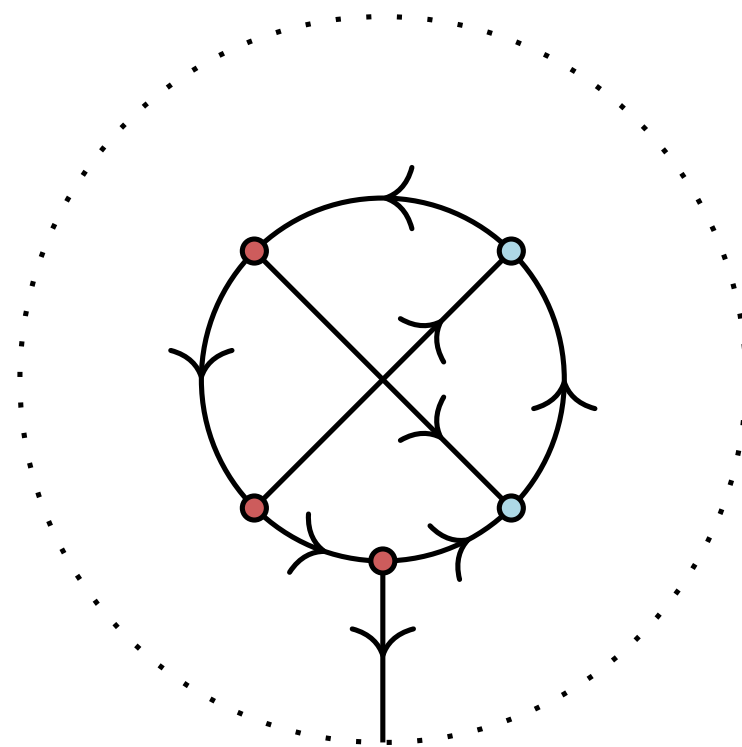


From linear λ -terms to rooted 3-valent maps



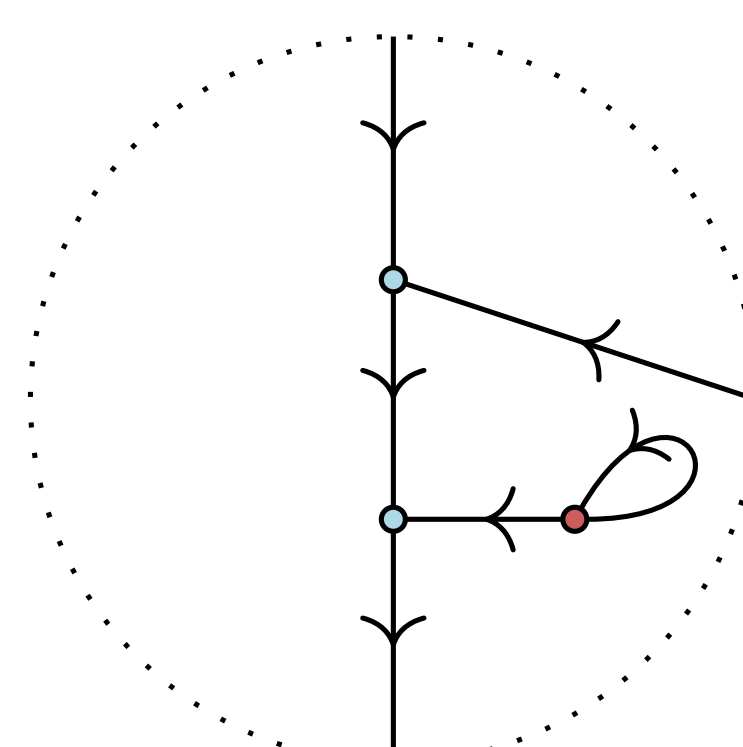
$\lambda x.\lambda y.\lambda z.x(yz)$

(B)

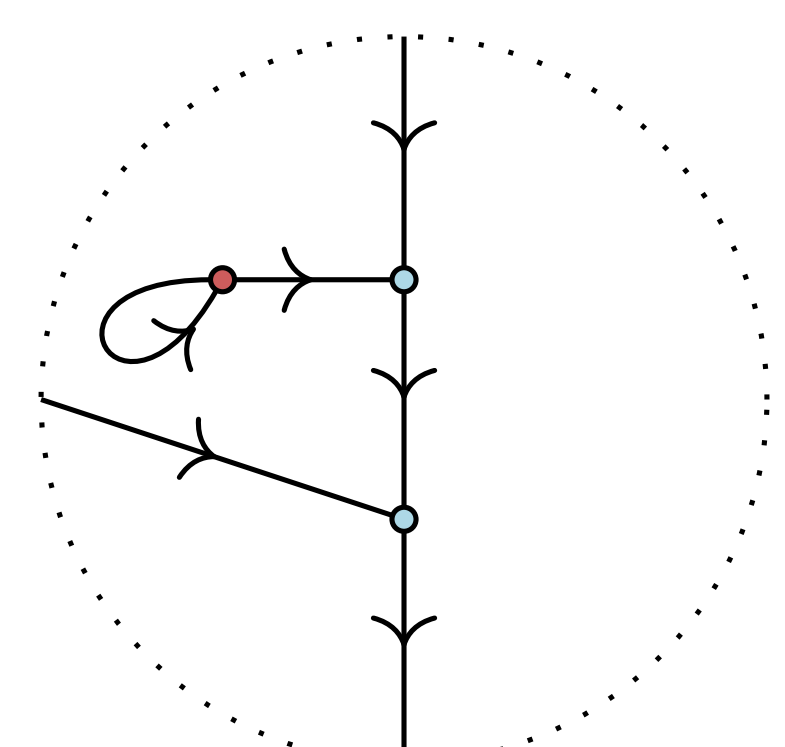


$\lambda x.\lambda y.\lambda z.(xz)y$

(C)

