A tutorial on (generalized) fibrations for logic, automata and language theory

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Ecole Polytechnique (Laboratoire d'Informatique de l'X)

Dagstuhl, 2 April 2025
Workshop on "Categories for Automata and Language Theory"

[1]Based on joint work with Paul-André Melliès. Main references: "Functors are type refinement systems" (POPL 2015) + "The categorical contours of the Chomsky-Schützenberger representation theorem" (LMCS, to appear, arXiv:2405.14703).

Some topological intuitions

Motivation

Imagine we are analyzing different spaces, more or less complicated.

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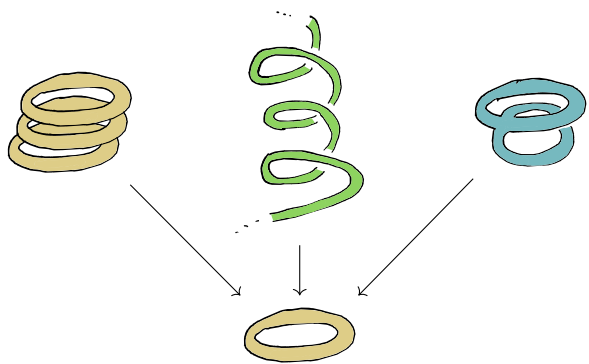

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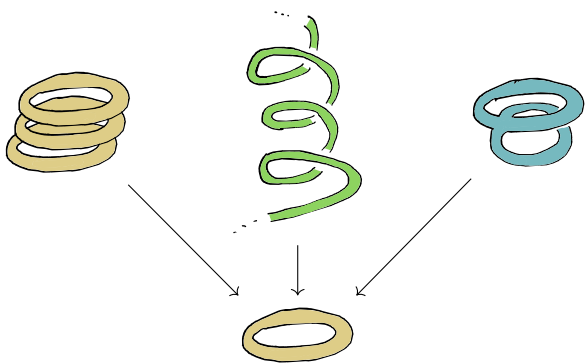

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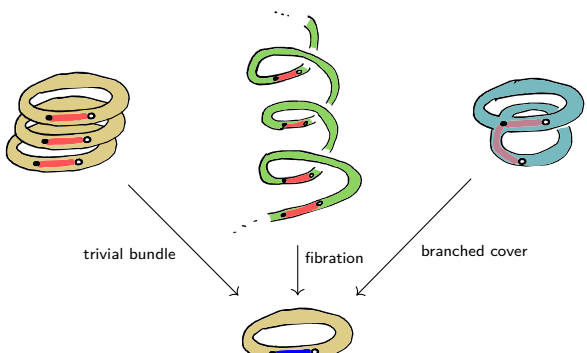


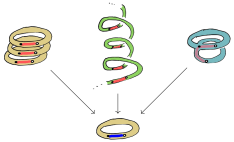
Basic terminology

A space E mapping down to a space B is sometimes called a "bundle" $p : E \to B$.

E is called the **total space**, B the **base space**, p the **projection**.

For any point $a \in B$, its inverse image $p^{-1}(a)$ is called the **fiber** over a. In general it is not just a set of isolated points, but a subspace of E.

Path lifting problems



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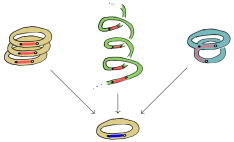

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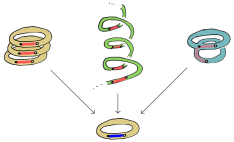

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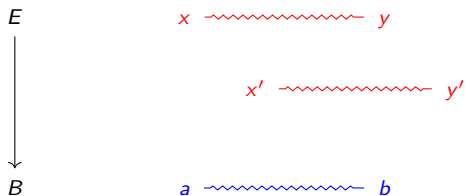

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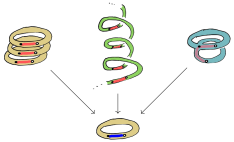

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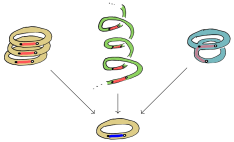

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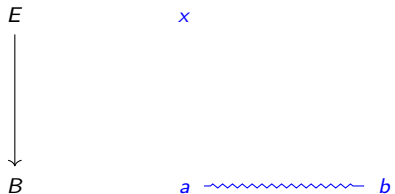

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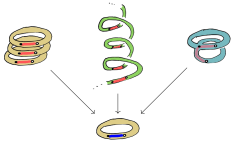

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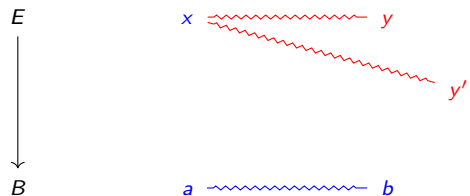

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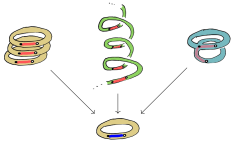

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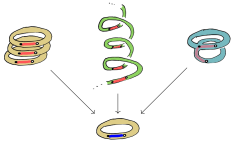

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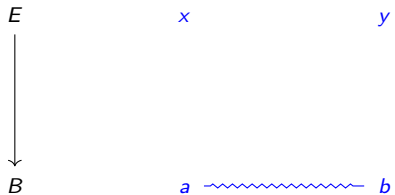

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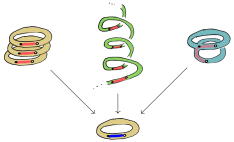

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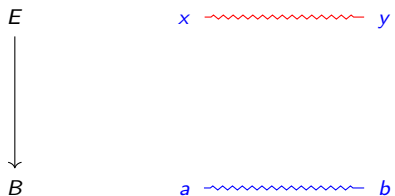

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Fibrational perspectives on deductive systems

Old idea (Lawvere 1969): the structure of predicate logic and the nature of quantifiers may be clarified by organizing proofs into bundles lying over spaces of formulas.

This idea may be extended to a variety of computational and deductive systems, including program logics, finite-state automata and context-free grammars.

Proof, recognition, and parsing may be thus reduced to lifting problems.

Plan

1. Functors as type refinement systems (aka, bundles of categories)
2. Finite-state automata as bundles I: determinism and codeterminism
3. Finite-state automata as bundles II: ULF and finitary functors
4. Finite-state automata as bundles III: NFAs over categories

Functors as type refinement systems (aka, bundles of categories)

Some terminology and notation

We like to think of a category $\mathcal{D}$ equipped with a functor $p : \mathcal{D} \to \mathcal{C}$ as defining a kind of abstract type system, or **type refinement system**. The idea is that the bundle $p : \mathcal{D} \to \mathcal{C}$ provides a source of *additional typing information* for the arrows of $\mathcal{C}$.

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A *typing judgment* is a triple (S, f, T) of an arrow $f : A \to B$ of $\mathcal{C}$ and a pair of objects $S \sqsubset A$ and $T \sqsubset B$ of $\mathcal{D}$ refining its domain and codomain.

