# A proof-theoretic analysis
# of the rotation lattice of binary trees[1]

Noam Zeilberger

University of Birmingham, School of Computer Science

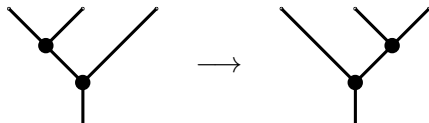Combi-Parsifal joint seminar @ LIX, 9 January 2019

# Introduction

**What is the "Tamari" order?**

Partial order on binary trees induced by **right rotation**[2].



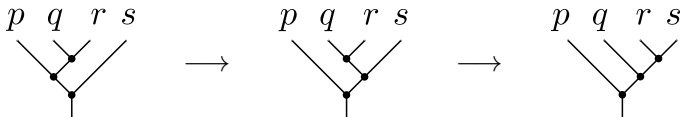Equivalently, ordering on fully bracketed words defined by

1. $(A \bullet B) \bullet C \leq A \bullet (B \bullet C)$ [**semi-associativity**]
2. if $A_1 \leq A_2$ and $B_1 \leq B_2$ then $A_1 \bullet B_1 \leq A_2 \bullet B_2$ [monotonicity]

(Named after Dov Tamari, who originally studied this ordering for motivations in algebra in a 1951 thesis from Université de Paris.)

---

[2](or alternatively left rotation)

Example: $(p \bullet (q \bullet r)) \bullet s \leq p \bullet (q \bullet (r \bullet s))$



Easy Fact: ordering only depends on shape of tree.

Let $Y_n$ be the set of $C_n = \binom{2n}{n}/(n+1)$ binary trees with $n$ internal nodes (or equivalently bracketings of $n+1$ letters), ordered by right rotation (semi-associativity + monotonicity).

Amazing Fact #1: each $Y_n$ is a <u>lattice</u>!
(Well it's kind of amazing. Certainly non-obvious.)

Amazing Fact #2: the Hasse diagram of $Y_n$ may be realized as the skeleton of an $n-1$ dimensional polytope! (The "associahedron".)
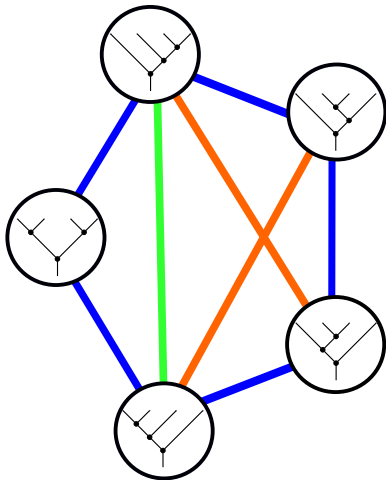
**A picture of $Y_3$**
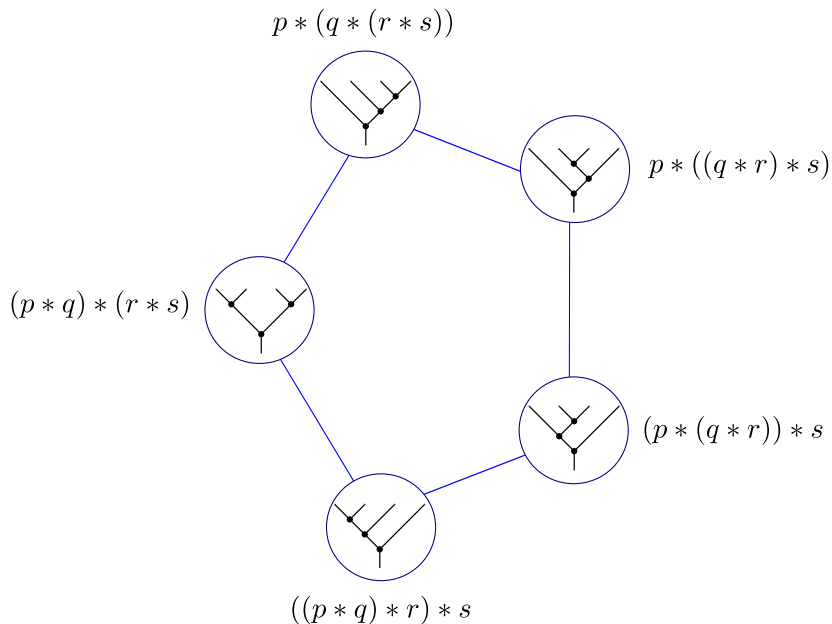
Amazing Fact #3: $Y_n$ contains exactly $\frac{2(4n+1)!}{(n+1)!(3n+2)!}$ intervals!