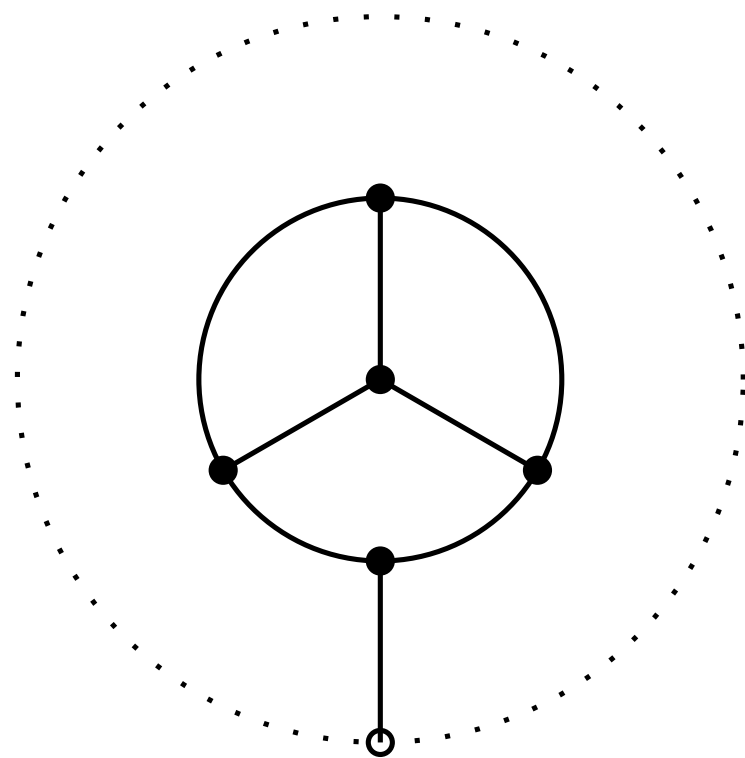


$x,y \vdash (xy)(\lambda z.z)$



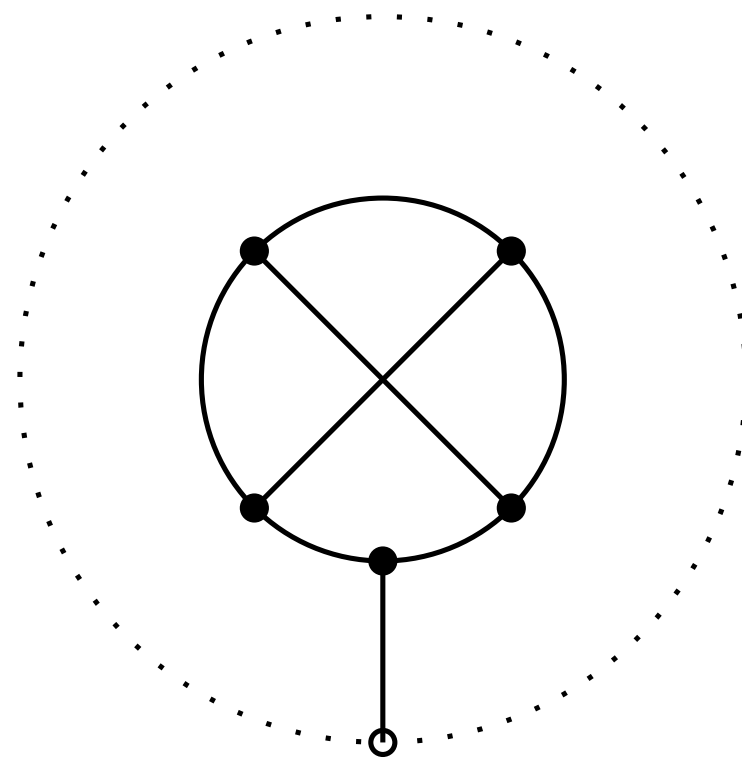
$x,y \vdash x((\lambda z.z)y)$

From linear λ -terms to rooted 3-valent maps



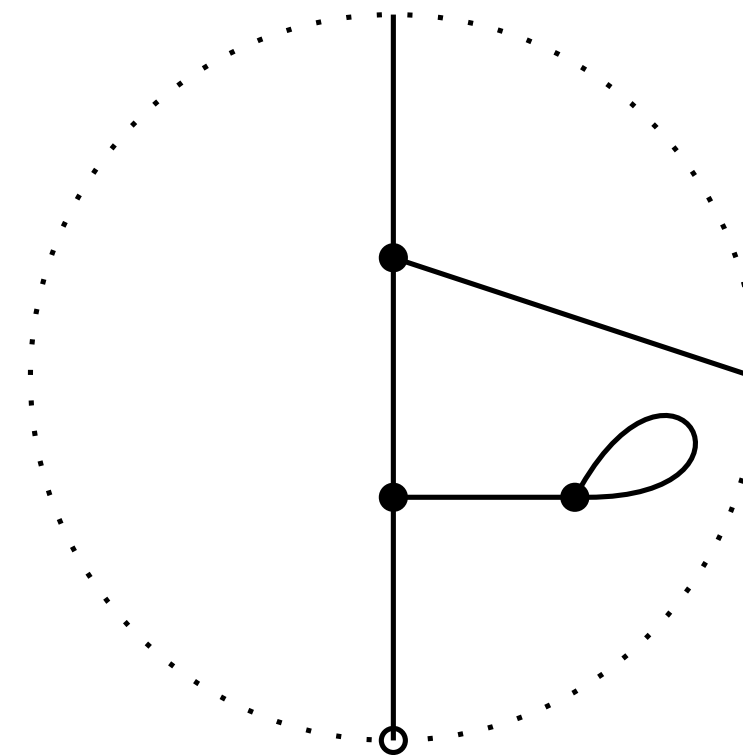
$\lambda x.\lambda y.\lambda z.x(yz)$

(B)

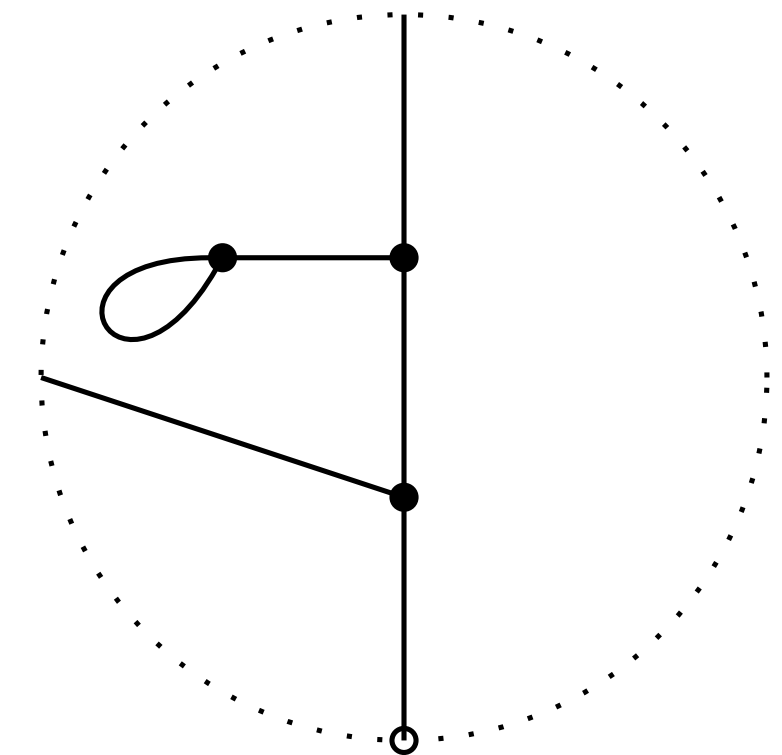


$\lambda x.\lambda y.\lambda z.(xz)y$

(C)