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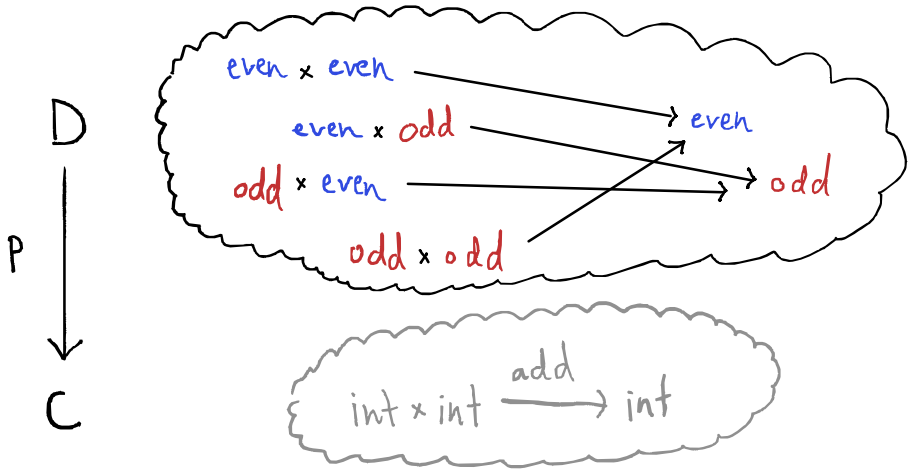
A *derivation* of a typing judgment (S, f, T) is an arrow $\alpha : S \to T$ of $\mathcal{D}$ s.t. $p(\alpha) = f$.

A picture to have in mind



Sequent calculus

We use sequent notation $S \Longrightarrow_f T$ for judgments, and write inference rules to stand for transformations from derivations of the premises to a derivation of the conclusion.

Proposition: the following rules are valid for any type refinement system $p : \mathcal{D} \to \mathcal{C}$.

$$\frac{S \underset{f}{\Longrightarrow} R \qquad R \underset{g}{\Longrightarrow} T}{S \underset{fg}{\Longrightarrow} T}\ \text{COMP} \qquad\qquad \frac{R \sqsubset A}{R \underset{id_A}{\Longrightarrow} R}\ \text{ID}$$

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Proof:

(COMP): Let α and β be derivations of the premises, meaning that there are arrows $\alpha : S \to R$ and $\beta : R \to T$ in $\mathcal{D}$ such that $p(\alpha) = f$ and $p(\beta) = g$. Then the arrow $\alpha\beta : S \to T$ is a derivation of the conclusion, since $p(\alpha\beta) = p(\alpha)p(\beta) = fg$.

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(ID): $p(id_R : R \to R) = id_{p(R)} = id_A$.

Subtyping

We define subtyping as a special case of typing:

$$R \leqslant_A R' \quad := \quad R \underset{id_A}{\Longrightarrow} R'$$

Note that subtyping is only meaningful between refinements of the same type.

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Proof:

$$\begin{array}{ccccccccc} \mathcal{D} & & S & \xrightarrow{\alpha} & S' & \xrightarrow{\beta} & T' & \xrightarrow{\gamma} & T \\ {\scriptstyle p}\downarrow & & & & & & & & \\ \mathcal{C} & & A & = & A & \xrightarrow{f} & B & = & B \end{array}$$

Subtyping

Restricting to subtyping derivations defines a family of *fiber categories*

$$\mathcal{D}_A = p^{-1}(id_A) = \{\, \alpha : R \to R' \mid p(\alpha) = id_A \,\} \subset \mathcal{D}.$$

If these contain only identity arrows (i.e., $p(\alpha) = id_A$ implies $\alpha = id_R$ for some R) then subtyping is trivial and p is said to have **discrete fibers**.

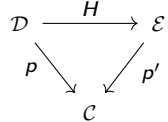
Ambiguity

A judgment $S \Longrightarrow_f T$ may in general have more than one derivation, in other words, a type refinement system may be *ambiguous*.

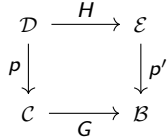
By definition, $p : \mathcal{D} \to \mathcal{C}$ is unambiguous iff p is **faithful**, i.e., for any pair of arrows $\alpha, \alpha' : S \to T$ with the same source and target, if $p(\alpha) = p(\alpha')$ then $\alpha = \alpha'$.

Morphism of type refinement systems

Two bundles over $\mathcal{C}$ may be related by a commutative triangle:



More generally, two bundles may be related by a commutative square:



Such a morphism maps p-derivations of $S \Longrightarrow_f T$ to p'-derivations of $H\,S \Longrightarrow_{G\,f} H\,T$.

General perspectives on type refinement systems

This perspective may be applied to *any* functor $p : \mathcal{D} \to \mathcal{C}$.

We are typically interested in functors with some additional properties/structure.

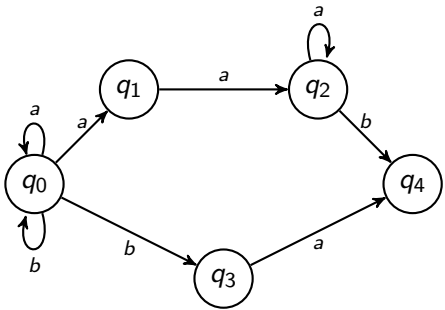
Considering different logico-computational systems as bundles, what are their fibrational characteristics?

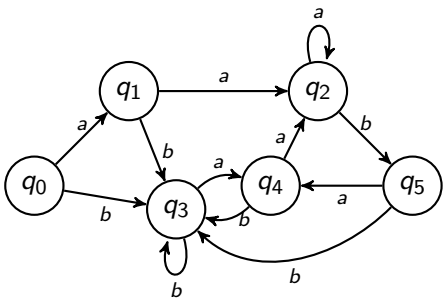
Finite-state automata as bundles I: determinism and codeterminism

Automata as graph homomorphisms

The underlying transition graph of any NFA (without ϵ-transitions) over the alphabet Σ is entirely described by a graph homomorphism $\phi : \mathbb{G} \to \mathbb{B}_\Sigma$ into the *bouquet graph* with one node $*$ and a loop $a : * \to *$ for every $a \in \Sigma$.

Automata as graph homomorphisms





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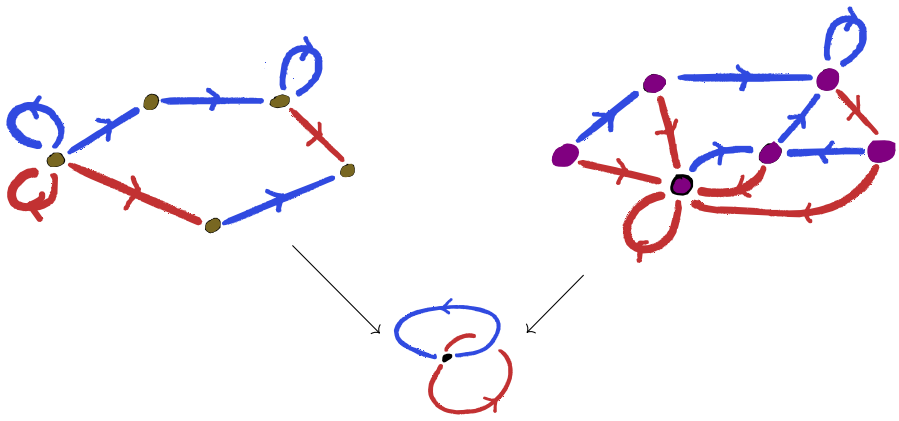