(Here an "interval" just means a pair $A$ and $B$ such that $A \leq B$. One can also think of it as the set $[A, B] = \{C \mid A \leq C \leq B\}$.)

For example, $Y_3$ contains $13 = \frac{2 \cdot 13!}{4! \cdot 11!}$ intervals:



$5 + 5 + 2 + 1 = 13$

Fréderic Chapoton proved this amazing formula in 2006, but even more amazing was how he found it. . .

In fact, this formula was first calculated by (legendary graph theorist and WW2 hero) W. T. Tutte in the early 1960s, but for a seemingly unrelated family of objects (a family of planar maps).

So in reality, Chapoton "discovered" the formula by counting the number of intervals in $Y_n$ for the first few values of $n$, and looking up the resulting sequence in the OEIS!

Later, sparked by Chapoton's observation, Bernardi and Bonichon (2009) gave an explicit bijection between Tamari intervals and the family of planar maps originally enumerated by Tutte.

My paper: new proofs of Amazing Facts #1 and #3.

Idea: consider the Tamari order as a very primitive *logic*, and apply old insights from proof theory (and category theory).

# Prologue to Introduction

**(Or, how does a logician get interested in this stuff?)**

It turns out that the study of *map enumeration* pioneered by Tutte also has connections to the *combinatorics of lambda calculus!*

A partial bibliography:

1. O. Bodini, D. Gardy, A. Jacquot (2013), Asymptotics and random sampling for BCI and BCK lambda terms, TCS 502: 227-238
2. Z, A. Giorgetti (2015), A correspondence between rooted planar maps and normal planar lambda terms, LMCS 11(3:22): 1-39
3. Z (2015), Counting isomorphism classes of beta-normal linear lambda terms, arXiv:1509.07596
4. Z (2016), Linear lambda terms as invariants of rooted trivalent maps, J. Functional Programming 26(e21)
5. J. Courtiel, K. Yeats, Z (2016), Connected chord diagrams and bridgeless maps, arXiv:1611.04611 (submitted to the Electronic Journal of Combinatorics)
6. Z (2018), A theory of linear typings as flows on 3-valent maps, LICS 2018