$x,y \vdash (xy)(\lambda z.z)$

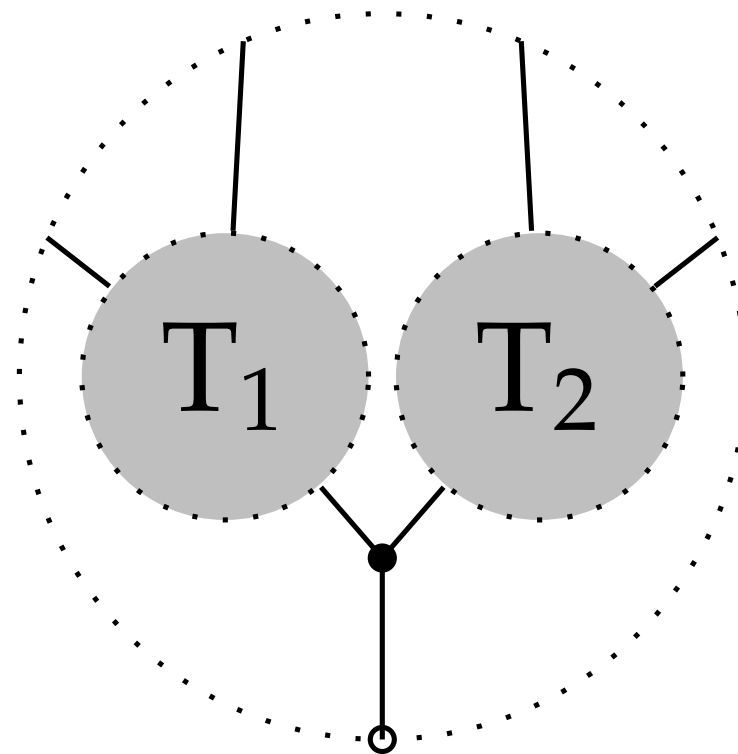


$x,y \vdash x((\lambda z.z)y)$

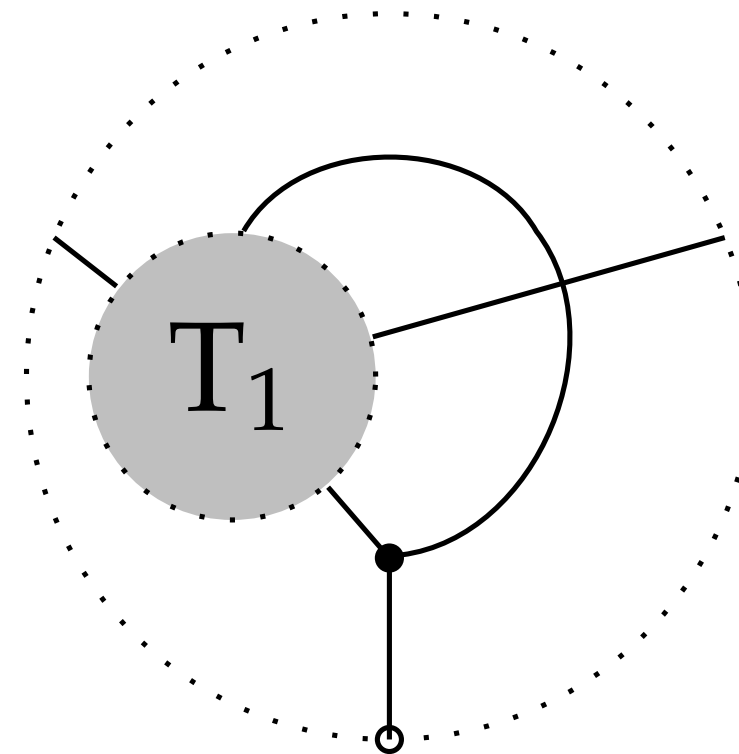
From rooted 3-valent maps to linear λ -terms

Step #1: generalize to 3-valent maps w/ ∂ of "free" edges, one marked as root.

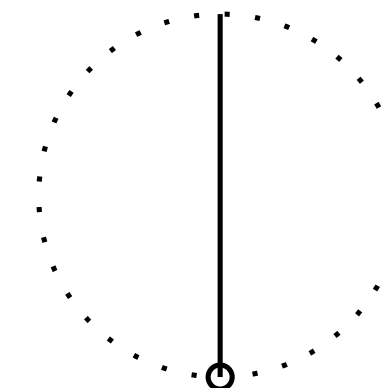
Step #2: observe any such map must have one of the following forms:



disconnecting
root vertex



connecting
root vertex

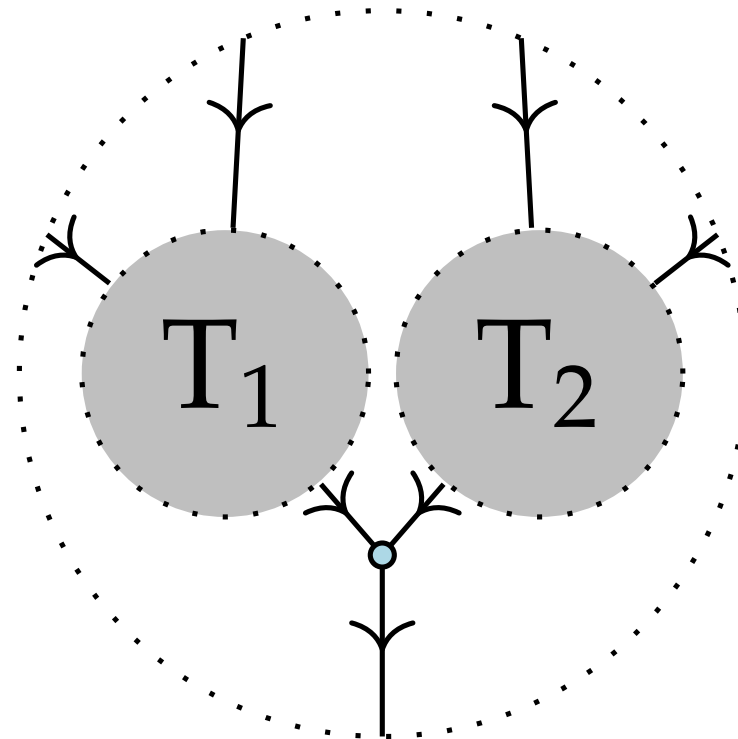


no
root vertex

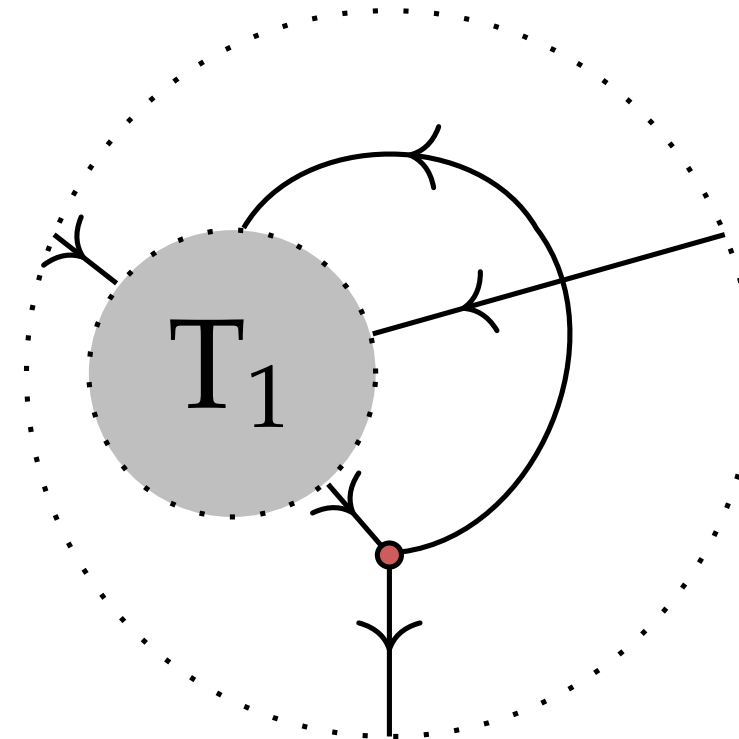
From rooted 3-valent maps to linear λ -terms