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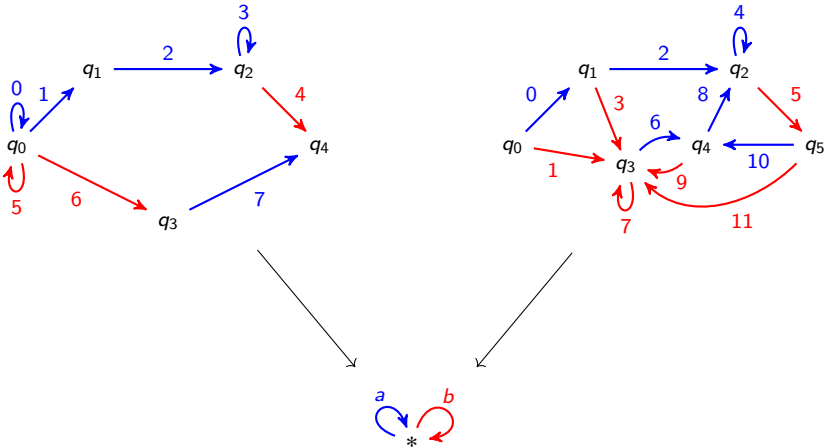
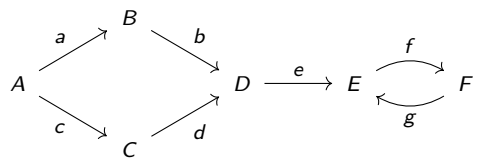


Free categories

To any graph $\mathbb{G}$ is associated a **free category** $\mathcal{F}\mathbb{G}$ whose objects are nodes and whose arrows are paths. For example, the free category over

$\mathbb{G}$ =



has $\hom(A, D) = \{\, ab, cd \,\}$ and $\hom(E, E) = (fg)^*$.

Universal property of free categories: any functor $\mathcal{F}\mathbb{G} \to \mathcal{C}$ into a category $\mathcal{C}$ is uniquely determined by a graph homomorphism $\mathbb{G} \to \mathcal{C}$ into the underlying graph of $\mathcal{C}$.

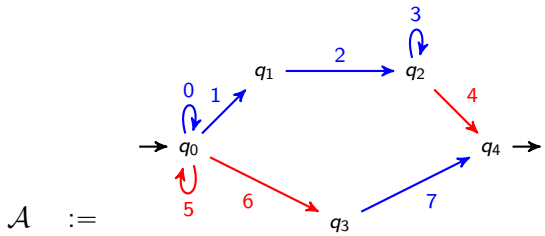
Recognition as a path lifting problem

Any *word* $w \in \Sigma^*$ corresponds to a *path* in $\mathbb{B}_\Sigma$, i.e., to an arrow $* \to *$ in $\mathcal{F}\mathbb{B}_\Sigma$.

Any graph homomorphism $\phi : \mathbb{G} \to \mathbb{H}$ induces a corresponding functor between free categories $\mathcal{F}\phi : \mathcal{F}\mathbb{G} \to \mathcal{F}\mathbb{H}$, sending paths in $\mathbb{G}$ to paths in $\mathbb{H}$ *of the same length*.

Let $\mathcal{A}$ be an NFA with transition graph $\phi : \mathbb{G} \to \mathbb{B}_\Sigma$ and associated functor $p = \mathcal{F}\phi$. Then $\mathcal{A}$ accepts $w \in \Sigma^*$ just in case there is an arrow $\alpha : q_0 \to q_f$ in $\mathcal{F}\mathbb{G}$ such that $p(\alpha) = w$, from an initial state q_0 to an accepting state q_f.

Recognition as a path lifting problem



q_0 q_4

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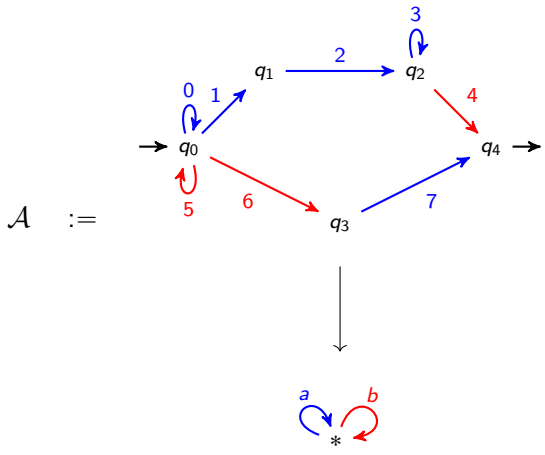

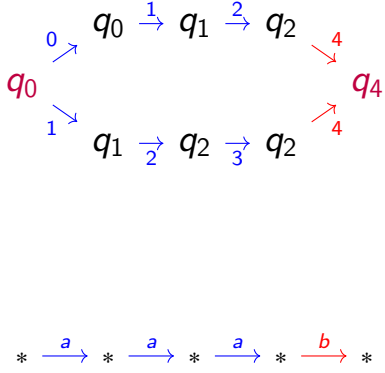

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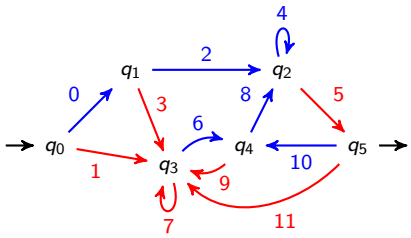

q_0

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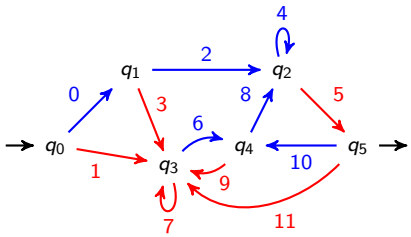

$q_0 \xrightarrow{0} q_1$

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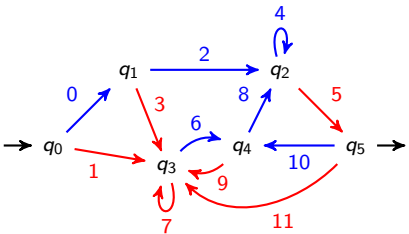

$\mathcal{A} \quad :=$

$q_0 \xrightarrow{0} q_1 \xrightarrow{2} q_2$

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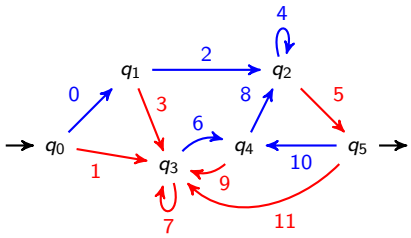
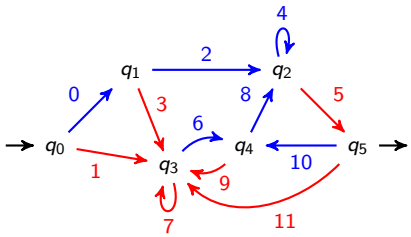