**A cube of sequences**
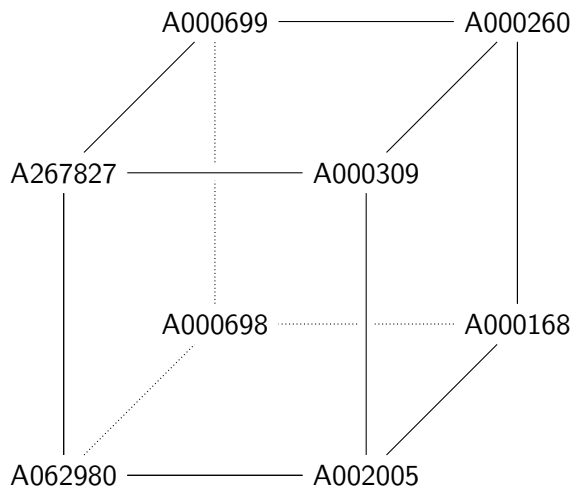


Linear terms: $[x : \text{ordered}, y : \text{unitless}, z : \beta\text{-normal}]$
Rooted maps: $[x : \text{planar}, y : \text{bridgeless}, 1\text{-}z : 3\text{-valent}]$
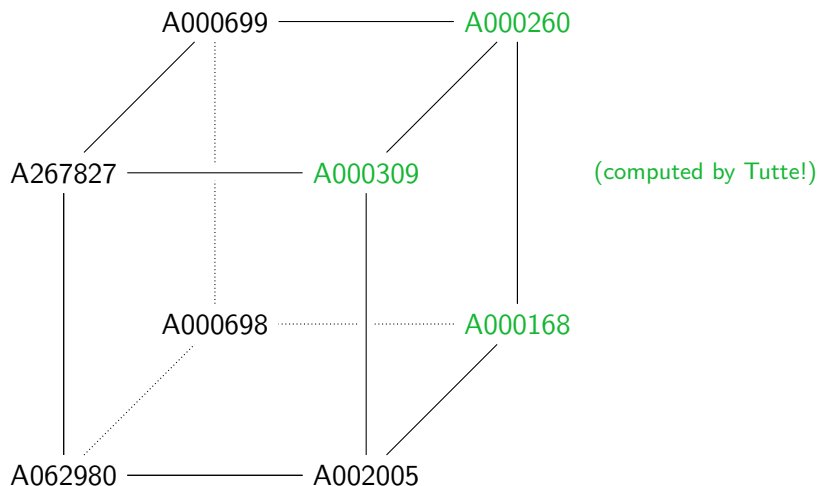
**A cube of sequences**



Linear terms: [$x$ : ordered, $y$ : unitless, $z$ : $\beta$-normal]
Rooted maps: [$x$ : planar, $y$ : bridgeless, 1-$z$ : 3-valent]

**A cube of sequences**
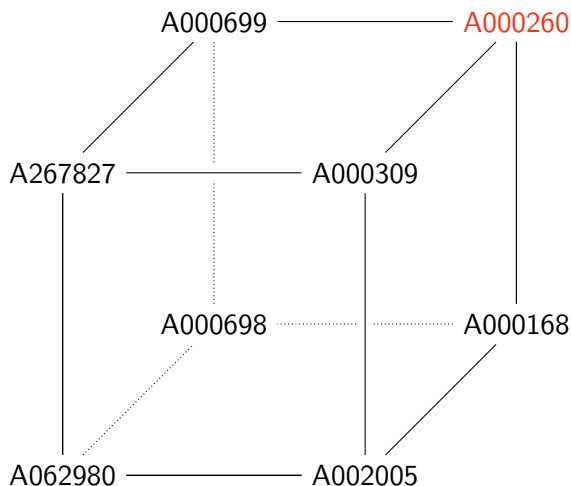


Linear terms: $[x$ : ordered, $y$ : unitless, $z$ : $\beta$-normal$]$
Rooted maps: $[x$ : planar, $y$ : bridgeless, $1$-$z$ : 3-valent$]$

# A sequent calculus for the Tamari order

**What is "sequent calculus"?**

An approach to the formal representation of logical inference, originally introduced by Gerhard Gentzen in the 1930s for the study of classical and intuitionistic logic.

Essentially, proofs are represented as (rooted planar) trees, where:

▶ nodes are labelled by "inference rules" chosen from a fixed set;

▶ edges are labelled by "sequents", subject to constraints imposed by the inference rules;

▶ the outgoing root edge marks the conclusion of the proof, and any incoming leaves correspond to premises.

Moreover, the set of inference rules is chosen in a clever way so that one can show nice "meta-theorems" (proofs about proofs).

Example of a proof with four premises:

$$\dfrac{\dfrac{A \longrightarrow C \quad B \longrightarrow C}{A \vee B \longrightarrow C} \vee L \quad \dfrac{A \longrightarrow D \quad B \longrightarrow D}{A \vee B \longrightarrow D} \vee L}{A \vee B \longrightarrow C \wedge D} \wedge R$$
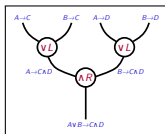


Example of a proof with no premises (i.e., of a valid sequent):

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \longrightarrow A} \; id \quad \overline{B \longrightarrow B} \; id}{A, B \longrightarrow A \wedge B} \wedge R}{A, B \longrightarrow (A \wedge B) \vee (A \wedge C)} \vee R_1 \quad \dfrac{\dfrac{\overline{A \longrightarrow A} \; id \quad \overline{C \longrightarrow C} \; id}{A, C \longrightarrow A \wedge C} \wedge R}{A, C \longrightarrow (A \wedge B) \vee (A \wedge C)} \vee R_2}{A, B \vee C \longrightarrow (A \wedge B) \vee (A \wedge C)} \vee L}{A \wedge (B \vee C) \longrightarrow (A \wedge B) \vee (A \wedge C)} \wedge L$$

Note use of "left rules" and "right rules", as well as the subtle
distinction between conjunction ($\wedge$) and concatenation (,)

