Lifting structure(s) from the base to the total category

Posetal closed (and $*$-autonomous) Grothendieck construction

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Joint work with
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Plan

A few theorems on Complete Lattices

Theorem (Egger, Kruml, Paseka ∼ 2008, Santocanale 2020)

Let L be a complete lattice. The following are equivalent:

- *L is a* **completely distributive lattice***.*
- *The* **quantale** *$L \multimap L$ of join-preserving endomaps of L is a Frobenius quantale.*

Theorem (Raney 1960, Higgs and Rowe 1989)

The **nuclear objects** *in* ***SLatt*** *are exactly the completely distributive lattice.*

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Constructing counter-examples in $*$-autonomous categories

Conjecture Let A be an object of a symmetric monoidal closed category. The following are equivalent:

1. A is nuclear.
2. The object $A \multimap A$ of endomorphisms of A is a **Frobenius monoid**.

Theorem (De Lacroix & S., CSL 2023)

If A is an object of $*$-autonomous category, then (1) implies (2). The converse implication holds if A is **pseudoaffine**, *that is, the tensor unit I is a retract of A.*

Counter-example (De Lacroix & S.): There exists a $*$-autonomous category and an object A (of this category) such that $A \multimap A$ is Frobenius monoid, which is not nuclear.

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Schalk-de Paiva category Q-Set

Let Q be **commutative quantale** (= posetal complete SMMC).

- An object of Q-**Set**:
  a pair (X, α) with X a set and $\alpha : X \longrightarrow Q$ a function.
- An arrow of Q-**Set** from (X, α) to (Y, β):
  a relation $R \subseteq X \times Y$ such that

$$xRy \implies \alpha(x) \leq \beta(y), \quad \forall x \in X, y \in Y.$$

Proposition Q-**Set** is SMMC. If Q is a Girard quantale, then Q-**Set** is $*$-autonomous.

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$*$-autonomous categories from Girard quantales?

- A Girard quantale is a posetal complete $*$-autonomous category.
- How do we lift properties from Q to Q-**Set**?

More general (and philosophical?) questions:

- How do Girard quantales relate to $*$-autonomous categories?
- Cf. Heyting algebras, CCCs, topoi.
- Is there a linear version of the notion of topos?

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The total (or Grothendieck) category $\int Q$ of a functor Q

For a functor

$$Q : \mathbf{B} \longrightarrow \mathbf{Pos}$$

its total category $\int \mathbf{Q}$ is defined as follows:

- an object of $\int \mathbf{Q}$:
  (X, α) with $X \in \mathrm{Obj}(\mathbf{B})$ and $\alpha \in Q(X)$;
- an arrow $(X, \alpha) \longrightarrow (Y, \beta)$:
  $f : X \longrightarrow Y$ such that $Q(f)(\alpha) \leq \beta$.

The first projection:

$$\pi : \int \mathbf{Q} \longrightarrow \mathbf{B}$$

is the standard example of an (op-)fibration (with posetal fibers).

Lemma [Folklore ?] If $\mathbf{B}$ and Q are monoidal:

$$1 \longrightarrow Q(I), \qquad \mu_{X,Y} : Q(X) \times Q(Y) \longrightarrow Q(X \otimes Y)$$

then $\int \mathbf{Q}$ is monoidal and π strictly preserves the tensor structure.

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Lemma [Folklore ?] If **B** and Q are monoidal:

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then $\int \mathbf{Q}$ is monoidal and π strictly preserves the tensor structure.

Q-Set as a total category

For $R \subseteq X \times Y$ and $\alpha \in Q^X$, define

$$Q^R(\alpha)(y) := \bigvee_{xRy} \alpha(x) .$$

Q^X is a functor **Rel** $\longrightarrow$ **Pos**.

Proposition Q-**Set** $= \int \mathbf{Q^X}$. Moreover, the functor Q^X is monoidal and, consequently, Q-**Set** is a monoidal category, and the first projection

$$Q\text{-}\mathbf{Set} \longrightarrow \mathbf{Rel}$$

strictly preserves the monoidal structure.

What more ?

Moral:

- Understanding why Q-**Set** $= \int \mathbf{Q}$ is monoidal is well-covered by the theory of monoidal (op-)fibrations.

Is it possible to have a theory explaining:

- when $\int \mathbf{Q}$ is closed?
- when $\int \mathbf{Q}$ is $*$-autonomous?
- which does not depend on specific properties of **Rel**
  (which is a **Cartesian bicategory** whence **dagger compact closed**)?

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Plan

Lifting functors from the base B

Let

$$Q : \mathbf{B} \longrightarrow \mathbf{Pos}\,, \qquad \text{so} \qquad \pi : \int \mathbf{Q} \longrightarrow \mathbf{B}$$

Definition Let $F : \mathbf{B} \longrightarrow \mathbf{B}$ be an endofuctor of $\mathbf{B}$. A lifting of F to $\int \mathbf{Q}$ is a functor $\overline{F} : \int \mathbf{Q} \longrightarrow \int \mathbf{Q}$ such that the following diagram commutes:

$$\begin{array}{ccc} \int \mathbf{Q} & \xrightarrow{\overline{F}} & \int \mathbf{Q} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbf{B} & \xrightarrow{F} & \mathbf{B} \end{array}$$

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That is, we want

$$\overline{F}(X, \alpha) = (F(X), \beta)$$

for some $\beta \in Q(F(X))$ which depends on $\alpha \in Q(X)$.