$q_0 \xrightarrow{0} q_1 \xrightarrow{2} q_2 \xrightarrow{4} q_2$

$* \xrightarrow{a} * \xrightarrow{a} * \xrightarrow{a} * \xrightarrow{b} *$

Recognition as a path lifting problem

$\mathcal{A}$:=



$q_0 \xrightarrow{0} q_1 \xrightarrow{2} q_2 \xrightarrow{4} q_2 \xrightarrow{5} q_5$

$* \xrightarrow{a} * \xrightarrow{a} * \xrightarrow{a} * \xrightarrow{b} *$

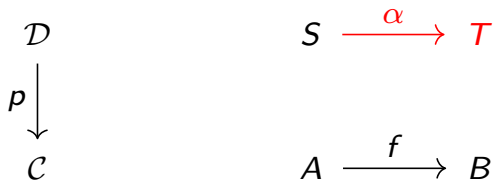
Determinism

Proposition: let $\phi : \mathbb{G} \to \mathbb{B}_\Sigma$ be a homomorphism of finite graphs. TFAE:

1. ϕ is the transition graph of a complete DFA.
2. for every node $q \in \mathbb{G}$ and every loop $a : * \to * \in \mathbb{B}_\Sigma$, there is a unique node q' and unique edge $e : q \to q'$ such that $\phi(e) = a$.
3. $\mathcal{F}\phi : \mathcal{F}\mathbb{G} \to \mathcal{F}\mathbb{B}_\Sigma$ is a *discrete opfibration*.

Definition

A functor $p : \mathcal{D} \to \mathcal{C}$ is a **discrete opfibration** if for any object $S \sqsubset A$ of $\mathcal{D}$ and any arrow $f : A \to B$ of $\mathcal{C}$ there exists a unique object $T \sqsubset B$ and arrow $\alpha : S \Longrightarrow_f T$.



Presheaves

A **covariant presheaf** on a category $\mathcal{C}$ is a functor $G : \mathcal{C} \to \mathrm{Set}$.

Explicitly, a covariant presheaf G consists of the following:

- a set G_A for every object A in $\mathcal{C}$;
- a function $G_f : G_A \to G_B$ for every arrow $f : A \to B$ in $\mathcal{C}$;
- such that $G_{fg} = G_f \,;G_g$ and $G_{id_A} = id_{G_A}$.

Category of elements

Given a presheaf $G : \mathcal{C} \to \mathrm{Set}$, the **category of elements** $\int G$ is defined like so:

- Objects are pairs (A, R) of an object A in $\mathcal{C}$ and an element $R \in G_A$.
- A morphism $(A, S) \to (B, T)$ is given by a morphism $f : A \to B$ in $\mathcal{C}$ such that the function $G_f : G_A \to G_B$ maps S to T.

This category is equipped with an evident projection functor $\pi_G : \int G \to \mathcal{C}$.

Proposition: $\pi_G : \int G \to \mathcal{C}$ is a discrete opfibration.

Example: monoid actions

Let M be a monoid, considered as a one-object category $\mathcal{B}_M =$ 

.

A presheaf $G : \mathcal{B}_M \to \mathrm{Set}$ consists of the following:

- a set $Q = G_*$;
- for every element $x \in M$, a function $G_x : Q \to Q$;
- such that $(q)G_{xy} = ((q)G_x)G_y$ and $(q)G_1 = q$.

This is equivalent to the data of a set Q equipped with a *right action* $Q \times M \to Q$.

To any such monoid action is associated a discrete opfibration $\pi_G : \int G \to \mathcal{B}_M$.

Example: monoid actions

When M is a free monoid $M = \Sigma^*$, then any any functor $\mathcal{B}_M \cong \mathcal{F}\mathbb{B}_\Sigma \to \mathrm{Set}$ is uniquely determined by a graph homomorphism $\mathbb{B}_\Sigma \to \mathrm{Set}$. (Equivalently: any monoid action $Q \times \Sigma^* \to Q$ is uniquely determined by its action on the generators $Q \times \Sigma \to Q$.)

Example: monoid actions

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For example, let $\Sigma = \{ a, b \}$, and consider the functor $G : \mathcal{F}\mathbb{B}_\Sigma \to \mathrm{Set}$ defined by

$$G_* = \{ q_0, \ldots, q_5 \}$$

$$G_a = \begin{pmatrix} q_0 & q_1 & q_2 & q_3 & q_4 & q_5 \\ q_1 & q_2 & q_2 & q_4 & q_2 & q_4 \end{pmatrix} \qquad G_b = \begin{pmatrix} q_0 & q_1 & q_2 & q_3 & q_4 & q_5 \\ q_3 & q_3 & q_5 & q_3 & q_3 & q_3 \end{pmatrix}$$

Then $\pi_G : \int G \to \mathcal{F}\mathbb{B}_\Sigma$ is exactly (isomorphic to) the DFA from a few slides back.

Duality of the fibered and indexed perspectives

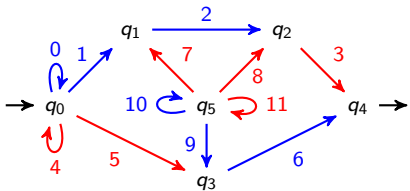
Conversely, to any discrete opfibration $p : \mathcal{D} \to \mathcal{C}$ is associated a **fiber functor** $G : \mathcal{C} \to \mathrm{Set}$ defined by taking $G_A = \{ R \in \mathcal{D} \mid R \sqsubset A \}$ and letting $G_f : G_A \to G_B$ be the function which maps any $R \sqsubset A$ to the unique $S \sqsubset B$ such that $R \Longrightarrow_f S$.

These two constructions

$$\text{discrete opfibration} \underset{\text{category of elements}}{\overset{\text{fiber functor}}{\rightleftarrows}} \text{covariant presheaf}$$

extend to an equivalence of categories between the category $\mathrm{DOpFib}(\mathcal{C})$ of discrete fibrations over $\mathcal{C}$ with commutative triangles as morphisms, and the category $[\mathcal{C}, \mathrm{Set}]$ of covariant presheaves on $\mathcal{C}$ with natural transformations as morphisms.

Running an automaton backwards

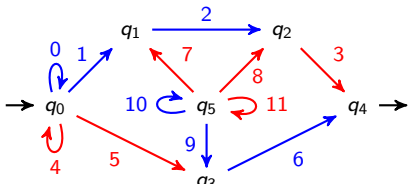


$\mathcal{A} \quad :=$

q_4

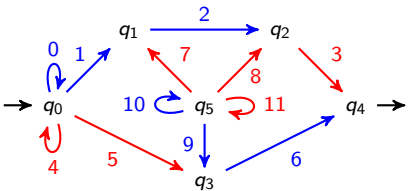
Running an automaton backwards

$\mathcal{A}$:=



$q_2 \xrightarrow{3} q_4$

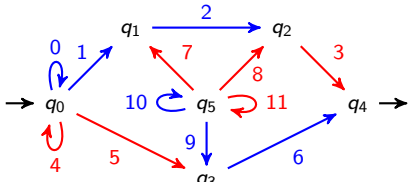
Running an automaton backwards



$\mathcal{A} \quad :=$

$q_5 \xrightarrow{8} q_2 \xrightarrow{3} q_4$

Running an automaton backwards

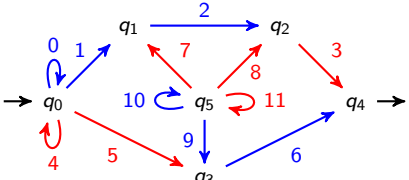


$\mathcal{A} \quad :=$

$$q_5 \xrightarrow{10} q_5 \xrightarrow{8} q_2 \xrightarrow{3} q_4$$

$$* \xrightarrow{a} * \xrightarrow{a} * \xrightarrow{b} * \xrightarrow{b} *$$

Running an automaton backwards



$\mathcal{A} \quad :=$

$$q_5 \xrightarrow{10} q_5 \xrightarrow{10} q_5 \xrightarrow{8} q_2 \xrightarrow{3} q_4$$