**Definition of the sequent calculus for the Tamari order**

Sequents of the form $A_0, \ldots, A_n \longrightarrow B$, where $A_0, \ldots, A_n, B$ are fully bracketed words (we'll refer to them as "logical formulas")

Four inference rules:

$$\frac{}{A \longrightarrow A} \ id \qquad \frac{\Theta \longrightarrow A \quad \Gamma, A, \Delta \longrightarrow B}{\Gamma, \Theta, \Delta \longrightarrow B} \ cut$$

$$\frac{A, B, \Delta \longrightarrow C}{A \bullet B, \Delta \longrightarrow C} \ \bullet L \qquad \frac{\Gamma \longrightarrow A \quad \Delta \longrightarrow B}{\Gamma, \Delta \longrightarrow A \bullet B} \ \bullet R$$

where $\Gamma, \Theta, \Delta$ range over lists of formulas.

Historical note: these sequent calculus rules are *almost* a direct copy of rules introduced 60 years ago[3] by Joachim Lambek! The only difference is a restriction on the left rule. . .

$$\frac{A, B, \Delta \longrightarrow C}{A \bullet B, \Delta \longrightarrow C} \ \bullet L \qquad versus \qquad \frac{\Gamma, A, B, \Delta \longrightarrow C}{\Gamma, A \bullet B, \Delta \longrightarrow C} \ \bullet L^{\mathrm{amb}}$$

. . . but this simple restriction accounts for semi-associativity!

---

[3] J. Lambek, "The mathematics of sentence structure," *The American Mathematical Monthly*, vol. 65, no. 3, pp. 154–170, 1958.

$$\boxed{\Y \leq \Y\Y}$$
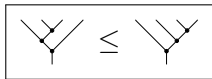
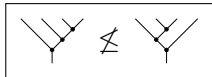Example: $(p \bullet (q \bullet r)) \bullet s \leq p \bullet (q \bullet (r \bullet s))$

$$
\cfrac{
  \cfrac{
    \overline{p \longrightarrow p}
    \qquad
    \cfrac{
      \overline{q \longrightarrow q}
      \quad
      \cfrac{
        \cfrac{\overline{r \longrightarrow r} \quad \overline{s \longrightarrow s}}{r,s \longrightarrow r \bullet s} R
      }{q,r,s \longrightarrow q \bullet (r \bullet s)}
      \;\;
      \cfrac{}{q \bullet r, s \longrightarrow q \bullet (r \bullet s)} L
    }{p,q \bullet r, s \longrightarrow p \bullet (q \bullet (r \bullet s))} R
  }{p \bullet (q \bullet r), s \longrightarrow p \bullet (q \bullet (r \bullet s))} L
}{(p \bullet (q \bullet r)) \bullet s \longrightarrow p \bullet (q \bullet (r \bullet s))} L
$$

Counterexample: $p \bullet (q \bullet (r \bullet s)) \not\leq (p \bullet (q \bullet r)) \bullet s$



$$\dfrac{\dfrac{\dfrac{}{p \longrightarrow p} \quad \dfrac{\dfrac{}{q \longrightarrow q} \quad \dfrac{}{r \longrightarrow r}}{q, r \longrightarrow q \bullet r} R}{p, q, r \longrightarrow p \bullet (q \bullet r)} R \quad \dfrac{}{s \longrightarrow s}}{\dfrac{\dfrac{\dfrac{p, q, r, s \longrightarrow (p \bullet (q \bullet r)) \bullet s}{p, q, r \bullet s \longrightarrow (p \bullet (q \bullet r)) \bullet s} L^{\mathrm{amb}}}{p, q \bullet (r \bullet s) \longrightarrow (p \bullet (q \bullet r)) \bullet s} L^{\mathrm{amb}}}{p \bullet (q \bullet (r \bullet s)) \longrightarrow (p \bullet (q \bullet r)) \bullet s} L} R$$

By abuse of notation, we write "$\Gamma \longrightarrow A$" to indicate that the sequent is valid, i.e., has a proof with no premises.