⋮

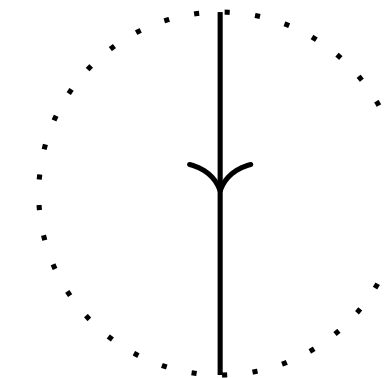
Step #3: observe this is exactly the inductive definition of linear λ -terms!



application



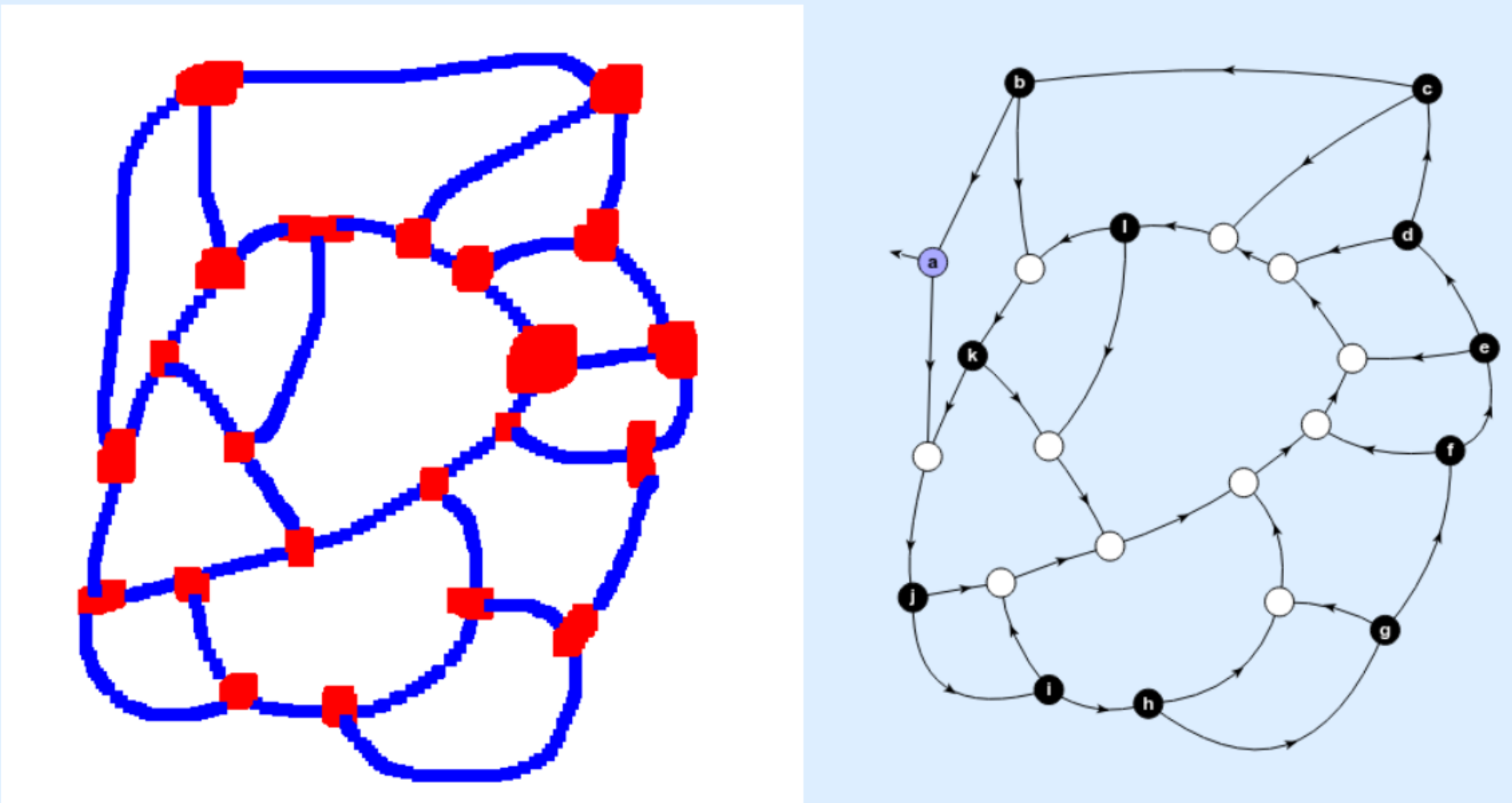
abstraction



variable

Demo: Jason Reed's "Interactive Lambda Maps Toy"

<https://jcreedcmu.github.io/demo/lambda-map-drawer/public/index.html>



The interface displays two visualizations side-by-side. On the left is a blue map with red square nodes and blue edges. On the right is a directed graph with black circular nodes labeled with letters (a, b, c, d, e, f, g, h, i, j, k, l) and white circular nodes. Arrows indicate the direction of the edges in the graph.

Clear
Export SVG

Examples:

Drawing Tools

- ☐ Node
- ☒ Edge
- ☐ Erase

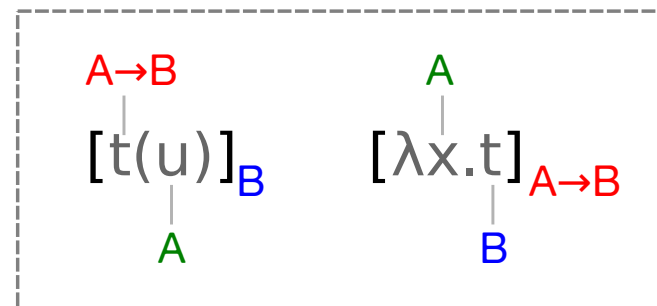
☐ Connectivity Debugging
☐ Cyclic Ordering
☐ Graph ☐ +Debugging Ids
☒ Lambda Graph
☐ +Root Choices
☒ +Variable Names
☐ +Variable Edges

$\lambda abcdefghij.a (\lambda k.b (\lambda l.c (d (e (f (g h (i j (k l))))))))$

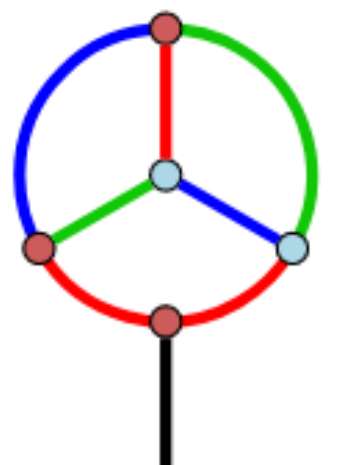
The Four Color Theorem as a typing problem

It is easy to check that the bijection sends planar 3-valent maps to ordered linear terms, and bridges to closed subterms. Now, note that any group G defines a closed multicategory where $A_1, \dots, A_n \rightarrow B$ iff $A_1 \cdots A_n = B$ and where $A \multimap B = B \cdot A^{-1}$. In particular, consider the Klein Four Group $\mathbb{V} = \mathbb{Z}_2 \times \mathbb{Z}_2$ as a closed multicategory.