$$* \xrightarrow{a} * \xrightarrow{a} * \xrightarrow{b} * \xrightarrow{b} *$$

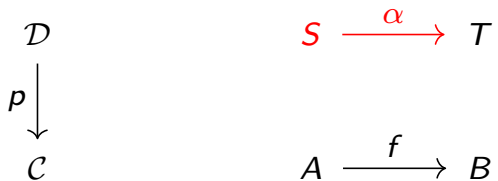
Codeterminism

Proposition: let $\phi : \mathbb{G} \to \mathbb{B}_\Sigma$ be a homomorphism of finite graphs. TFAE:

1. ϕ is the transition graph of a complete codeterministic automaton.
2. for every node $q \in \mathbb{G}$ and every loop $a : * \to * \in \mathbb{B}_\Sigma$, there is a unique node q' and unique edge $e : q' \to q$ such that $\phi(e) = a$.
3. $\mathcal{F}\phi : \mathcal{F}\mathbb{G} \to \mathcal{F}\mathbb{B}_\Sigma$ is a *discrete fibration*.

Definition

A functor $p : \mathcal{D} \to \mathcal{C}$ is a **discrete fibration** if for any object $T \sqsubset B$ of $\mathcal{D}$ and any arrow $f : A \to B$ of $\mathcal{C}$ there exists a unique object $S \sqsubset A$ and arrow $\alpha : S \Longrightarrow_f T$.



Discrete fibrations as contravariant presheaves

A **contravariant presheaf** on a category $\mathcal{C}$ is a functor $G : \mathcal{C} \to \mathrm{Set}^{\mathrm{op}}$.

Explicitly, a contravariant presheaf G consists of the following:

- a set G_A for every object A in $\mathcal{C}$;
- a function $G_f : G_B \to G_A$ for every arrow $f : A \to B$ in $\mathcal{C}$;
- such that $G_{fg} = G_f \circ G_g$ and $G_{id_A} = id_{G_A}$.

Symmetric versions of the category of elements and fiber functor constructions realize an equivalence of categories $\mathrm{DFib}(\mathcal{C}) \cong [\mathcal{C}, \mathrm{Set}^{\mathrm{op}}]^{\mathrm{op}}$.

Finite-state automata as bundles II: ULF and finitary functors

Characterizing nondeterministic finite-state automata

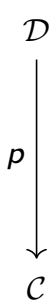
Recall that any NFA $\mathcal{A}$ over an alphabet Σ has an underlying transition graph $\phi : \mathbb{G} \to \mathbb{B}_\Sigma$ that induces an associated functor $p = \mathcal{F}\phi : \mathcal{F}\mathbb{G} \to \mathcal{F}\mathbb{B}_\Sigma$.

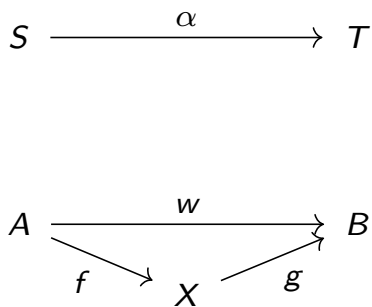
As we saw, $\mathcal{A}$ is deterministic (resp. codeterministic) iff p is a discrete opfibration (resp. discrete fibration). Either condition implies that p is faithful, i.e., unambiguous.

But what about in general? Can we characterize those functors $p : \mathcal{D} \to \mathcal{F}\mathbb{B}_\Sigma$ that are generated by the transition graph of an arbitrary (potentially ambiguous) NFA?

ULF functors (aka discrete Conduché fibrations)

A functor $p : \mathcal{D} \to \mathcal{C}$ has **unique lifting of factorizations** if for every arrow α in $\mathcal{D}$ and pair of arrows f, g in $\mathcal{C}$ such that $p(\alpha) = fg$, there exist unique β and γ in $\mathcal{D}$ such that $\alpha = \beta\gamma$ and $p(\beta) = f$ and $p(\gamma) = g$.





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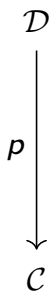
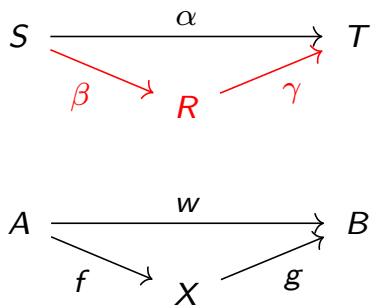




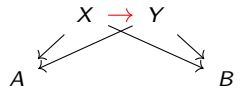
ULF $\Rightarrow$ discrete fibers

Proposition: if $p : \mathcal{D} \to \mathcal{C}$ is ULF then it has discrete fibers.

Proof: Suppose that $\alpha : R \to R'$ were a non-identity arrow such that $p(\alpha) = id_A$. Then (id_R, α) and $(\alpha, id_{R'})$ would be two distinct liftings of the factorization (id_A, id_A) of id_A, a contradiction.

(Hence ULF functors have no ϵ-transitions!)

Another equivalent view of ULF functors

Let Span be the bicategory whose objects are sets, whose 1-cells are spans $A \leftarrow X \rightarrow B$, and whose 2-cells are morphisms of spans 

.

To any functor whatsoever $p : \mathcal{D} \rightarrow \mathcal{C}$ is associated a *lax fiber functor* $G : \mathcal{C} \rightarrow \mathrm{Span}$ that sends every arrow $f : A \rightarrow B$ of $\mathcal{C}$ to the following span of sets:

$$p^{-1}(A) = \{\, S \mid S \sqsubset A \,\} \longleftarrow p^{-1}(f) = \{\, \alpha \mid \alpha : S \Longrightarrow_f T \,\} \longrightarrow p^{-1}(B) = \{\, T \mid T \sqsubset B \,\}$$

In general this fiber functor is only a lax functor, equipped with structure maps $\textsc{comp} : G_f\, G_g \Rightarrow G_{fg}$ and $\textsc{id} : id_{G_A} \Rightarrow G_{id_A}$ that are not necessarily invertible.

Proposition: p is ULF iff $\textsc{comp}$ and $\textsc{id}$ are invertible, i.e., iff G is a pseudofunctor.

ULF into free = free

Proposition (Street 1996): Let $p : \mathcal{D} \to \mathcal{C}$ be a functor into a category $\mathcal{C} = \mathcal{F}\mathbb{H}$ freely generated by some graph $\mathbb{H}$. Then p is ULF iff $\mathcal{D} = \mathcal{F}\mathbb{G}$ is free over some graph $\mathbb{G}$ and $p = \mathcal{F}\phi$ is generated by a graph homomorphism $\phi : \mathbb{G} \to \mathbb{H}$.