Lifting functors from the base B

Let

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Definition Let

$$F : (\mathbf{B}^{op})^n \times \mathbf{B}^m \longrightarrow \mathbf{B}$$

be functor. A lifting of F to $\int \mathbf{Q}$ is a functor

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such that the following diagram commutes:

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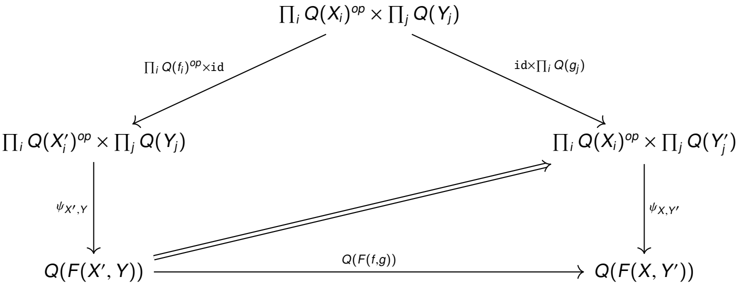
(\mathbf{B}^{op})^n \times \mathbf{B}^n & \xrightarrow{\quad F \quad} & \mathbf{B}
\end{array}$$

Proposition Liftings of a functor $F : (\mathbf{B}^{op})^n \times \mathbf{B}^m \longrightarrow \mathbf{B}$ to $\int \mathbf{Q}$ bijectively correspond to collections of order-preserving maps

$$\psi_{X,Y} : \prod_i Q(X_i)^{op} \times \prod_j Q(Y_j) \longrightarrow Q(F(X,Y))$$

such that, for each pair of maps $f : X \longrightarrow X'$ in $\mathbf{B}^n$ and $g : Y \longrightarrow Y'$ in $\mathbf{B}^m$, the following diagram half-commutes:



Lifting monoidal structures

Proposition There is a bijection between the following kind of data:

- a lifting of a symmetric monoidal structure $(I, \otimes, \alpha, \lambda, \rho, \sigma)$ from **B** to $\int Q$,
- a collection of order-preserving maps

$$1 \xrightarrow{u} Q(1)\,, \qquad \{\ \mu_{X,Y} : Q(X) \times Q(Y) \longrightarrow Q(X \otimes Y)\ \}_{X,Y \in \mathrm{Obj}(\mathbf{B})}\,,$$

such that

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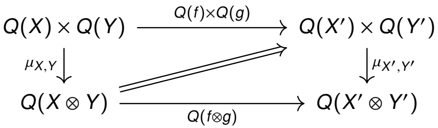

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2. and

$$\alpha : ((X,x) \otimes (Y,y)) \otimes (Z,z) \longrightarrow (X,x) \otimes ((Y,y) \otimes (Z,z))$$

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2. and

$$\alpha : ((X \otimes Y) \otimes Z, \mu_{X \otimes Y, Z}(\mu_{X,Y}(x,y), z)) \longrightarrow (X \otimes (Y \otimes Z), \mu_{X, Y \otimes Z}(x, \mu_{Y,Z}(y,z)))$$

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$$\begin{aligned} Q(\alpha)(\mu_{X \otimes Y, Z}(\mu_{X,Y}(x,y),z)) &= \mu_{X, Y \otimes Z}(x, \mu_{Y,Z}(y,z))\,, \\ Q(\lambda)(\mu_{I,Y}(u,y)) &= y\,, \\ Q(\rho)(\mu_{X,I}(x,u)) &= u\,, \\ Q(\sigma)(\mu_{X,Y}(x,y)) &= \mu_{Y,X}(y,x)\,. \end{aligned}$$

$$
\begin{array}{ccc}
(Q(X)\times Q(Y))\times Q(Z) & \xrightarrow{\alpha_Q} & Q(X)\times(Q(Y)\times Q(Z)) \\
\downarrow{\scriptstyle \mu\times id} & & \downarrow{\scriptstyle id\times\mu} \\
Q(X\otimes Y)\times Q(Z) & & Q(X)\times Q(Y\otimes Z) \\
\downarrow{\scriptstyle id\times\mu} & & \downarrow{\scriptstyle \mu\otimes id} \\
Q((X\otimes Y)\otimes Z) & \xrightarrow{Q(\alpha)} & Q(X\otimes(Y\otimes Z))
\end{array}
$$