Claim: every unitless ordered linear term has a \mathbb{V} -typing such that no subterm is assigned the unit type $(0,0) \in \mathbb{V}$. More generally, every ordered linear term has a \mathbb{V} -typing such that a subterm u is assigned the unit type iff u is closed.



$$\begin{array}{c}
 \frac{x : \beta \multimap \gamma \vdash x : \beta \multimap \gamma}{x : \beta \multimap \gamma, y : \alpha \multimap \beta, z : \alpha \vdash x(yz) : \gamma} \quad \frac{\frac{\overline{y : \alpha \multimap \beta \vdash y : \alpha \multimap \beta} \quad \overline{z : \alpha \vdash z : \alpha}}{y : \alpha \multimap \beta, z : \alpha \vdash y(z) : \beta}}{x : \beta \multimap \gamma, y : \alpha \multimap \beta, z : \alpha \vdash x(yz) : \gamma} \\
 \frac{x : \beta \multimap \gamma, y : \alpha \multimap \beta \vdash \lambda z. x(yz) : \alpha \multimap \gamma}{x : \beta \multimap \gamma \vdash \lambda y. \lambda z. x(yz) : (\alpha \multimap \beta) \multimap (\alpha \multimap \gamma)} \\
 \vdash \lambda x. \lambda y. \lambda z. x(yz) : (\beta \multimap \gamma) \multimap ((\alpha \multimap \beta) \multimap (\alpha \multimap \gamma))
 \end{array}$$





5. In pursuit of our
cartographic quest...

The LambdaComb Project

www.lix.polytechnique.fr/LambdaComb/

The project aims to:

- develop rigorous logical perspectives on maps and related combinatorial objects
- develop precise quantitative perspectives on lambda calculus and related systems

We proposed six high-level "workpackages"...

WP1a: A bilingual dictionary between graph theory and lambda calculus

higher-connectivity of λ -terms?

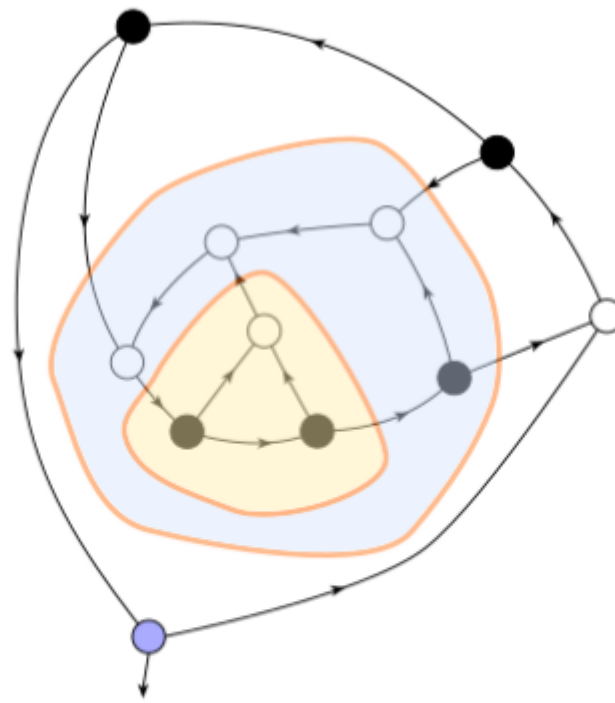


Figure 6: Example of a 3-edge-connected planar (= ordered linear) term $t = \lambda a. \lambda b. \lambda c. a(\lambda d. \lambda e. \lambda f. (b(cd))(ef))$. We have highlighted two different 3-cuts in t of type $(U \rightarrow U) \rightarrow U$ (in yellow) and of type $U \rightarrow (U \rightarrow U)$ (in blue).

complexity of planar/bridgeless normalization*?



Figure 7: β -reduction and η -expansion of linear lambda terms as certain natural surgeries on trivalent graphs (corresponding to the “unzipping” and “bubbling” moves of [84]).

*already have some results with A. Das, D. Mazza, and L. T. D. Nguyễn...

↑
(Tito, new postdoc @ LIX!)