Proof: The direction ($\Leftarrow$) is immediate. For ($\Rightarrow$), take $\mathbb{G} = p^{-1}(\mathbb{H})$. Since the image of any arrow α in $\mathcal{D}$ uniquely decomposes as a path $p(\alpha) = e_1 \cdots e_n$ in $\mathbb{H}$, the ULF property implies that α uniquely decomposes as a composition of edges in $\mathbb{G}$, and p is generated by its restriction to the generators $\phi : \mathbb{G} \to \mathbb{H}$.

Finitary functors

We say that a functor $p : \mathcal{D} \to \mathcal{C}$ is **finitary** if the sets $p^{-1}(A)$ and $p^{-1}(f)$ are finite for every object A and arrow $f : A \to B$ of $\mathcal{C}$.

Equivalently, p is finitary iff the lax fiber functor $G : \mathcal{C} \to \mathrm{Span}$ factors via $\mathrm{FinSpan}$.

Proposition: Let $\phi : \mathbb{G} \to \mathbb{H}$ be a homomorphism into a finite graph $\mathbb{H}$. Then $\mathcal{F}\phi : \mathcal{F}\mathbb{G} \to \mathcal{F}\mathbb{H}$ is finitary iff $\mathbb{G}$ is finite.

The characterization

Corollary: Let $p : \mathcal{D} \to \mathcal{F}\mathbb{B}_\Sigma$ be a functor into the free category on a bouquet. TFAE:

1. $\mathcal{D} = \mathcal{F}\mathbb{G}$ and $p = \mathcal{F}\phi$ where $\phi : \mathbb{G} \to \mathbb{B}_\Sigma$ is the transition graph of a NFA.
2. p is finitary and ULF.

Finite-state automata as bundles III: NFAs over categories

Definition

A **NFA over a category** $\mathcal{C}$ is a tuple $\mathcal{A} = (\mathcal{Q}, p, q_0, q_f)$ consisting of

- a category $\mathcal{Q}$
- a finitary ULF functor $p : \mathcal{Q} \to \mathcal{C}$
- a pair of objects $q_0, q_f \in \mathcal{Q}$.

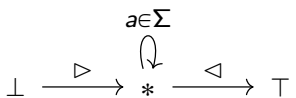
The automaton recognizes a **regular language of arrows** $\mathcal{L}_{\mathcal{A}}$ defined as the image of the homset $\mathcal{Q}(q_0, q_f)$ along p, that is,

$$\mathcal{L}_{\mathcal{A}} \stackrel{\text{def}}{=} \{\, p(\alpha) \mid \alpha : q_0 \to q_f \,\} \subseteq \mathcal{C}(A, B)$$

where $q_0 \sqsubset A$, $q_f \sqsubset B$.

Example: automata over free categories

Let $\mathbb{B}_{\Sigma}^{\triangleright\triangleleft}$ be the graph obtained from $\mathbb{B}_{\Sigma}$ by adjoining a pair of initial and final nodes:

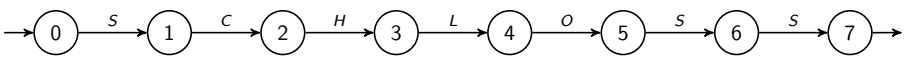


An automaton over $\mathcal{F}\mathbb{B}_{\Sigma}^{\triangleright\triangleleft}$ processes a word with explicit begin/end markers.

More generally, an automaton over the free category $\mathcal{F}\mathbb{H}$ on an arbitrary graph $\mathbb{H}$ recognizes a language of paths, which may be considered as "typed words".

Example: singleton automaton

For any $w \in \Sigma^*$ s.t. $|w| = n$, there is an $(n+1)$-state automaton recognizing $\{ w \}$:

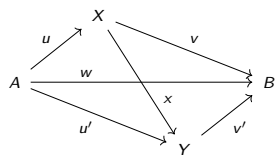


This is a special case of the following general construction...

Example: singleton automaton

Let $\mathcal{C}$ be a category and let $w : A \to B$ be an arrow of $\mathcal{C}$. We define an automaton $\mathcal{A}_w = (\mathcal{Q}, p, q_0, q_f)$ recognizing the singleton language $\{ w \} \subseteq \mathcal{C}(A, B)$ as follows:

- $\mathcal{Q} = \mathrm{Fact}_w$ is the *category of factorizations* of w.



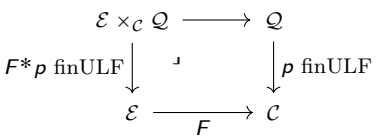
- $p = p_w : \mathrm{Fact}_w \to \mathcal{C}$ is the projection returning the middle of a factorization.
- the initial and final states are $q_0 = (id_A, w)$ and $q_f = (w, id_B)$.

Key fact: $p_w : \mathrm{Fact}_w \to \mathcal{C}$ is ULF. (But not necessarily finitary.)

If the category $\mathcal{C}$ has *finitary factorizations* then $\mathcal{A}_w$ defines a NFA.

Example: pullback automaton

Proposition: Finitary ULF functors are preserved by pullback along arbitrary functors.



(Easy way to see this: if $G : \mathcal{C} \to \mathrm{FinSpan}$ is the fiber functor of p, then $G \circ F$ is the fiber functor of F^*p.)

Corollary: Regular languages of arrows are closed under inverse image along functors, and under intersection.

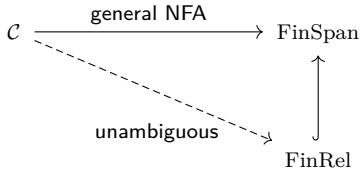
Classifying automata by their fibrational properties

We say that a categorical automaton is unambiguous/deterministic/codeterministic if p is faithful/opfibration/fibration, or equivalently, just in case its fiber functor factors as follows...

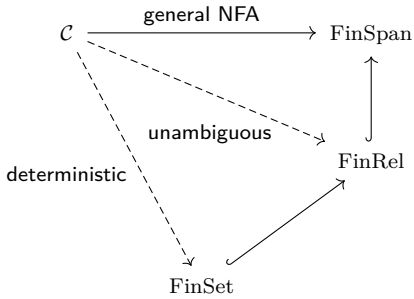
Classifying automata by their fibrational properties

$$\mathcal{C} \xrightarrow{\text{general NFA}} \mathrm{FinSpan}$$

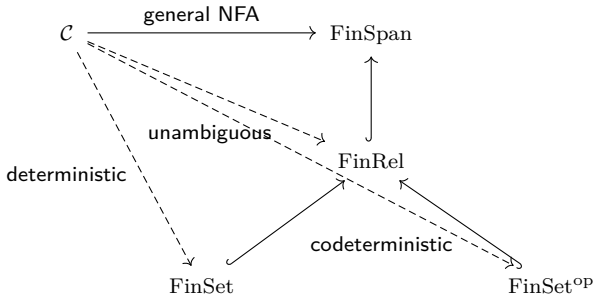
Classifying automata by their fibrational properties