Two basic "meta-theorems":

- Completeness: if $A \leq B$ then $A \longrightarrow B$.
- Soundness: if $\Gamma \longrightarrow B$ then $\phi[\Gamma] \leq B$, where $\phi[A_0, \ldots, A_n] \stackrel{\text{def}}{=} ((A_0 \bullet A_1) \cdots) \bullet A_n$.

The (meta-)proof of completeness is very easy (give proofs of reflexivity + transitivity + monotonicity + semi-associativity).

The (meta-)proof of soundness is mildly satisfying (key lemma: $\phi[\Gamma, \Delta] \leq \phi[\Gamma] \bullet \phi[\Delta]$), but still a quick induction on proofs.

Gentzen's original genius was his insight about the importance of "cut-elimination", which is essentially a normal form theorem for sequent calculus proofs.

Classically, the cut-elimination theorem has all sorts of logical applications, and is the fundamental reason why sequent calculus may be used as a basis for automated proof search.

The sequent calculus for the Tamari order admits a strong form of cut-elimination that likewise will have some nice applications...

We say that a proof is **focused** (or **normal**) if it only contains uses of •*L* and the following restricted forms of •*R* and *id* (and no *cut*):

$$\frac{\Gamma^{\mathrm{irr}} \longrightarrow A \quad \Delta \longrightarrow B}{\Gamma^{\mathrm{irr}}, \Delta \longrightarrow A \bullet B} \; \bullet R^{\mathrm{foc}} \qquad \frac{}{p \longrightarrow p} \; id^{\mathrm{atm}}$$

where $\Gamma^{\mathrm{irr}}$ "irreducible" means its leftmost formula is not $C \bullet D$

**Theorem (Focusing completeness/cut-elimination)**

*Every valid sequent has a focused proof.*

The (meta-)proof is relatively long ($\sim$ 2 pages), but essentially follows a standard pattern (no surprises other than that it works!).

Example reductions on proof trees:

$$\cfrac{\cfrac{C_1, C_2, \Gamma \longrightarrow A}{C_1 \bullet C_2, \Gamma \longrightarrow A} \bullet L \quad \Delta \longrightarrow B}{C_1 \bullet C_2, \Gamma, \Delta \longrightarrow A \bullet B} \bullet R \quad \Longrightarrow \quad \cfrac{\cfrac{C_1, C_2, \Gamma \longrightarrow A \quad \Delta \longrightarrow B}{C_1, C_2, \Gamma, \Delta \longrightarrow A \bullet B} \bullet R}{C_1 \bullet C_2, \Gamma, \Delta \longrightarrow A \bullet B} \bullet L$$

---

$$\cfrac{\cfrac{\Gamma_1^{\mathrm{irr}} \longrightarrow A_1 \quad \Gamma_2 \longrightarrow A_2}{\Gamma_1^{\mathrm{irr}}, \Gamma_2 \longrightarrow A_1 \bullet A_2} \bullet R \quad \cfrac{A_1, A_2, \Delta \longrightarrow B}{A_1 \bullet A_2, \Delta \longrightarrow B} \bullet L}{\Gamma_1^{\mathrm{irr}}, \Gamma_2, \Delta \longrightarrow B} \; cut$$

$$\Longrightarrow$$

$$\cfrac{\Gamma_1^{\mathrm{irr}} \longrightarrow A_1 \quad \cfrac{\Gamma_2 \longrightarrow A_2 \quad A_1, A_2, \Delta \longrightarrow B}{A_1, \Gamma_2 \Delta \longrightarrow B} \; cut}{\Gamma_1^{\mathrm{irr}}, \Gamma_2, \Delta \longrightarrow B} \; cut$$

Conversely, it is not hard to show that any sequent has <u>at most</u> one focused proof. We therefore obtain the following

**Theorem (Coherence)**

*Every valid sequent has* **exactly one** *focused proof.*

With soundness and completeness, coherence says in a sense that focused proofs are a canonical representation of Tamari intervals.

[The name is inspired by Mac Lane's coherence theorem in category theory, which is in fact related, cf. MFPS 2018 paper by Uustalu, Veltri, and Z, "The sequent calculus of skew monoidal categories".]

# Application to counting intervals

Idea: by the coherence theorem, the problem of counting intervals is equivalent to the problem of counting focused proofs – but since the latter are just special kinds of trees, this is easy!