WP1b: Bijections with blossoming trees, walks in the quarter-plane, and more

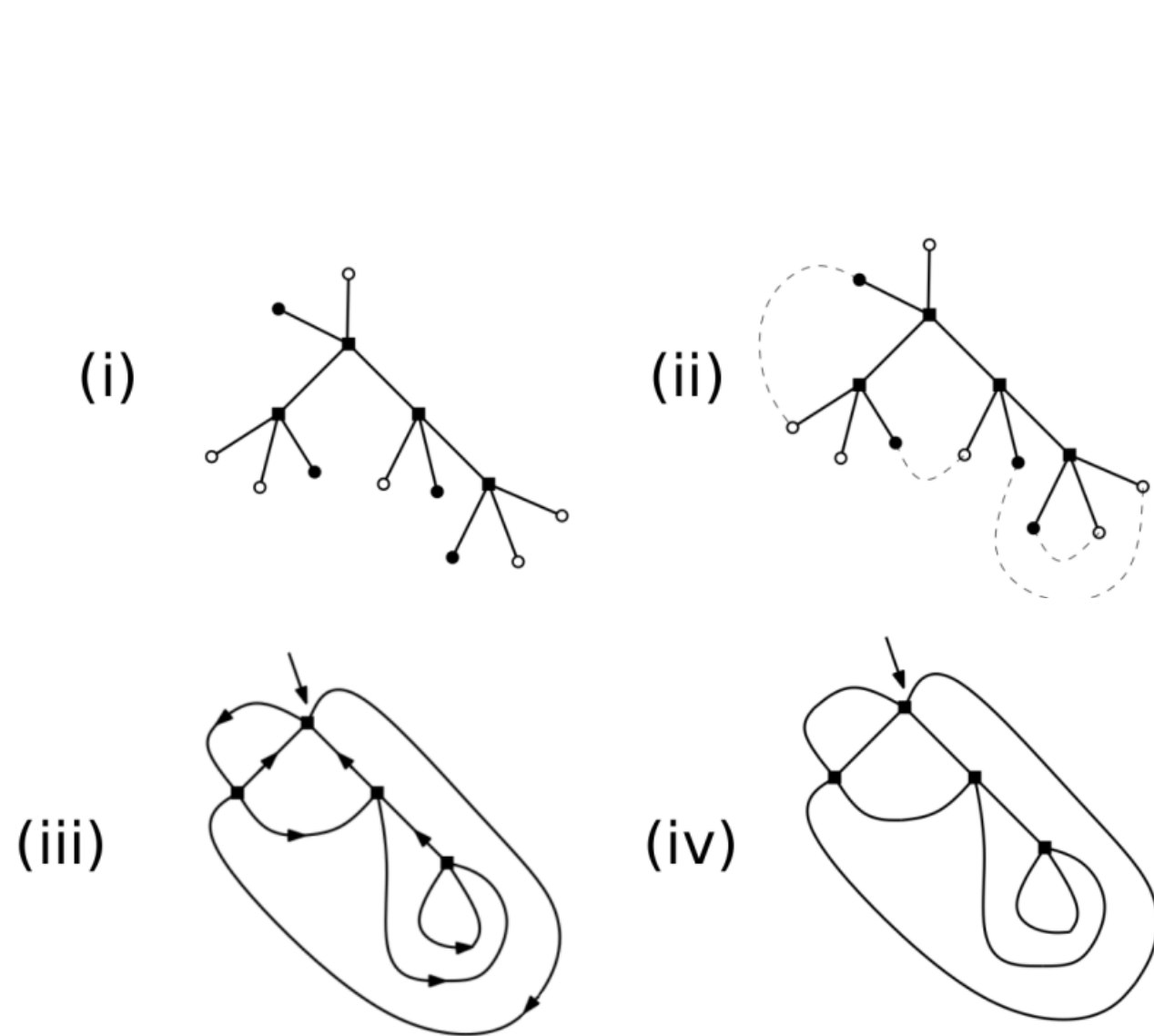


Figure 8: (i) An unrooted balanced blossoming tree T . (ii) The canonical matching of black and white leaves of T . (iii) The associated rooted 4-valent planar map equipped with a canonical 2-orientation. (iv) The result of forgetting the orientation. (Diagrams taken from Figure 1.17 of [77].)

[77] G. Schaeffer. Planar maps. In M. Bóna, editor, *Handbook of Enumerative Combinatorics*. CRC, 2015. [oa](#):

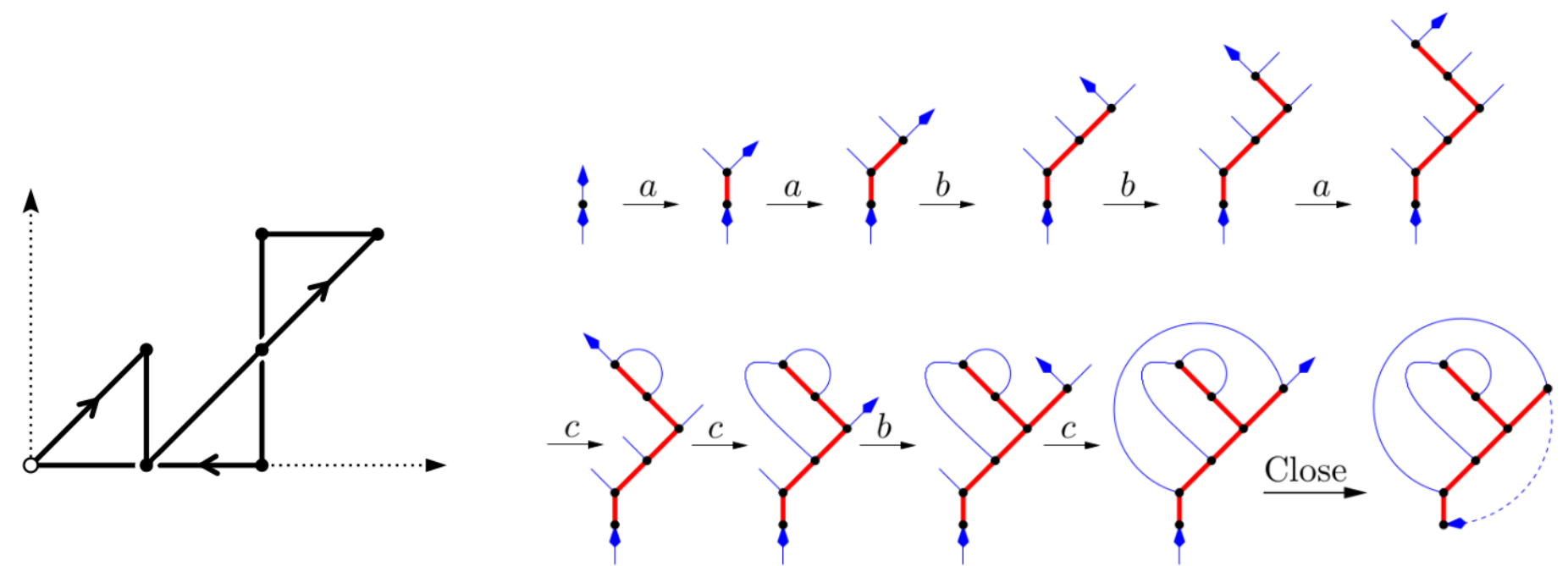


Figure 9: A Kreweras walk in the quarter-plane, and the construction of the corresponding bridgeless planar 3-valent map equipped with a distinguished depth tree. (Diagrams on the right taken from Figure 19 of [9]. Note that the map is constructed by reading the walk in reverse, with the labels indicating the types of steps $a = \leftarrow$, $b = \downarrow$, and $c = \nearrow$.)

[9] O. Bernardi. Bijective counting of Kreweras walks and loopless triangulations. *J. Comb. Th. A*, 114(5):931–956, 2007. [oa](#):

WP1c: Category-theoretic and operadic views of combinatorics

This WP is more of a "unifying outlook" rather than a specific set of problems.

Still, we can identify one particularly natural group of problems.

These bijections suggest the existence of a strong connection between various types of geometric and combinatorial objects (maps, trees, walks, ...) and various theories of closed and monoidal categories. Can these connections be strengthened to *full and faithful functors*?

This kind of research is somewhat parallel to and inspired from the foundational work by Joyal and Street relating string diagrams to braided monoidal categories. Does this analogy extend to *a continuous path from their work to our research*?

WP2a: Asymptotic analysis of parameters in lambda calculus and maps

can we apply these connections + techniques of analytic combinatorics to estimate distributions of parameters in large random maps and λ -terms? (cf. A. Singh's PhD work)

can we develop a quantitative perspective on the combinatorics of reduction?