Classifying automata by their fibrational properties



Classifying automata by their fibrational properties



Transducers between categories

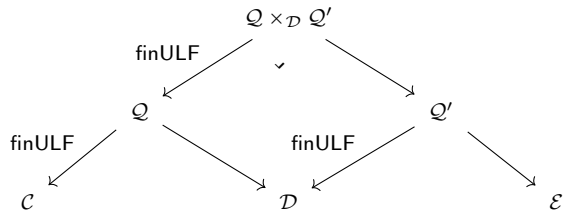
A **nondeterministic finite-state transducer** $\mathcal{M} : \mathcal{C} \nrightarrow \mathcal{D}$ is a pair $\mathcal{M} = (\mathcal{A}, O)$ consisting of an NFA $\mathcal{A} = (\mathcal{Q}, p, q_0, q_f)$ over $\mathcal{C}$ and a functor $O : \mathcal{Q} \to \mathcal{D}$. By taking the joint image of the homset $\mathcal{Q}(q_0, q_f)$, a transducer induces a relation

$$\{ (p(\alpha), O(\alpha)) \mid \alpha : q_0 \to q_f \} \subseteq \mathcal{C}(A, B) \times \mathcal{D}(X, Y)$$

where $(A, B) = (p(q_0), p(q_f))$ and $(X, Y) = (O(q_0), O(q_f))$.

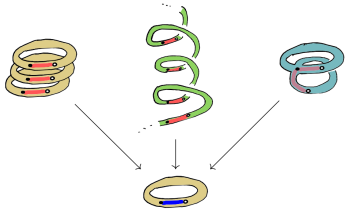
Transducers between categories

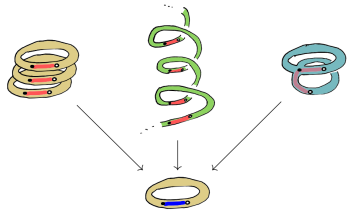
Transducers $\mathcal{M} : \mathcal{C} \multimap \mathcal{D}$ and $\mathcal{M}' : \mathcal{D} \multimap \mathcal{E}$ can be composed as follows:



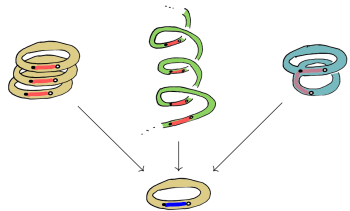
Unambiguous/deterministic/codeterministic transducers are closed under composition.

Concluding perspectives



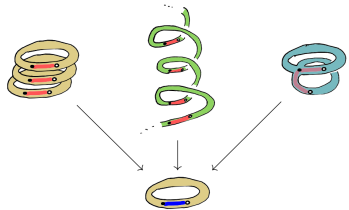


Finite-state automata may be considered as bundles of categories (without violence!), in a way that allows to treat them uniformly like other type refinement systems.



Finite-state automata may be considered as bundles of categories (without violence!), in a way that allows to treat them uniformly like other type refinement systems.

Finitary ULF functors play an important role, taking nondeterminism with potential ambiguity as a starting point. (Accounting for ϵ-transitions is work-in-progress.)



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Moving from categories to operads (aka multicategories) allows to treat regular languages of trees and context-free languages of arrows, including an account of the Chomsky-Schützenberger representation theorem.

Extra slides

Traditionally (e.g. [2]) one looks at a finite-state machine as processing sequences of inputs drawn from a finite set A, the input alphabet: one then considers the free monoid A [...]. To the automaton can be associated a congruence γ of a finite index on A*. A*/γ being a finite monoid, one is then led to investigate relationships between the structure of this algebraic system and the combinatorial processing of input sequences.*

Recent research has established the possibility and the necessity of generalizing this model. For example, we are often interested in decompositions of automata. In such situations a component may receive its input from the output of some other component. This "preprocessing" imposes restrictions on the possible input sequences that need to be considered. A simple way to take into account these restrictions is to view a machine as processing input sequences that are paths in a finite directed multigraph.

D. Thérien and M. Sznajder-Glodowski (1988), "Finite categories and regular languages".

Categorical bifibrations for logic and languages

A functorial view of Hoare logic

Let $\mathcal{S}$ be the one-object category whose arrows $c : * \to *$ are sequential compositions of program commands defined over some global set of variables.

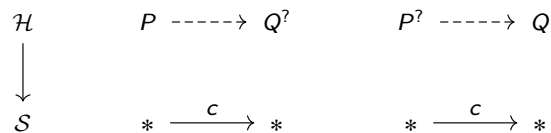
Suppose we have fixed some interpretation $[c] \subseteq S \times S$ of program commands as (potentially partial or nondeterministic) relations from input states to output states. (This interpretation should correspond to a functor $[-] : \mathcal{S} \to \mathrm{Rel}$ with $S = [*]$.)

Let $\mathcal{H}$ be the category whose objects are predicates on the global state, and whose arrows $P \to Q$ are given by commands c satisfying the Hoare triple $\{P\}c\{Q\}$, i.e., such that for all pairs of states s, s', if $P(s)$ and $(s, s') \in [c]$ then $Q(s')$.

Let $p : \mathcal{H} \to \mathcal{S}$ be the evident forgetful functor.

Two kinds of lifting problems

Given a command c and a predicate P (resp., a predicate Q), does there exists a predicate Q (resp. P) such that a $\{P\}c\{Q\}$?



In general there can be many solutions! (e.g., $Q^? = \mathit{true}$, $P^? = \mathit{false}$)

Weakest preconditions and strongest postconditions

Given a command c and a predicate Q, a *weakest precondition* is a predicate $wp(c, Q)$ such that for any predicate P, the triple $\{P\}c\{Q\}$ is valid iff $P \Rightarrow wp(c, Q)$.

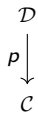
Dually, given a command c and a predicate P, a *strongest postcondition* is a predicate $sp(c, P)$ such that for any predicate Q, the triple $\{P\}c\{Q\}$ is valid iff $sp(c, P) \Rightarrow Q$.

TFAE:

1. Weakest preconditions and strongest postconditions exist for all P, c, Q.
2. $p : \mathcal{H} \to \mathcal{S}$ is a *bifibration*

What is a bifibration?

Formally:



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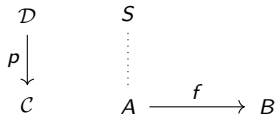

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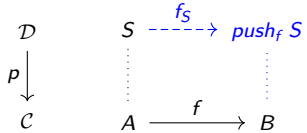

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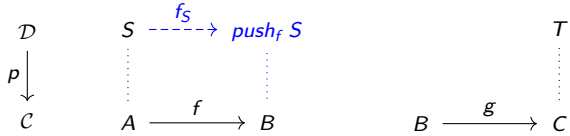

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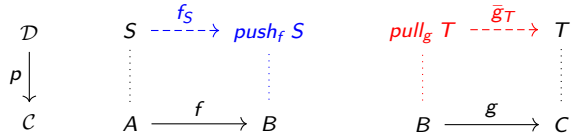

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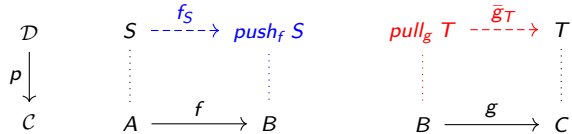

...and these liftings should be universal...