Formally, consider the bivariate OGFs $L(z, x)$ and $R(z, x)$ where

$[z^n x^k]L(z, x) = \#$ focused proofs of sequents $\Gamma \longrightarrow A$ where $len(\Gamma) = k$ and $size(A) = n$.

$[z^n x^k]R(z, x) = \#$ focused proofs of sequents $\Gamma^{\mathrm{irr}} \longrightarrow A$ where $len(\Gamma^{\mathrm{irr}}) = k$ and $size(A) = n$.

(here the size of a word is defined as the number of •s).

An immediate corollary of the coherence theorem is that

$$\# \text{ intervals in } Y_n = [z^n]L_1(z)$$

where $L_1(z) = [x^1]L(z, x)$.

Moreover, from the definition of the inference rules. . .

$$\frac{A, B, \Delta \longrightarrow C}{A \bullet B, \Delta \longrightarrow C} \; \bullet L \qquad \frac{\Gamma^{\mathrm{irr}} \longrightarrow A \quad \Delta \longrightarrow B}{\Gamma^{\mathrm{irr}}, \Delta \longrightarrow A \bullet B} \; \bullet R^{\mathrm{foc}} \qquad \frac{}{p \longrightarrow p} \; id^{\mathrm{atm}}$$

we immediately obtain the following functional equations:

$$\begin{aligned}
L(z, x) &= (L(z, x) - x L_1(z))/x + R(z, x) \\
&= x \frac{R(z, x) - R(z, 1)}{x - 1} \qquad &(1) \\
R(z, x) &= z R(z, x) L(z, x) + x \qquad &(2)
\end{aligned}$$

$$L(z, x) = x \frac{R(z, x) - R(z, 1)}{x - 1} \qquad (1)$$

$$R(z, x) = zR(z, x)L(z, x) + x \qquad (2)$$

Finally, we observe that these same equations (1) and (2) were already considered by Cori and Schaeffer (2003) as counting "(1,1)-description trees". They explained how these equations may be solved using the "quadratic method" to obtain the desired result

$$[z^n]L_1(z) = [z^n]R(z, 1) = \frac{2(4n + 1)!}{(n + 1)!(3n + 2)!}.$$

## Comparison to Chapoton (2006)

Chapoton likewise defined a bivariate OGF $\Phi(z, x)$, where $x$ keeps track of "the number of segments along the left border" of the tree at the lower end of the interval,[4] and obtains by hand that

$$\Phi(z,x) = x^2 z(1 + \Phi(z,x)/x)\left(1 + \frac{\Phi(z,x) - \Phi(z,1)}{x-1}\right) \qquad (3)$$

which he ultimately shows how to solve at $x = 1$ by appeal to a different result of Cori and Schaeffer (2003).

In fact (3) can be derived from (1) and (2) by taking

$$\Phi(z,x) = R(z,x) - x$$

(since Chapoton ignores the case of $Y_0$).

... In other words, the two proofs are quite closely related!

---

[4]Or rather at the upper end, since he uses the dual convention for $Y_n$.

**Application to the lattice property**

The observation that each $Y_n$ is a lattice was made already in Tamari's thesis, but sans proof.[5] A first proof was worked out by his student Haya Friedman in 1958, although this did not appear in print until 1967.[6] That was followed by another, simpler proof with Tamari's student Samuel Huang.[7]

For more recent proofs of the lattice property of the rotation order, see for example Knuth's 1993 lecture (available on YouTube!), or the textbook by Caspard, Santocanale, and Wehrung (2016).

---

[5]Dov Tamari. *Monoïdes préordonnés et chaînes de Malcev*. Thèse, Université de Paris, 1951.

[6]H. Friedman and D. Tamari. Problèmes d'associativité: une structure de treillis finis induite par une loi demi-associative. *Journal of Combinatorial Theory*, 2:215–242, 1967.

[7]S. Huang and D. Tamari. Problems of associativity: A simple proof for the lattice property of systems ordered by a semi-associative law. *Journal of Combinatorial Theory Series A*, 13(1):7–13, 1972.

We give a new proof of the lattice property via sequent calculus.

Our initial step is to define a canonical ordering on *lists* of words by $\Gamma \leq \Delta$ iff splittings $\Gamma = \Gamma_1, \ldots, \Gamma_n$ and $\Delta = A_1, \ldots, A_n$ where $\Gamma_1 \longrightarrow A_1, \ldots, \Gamma_n \longrightarrow A_n$.