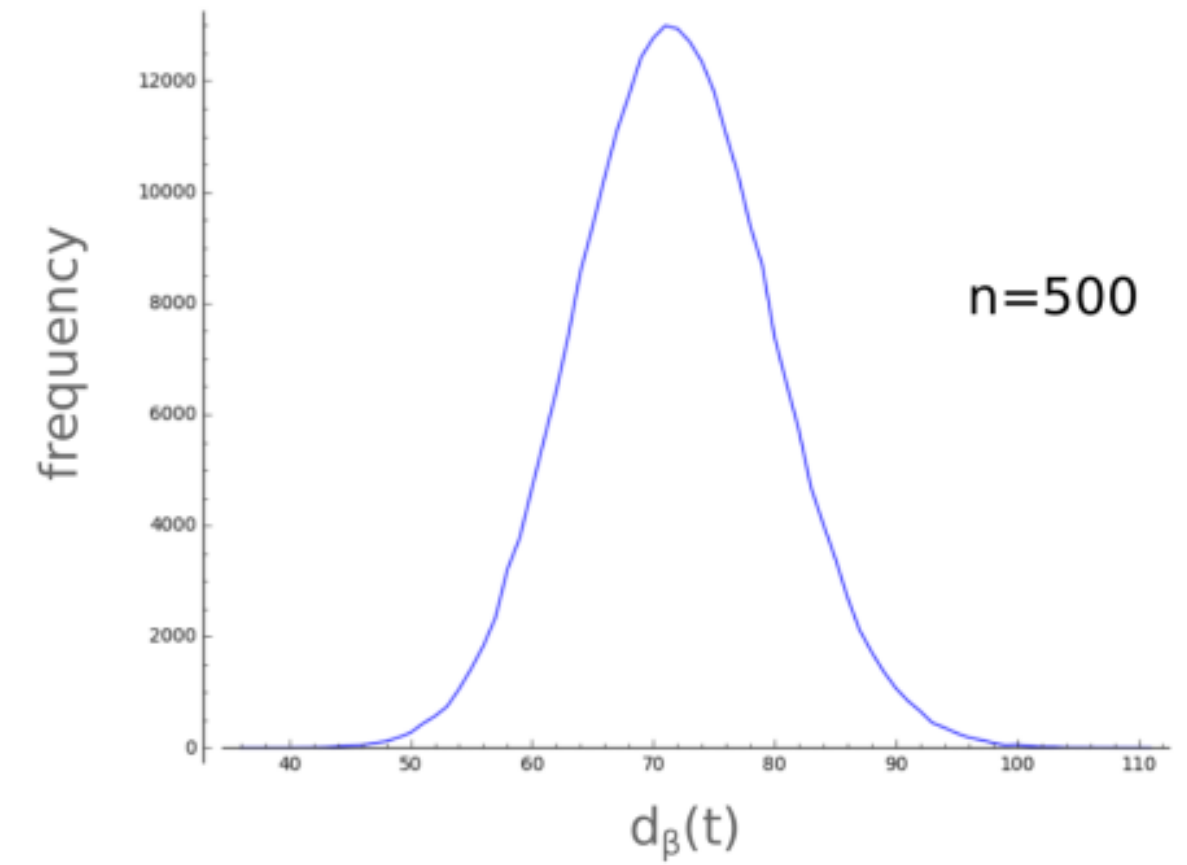


Figure 10: Histogram of distance to β -normal form (or equivalently length of longest β -reduction sequence), for randomly sampled closed linear terms of size $3 \cdot 500 + 2$.

WP2b: Random sampling and experimental lambda calculus

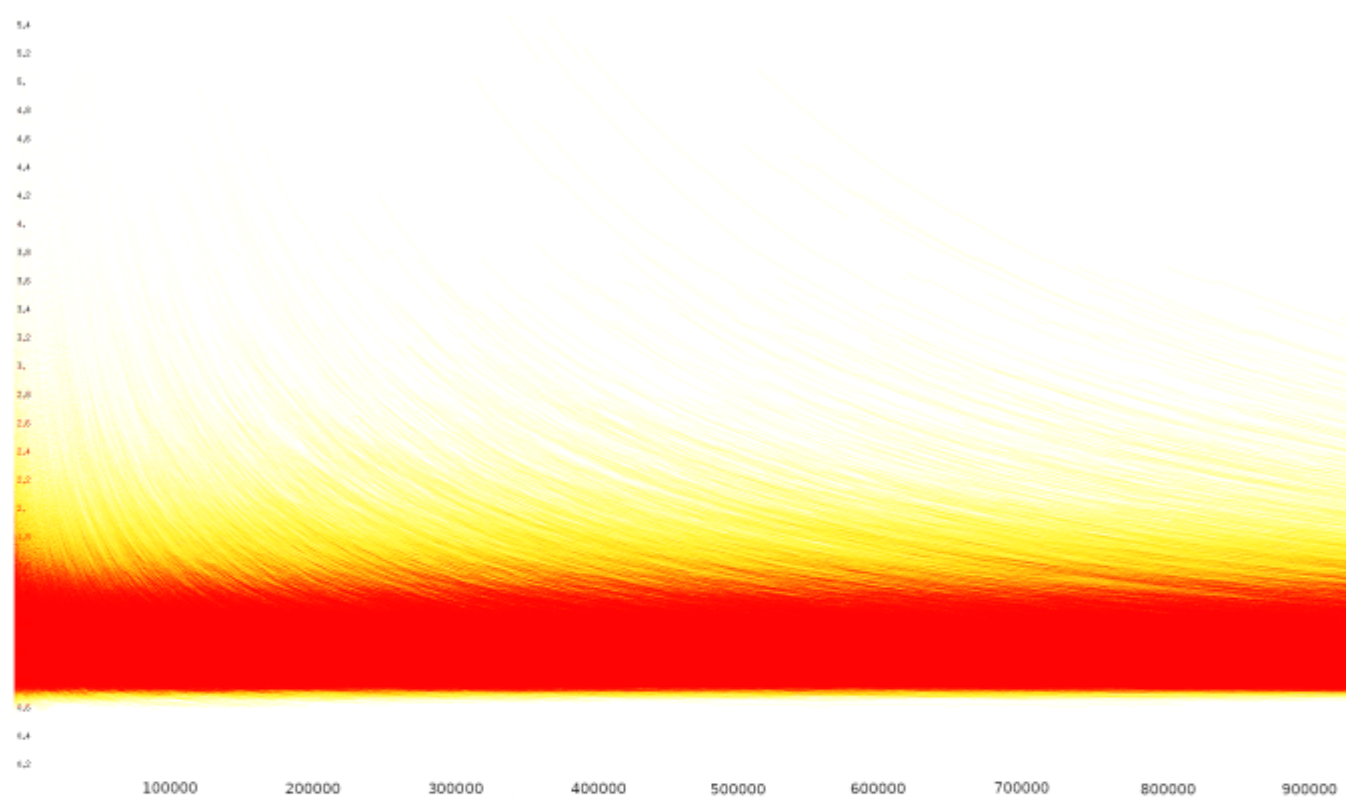


Figure 11: Experimental distribution of the size of the non-linear term $Y \lambda c. \lambda x. \lambda y. c(cxy)(cxy)$ (where $Y = \lambda f. (\lambda x. f(xx)) (\lambda x. f(xx))$ is a standard fixed-point combinator) over a million iterations of randomized evaluation; the graph is renormalized by a factor of $n \log(n) \log^2(n)$, the conjectured asymptotic size.