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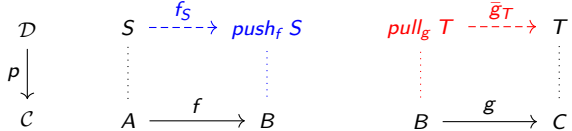

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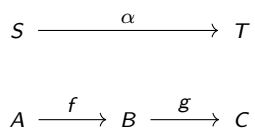

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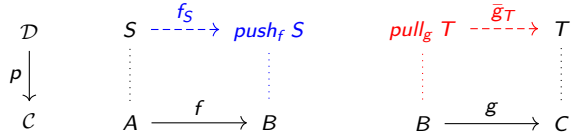


...and these liftings should be universal...

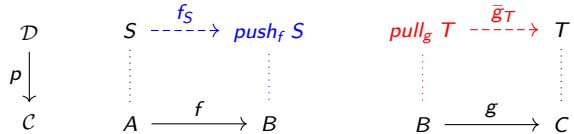
$$S \xrightarrow{f_S} push_f\ S \xrightarrow{f \setminus_g \alpha} T \qquad S \xrightarrow{\quad\alpha\quad} T$$

$$=$$

$$A \xrightarrow{f} B \xrightarrow{g} C \qquad A \xrightarrow{f} B \xrightarrow{g} C$$

What is a bifibration?

Formally:



...and these liftings should be universal...

$$S \xrightarrow{f_S} push_f\ S \xrightarrow{f \setminus_g \alpha} T \quad = \quad S \xrightarrow{\alpha} T \quad = \quad S \xrightarrow{\alpha\, f / \bar{g}} pull_g\ T \xrightarrow{\bar{g}_T} T$$

$$A \xrightarrow{f} B \xrightarrow{g} C \qquad A \xrightarrow{f} B \xrightarrow{g} C \qquad A \xrightarrow{f} B \xrightarrow{g} C$$

Definition of a bifibration in inference rules

$$\frac{S \sqsubset A \qquad f : A \to B}{push_f\ S \sqsubset B} \qquad \frac{f : A \to B \qquad T \sqsubset B}{pull_f\ T \sqsubset A}$$

$$\frac{S \underset{fg}{\Longrightarrow} T}{push_f\ S \underset{g}{\Longrightarrow} T} \qquad \frac{S' \underset{f'}{\Longrightarrow} S}{S' \underset{f'f}{\Longrightarrow} push_f\ S}$$

$$\frac{T \underset{g'}{\Longrightarrow} T'}{pull_g\ T \underset{gg'}{\Longrightarrow} T'} \qquad \frac{S \underset{fg}{\Longrightarrow} T}{S \underset{f}{\Longrightarrow} pull_g\ T}$$

(+ some equations)

Some consequences of the definitions

For all S, f, g, T of the appropriate type, we have

$$push_{fg}\, S \equiv push_g\, push_f\, S \qquad push_{id}\, S \equiv S$$

$$pull_{fg}\, T \equiv pull_f\, pull_g\, T \qquad pull_{id}\, T \equiv T$$

where "$\equiv$" denotes iso in $\mathcal{D}$ lying over an identity in $\mathcal{C}$ ("vertical isomorphism").

Some consequences of the definitions

The following subtyping rules are derivable:

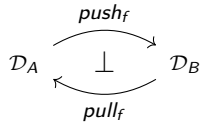
$$\frac{S \leqslant_A S'}{push_f\ S \leqslant_B push_f\ S'} \qquad \frac{T \leqslant_B T'}{pull_f\ T \leqslant_A pull_f\ T'}$$

The following judgments are interderivable:

$$S \leqslant_A pull_f\ T \quad \text{iff} \quad S \underset{f}{\Longrightarrow} T \quad \text{iff} \quad push_f\ S \leqslant_B T$$

Bifibrations as indexed adjunctions

Summarizing the last two slides, pushing and pulling along any given arrow $f : A \to B$ in the base of a bifibration[2] $p : \mathcal{D} \to \mathcal{C}$ induces an *adjunction of fiber categories*



and taking fibers lets us define a *pseudofunctor* $\mathcal{C} \to \mathrm{Adj}$ *into the category of small categories and adjunctions.* Conversely, any such pseudofunctor $G : \mathcal{C} \to \mathrm{Adj}$ induces a bifibration $\pi_G : \int G \to \mathcal{C}$ via a generalization of the category of elements construction.

[2] Technically the bifibration should be equipped with a choice of lifts, i.e., be "cloven".

Example #1: functional image and inverse image

Let SubSet be the category whose objects are pairs $(A, R \subseteq A)$, and whose arrows $(A, S) \to (B, T)$ are functions $f : A \to B$ such that $\forall a, a \in S \Rightarrow f\, a \in T$.

The projection functor $\pi : \mathrm{SubSet} \to \mathrm{Set}$ is a bifibration with

$$push_f\,(A, S) \stackrel{\mathrm{def}}{=} (B, f(S)) \qquad pull_f\,(B, T) \stackrel{\mathrm{def}}{=} (A, f^{-1}(T))$$

Example #2: existential and universal quantification along a relation

Let SubRel be the category with same objects as SubSet but whose arrows $(A, S) \to (B, T)$ are *relations* $r \subseteq A \times B$ such that $\forall ab, a \in S \wedge (a, b) \in r \Rightarrow b \in T$.

The projection functor $\pi : \mathrm{SubRel} \to \mathrm{Rel}$ is a bifibration with

$$push_r\,(A, S) \stackrel{\text{def}}{=} (B, \{\, b \mid \exists a, a \in S \wedge (a, b) \in r \,\})$$
$$pull_r\,(B, T) \stackrel{\text{def}}{=} (A, \{\, a \mid \forall b, (a, b) \in r \Rightarrow b \in T \,\})$$

Monoidal closed bifibrations

$\mathrm{SubSet} \to \mathrm{Set}$ and $\mathrm{SubRel} \to \mathrm{Rel}$ are examples of *monoidal closed bifibrations*, in the sense that the categories are monoidal closed and the projection functors strictly preserve the monoidal closed structure, in addition to being bifibrations.

Monoidal closed bifibrations

$\mathrm{SubSet} \to \mathrm{Set}$ and $\mathrm{SubRel} \to \mathrm{Rel}$ are examples of *monoidal closed bifibrations*, in the sense that the categories are monoidal closed and the projection functors strictly preserve the monoidal closed structure, in addition to being bifibrations.

Proposition: if $p : \mathcal{D} \to \mathcal{C}$ is a monoidal closed bifibration, and M is a monoid in $\mathcal{C}$, then the fiber category $\mathcal{D}_M$ is monoidal closed, with tensor and internal hom defined by

$$R \otimes_M S \stackrel{\text{def}}{=} \mathit{push}_m\,(R \otimes_{\mathcal{D}} S) \qquad R \multimap_M S \stackrel{\text{def}}{=} \mathit{pull}_{\lambda m}\,(R \multimap_{\mathcal{D}} S)$$

where $m : M \otimes_{\mathcal{C}} M \longrightarrow M$ is the monoid multiplication map and λm is its currying.