Let $F(Y)_{[n]}$ be the induced poset of "forests" with $n+1$ leaves.

The proof now relies on two key observations.

First, the evident embedding $i : Y_n \to F(Y)_{[n]}$ forms the right end of an *adjoint triple*,

$$
\begin{array}{ll}
\phi[\Gamma] \leq A & \iff \quad \Gamma \leq i[A] \\
\psi[A] \leq \Delta & \iff \quad A \leq \phi[\Delta]
\end{array}
$$

$$Y_n \xrightarrow{\ i\ } F(Y)_{[n]}$$

where $\phi$ is the left-associated product and $\psi$ is the *maximal left decomposition* (e.g., $\psi[(p \bullet (q \bullet r)) \bullet s] = (p, q \bullet r, s)$).

For completely general reasons, this allows us to reduce any join of trees to a join of forests:

$$A \vee B = \phi[\psi[A] \sqcup \psi[B]]$$

Second, any forest with $n+1$ leaves divided among $k$ trees implicitly contains a *composition* of $n+1$ into $k$ parts, inducing a "forgetful" monotone function $\alpha : \mathsf{F}(\mathsf{Y})_{[n]} \to \mathrm{O}_{[n]}$ from the poset of forests to the *lattice of compositions ordered by refinement*.

We can use this to reduce a join of forests in $\mathsf{F}(\mathsf{Y})_{[n]}$ to a join of compositions in $\mathrm{O}_{[n]} \cong 2^n$ plus joins of trees in $\mathsf{Y}_m$ for $m < n$.

Tying everything together, we get a constructive proof of the existence of joins that corresponds to a simple recursive algorithm.

(See paper for details. Of course meets can be computed dually!)

Input:

$$A = p \bullet ((q \bullet (r \bullet ((s \bullet t) \bullet u))) \bullet v)$$
$$B = (p \bullet (q \bullet r)) \bullet ((s \bullet t) \bullet (u \bullet v))$$

Computation:

| round | $A$ | $\psi[A]$ | $B$ | $\psi[B]$ | $\psi[A] \sqcup \psi[B]$ |
|-------|-----|-----------|-----|-----------|--------------------------|
| 1 | $p((q(r((st)u)))v)$ | $p, (q(r((st)u)))v$ | $(p(qr))((st)(uv))$ | $p, qr, (st)(uv)$ | $p, A_2 \vee B_2$ |
| 2 | $(q(r((st)u)))v$ | $q, r((st)u), v$ | $(qr)((st)(uv))$ | $q, r, (st)(uv)$ | $q, A_3 \vee B_3$ |
| 3 | $(r((st)u))v$ | $r, (st)u, v$ | $r((st)(uv))$ | $r, (st)(uv)$ | $r, A_4 \vee B_4$ |
| 4 | $((st)u)v$ | $s, t, u, v$ | $(st)(uv)$ | $s, t, uv$ | $s, t, A_5 \vee B_5$ |
| 5 | $uv$ | $u, v$ | $uv$ | $u, v$ | $u, v$ |

Output:
$$A \vee B = (p \bullet (q \bullet (r \bullet ((s \bullet t) \bullet (u \bullet v)))))$$

**Conclusion**

We've seen a display of the *surprisingly productive interaction* between **proof theory** and **combinatorics**:

▶ Used proof theory to develop simple and systematic explanations for "Amazing Facts" #1 and #3.

▶ Conversely, combinatorics provided the original impetus for studying this apparently very natural sequent calculus, which also has independent applications (e.g., to category theory).

Questions:

▶ Does this help in understanding Amazing Fact #2?

▶ Can related posets/lattices (such as the weak order on permutations) be considered similarly?

▶ What else can we learn by counting proofs?. . .

# 14th Workshop
# Computational Logic and Applications
Versailles, France, 1–2 July 2019

It is our pleasure to announce that the 14th workshop Computational Logic and Applications CLA'19 will be held on the 1st and 2nd of July 2019 in Versailles, France. The main purpose of CLA is to provide an open, free access forum for scientific research concentrated around combinatorial and quantitative aspects of mathematical logic and their applications in computer science.