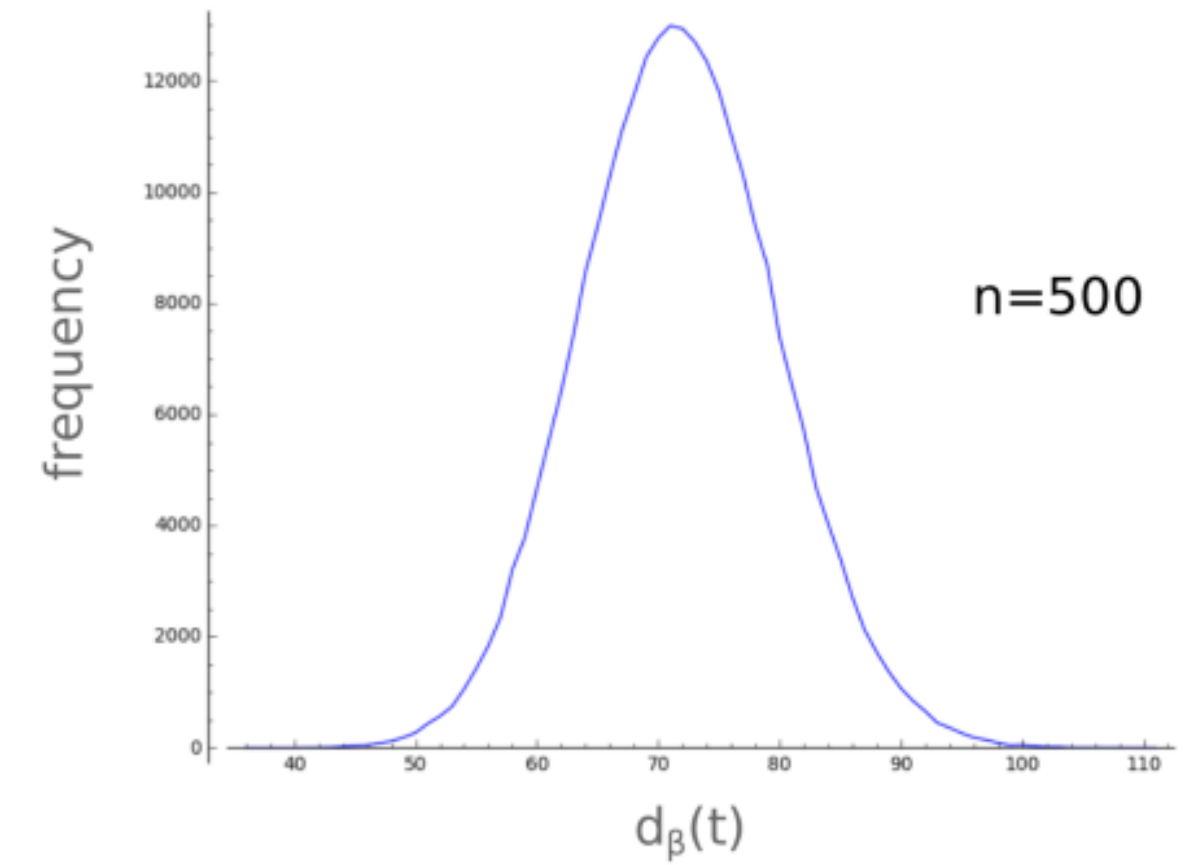


Figure 10: Histogram of distance to β -normal form (or equivalently length of longest β -reduction sequence), for randomly sampled closed linear terms of size $3 \cdot 500 + 2$.

WP2c: Typed enumeration

What should be the role of types in these connections?

Can the analogy between typing and coloring be pursued?

Is there a relationship with Tutte's original work (and Bousquet-Mélou's) on enumeration of colored maps?

$$\begin{array}{c}
 \frac{}{x : \beta \multimap \gamma \vdash x : \beta \multimap \gamma} \quad \frac{}{y : \alpha \multimap \beta \vdash y : \alpha \multimap \beta} \quad \frac{}{z : \alpha \vdash z : \alpha} \\
 \hline
 \frac{}{x : \beta \multimap \gamma, y : \alpha \multimap \beta, z : \alpha \vdash x(yz) : \gamma} \\
 \hline
 \frac{}{x : \beta \multimap \gamma, y : \alpha \multimap \beta \vdash \lambda z. x(yz) : \alpha \multimap \gamma} \\
 \hline
 \frac{}{x : \beta \multimap \gamma \vdash \lambda y. \lambda z. x(yz) : (\alpha \multimap \beta) \multimap (\alpha \multimap \gamma)} \\
 \hline
 \vdash \lambda x. \lambda y. \lambda z. x(yz) : (\beta \multimap \gamma) \multimap ((\alpha \multimap \beta) \multimap (\alpha \multimap \gamma))
 \end{array}$$

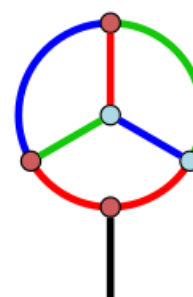


Figure 5: Principal typing derivation for the B term (cf. Figure 4), and the corresponding edge-coloring obtained by taking α, β, γ to be three distinct non-zero values of the Klein Four Group (here colored α = red, β = blue, γ = green) and interpreting $A \multimap B := -A + B$.

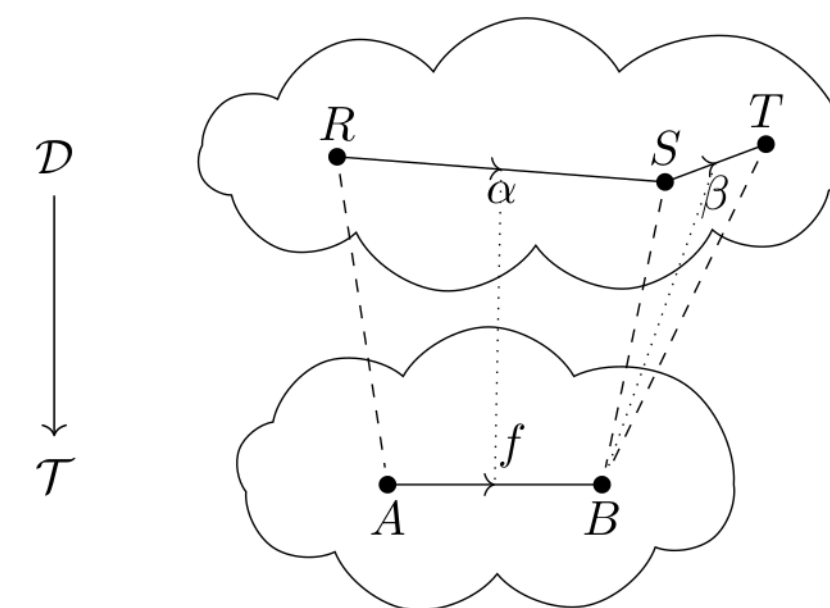


Figure 12: Type systems as functors. Here the morphism $\alpha : R \rightarrow S$ in \mathcal{D} may be considered abstractly as a “typing derivation” for the morphism $f : A \rightarrow B$, and the morphism $\beta : S \rightarrow T$ as a subtyping derivation over the identity morphism on B , cf. [62].

The LambdaComb Project

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summary of proposed work:

WP1a: A bilingual dictionary between graph theory and lambda calculus

WP1b: Bijections with blossoming trees, walks in the quarter-plane, and more

WP1c: Category-theoretic and operadic views of combinatorics

WP2a: Asymptotic analysis of parameters in lambda calculus and maps

WP2b: Random sampling and experimental lambda calculus

WP2c: Typed enumeration

kick-off meeting: 11 April @ LIX!