$$
\begin{array}{ccc}
1\times Q(X) & \xrightarrow{u\times id} & Q(I)\times Q(X) \\
\downarrow{\scriptstyle \lambda_Q} & & \downarrow{\scriptstyle \mu} \\
Q(X) & \xleftarrow[Q(\lambda)]{} & Q(I\otimes X)
\end{array}
\qquad
\begin{array}{ccc}
Q(X)\times 1 & \xrightarrow{id\times u} & Q(X)\times Q(I) \\
\downarrow{\scriptstyle \rho_Q} & & \downarrow{\scriptstyle \mu} \\
Q(X) & \xleftarrow[Q(\rho)]{} & Q(X\otimes I)
\end{array}
$$

$$
\begin{array}{ccc}
Q(X)\times Q(Y) & \xrightarrow{\sigma_Q} & Q(Y)\times Q(X) \\
\downarrow{\scriptstyle \mu_{X,Y}} & & \downarrow{\scriptstyle \mu_{Y,X}} \\
Q(X\otimes Y) & \xrightarrow{Q(\sigma)} & Q(X\otimes Y)
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Lifting the closed structure

Let **B** be SMC, with

$$ev_{X,Y} : X \otimes (X \multimap Y) \longrightarrow Y\,, \qquad \eta_{X,Y} : Y \longrightarrow X \multimap (X \otimes Y)\,.$$

Suppose μ is used to lift $\otimes$ to $\int \mathbf{Q}$.

Proposition $\int \mathbf{Q}$ is closed if and only if we have are given a collection of order-preserving maps

$$\{\ \iota_{X,Y} : Q(X)^{op} \times Q(Y) \longrightarrow Q(X \multimap Y)\ \}_{X,Y \in \mathrm{Obj}(\mathbf{B})}\,,$$

such that

1. for $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$, the following diagram semi-commutes:

$$\begin{array}{ccc}
Q(X)^{op} \times Q(Y) & \xrightarrow{\mathrm{id} \times Q(g)} & Q(X)^{op} \times Q(Y') \\
\downarrow{\scriptstyle Q(f)^{op} \times \mathrm{id}} & & \\
Q(X')^{op} \times Q(Y) & \Nearrow & \downarrow{\scriptstyle \iota_{X,Y'}} \\
\downarrow{\scriptstyle \iota_{X',Y}} & & \\
Q(X' \multimap Y) & \xrightarrow{Q(f \multimap g)} & Q(X \multimap Y')
\end{array}$$

2. and

$$Q(\eta_{X,Y})(y) \leq \iota_{X,X \otimes Y}(x, \mu_{X,Y}(x,y))\,,$$

$$Q(ev_{X,Y})(\mu_{X,X \multimap Y}(x, \iota_{X,Y}(x,y))) \leq y\,.$$

Lifting the closed structure

Let **B** be SMC, with

$$ev_{X,Y} : X \otimes (X \multimap Y) \longrightarrow Y\,, \qquad \eta_{X,Y} : Y \longrightarrow X \multimap (X \otimes Y)\,.$$

Suppose μ is used to lift $\otimes$ to $\int \mathbf{Q}$.

Proposition $\int \mathbf{Q}$ is closed if and only if we have are given a collection of order-preserving maps

$$\{\ \iota_{X,Y} : Q(X)^{op} \times Q(Y) \longrightarrow Q(X \multimap Y)\ \}_{X,Y \in \mathrm{Obj}(\mathbf{B})}\,,$$

such that

1. for $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$, the following diagram semi-commutes:

$$\begin{array}{ccc} Q(X)^{op} \times Q(Y) & \xrightarrow{id \times Q(g)} & Q(X)^{op} \times Q(Y') \\ \downarrow{\scriptstyle Q(f)^{op} \times id} & & \\ Q(X')^{op} \times Q(Y) & & \downarrow{\scriptstyle \iota_{X,Y'}} \\ \downarrow{\scriptstyle \iota_{X',Y}} & & \\ Q(X' \multimap Y) & \xrightarrow{Q(f \multimap g)} & Q(X \multimap Y') \end{array}$$

2. and

$$Q(\eta_{X,Y})(y) \leq \iota_{X,X \otimes Y}(x, \mu_{X,Y}(x,y))\,,$$

$$Q(ev_{X,Y})(\,\mu_{X,X \multimap Y}(x, \iota_{X,Y}(x,y))\,) \leq y\,.$$

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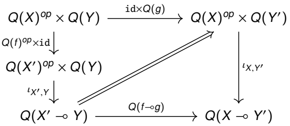

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...a more readable characterisation

Proposition $\int \mathbf{Q}$ is closed if and only if for each pair of objects X, Y, the following diagram

$$\begin{array}{ccc} Q(X) \times Q(Y) & \xrightarrow{\mu_{X,Y}} & Q(X \otimes Y) \\ \downarrow{\scriptstyle Q(X)\otimes Q(\eta_{X,Y})} & & \uparrow{\scriptstyle Q(ev_{X,X\otimes Y})} \\ Q(X) \times Q(X \multimap (X \otimes Y)) & \xrightarrow{\mu_{X,X\multimap(X\otimes Y)}} & Q(X \otimes X \multimap (X \otimes Y)) \end{array}$$

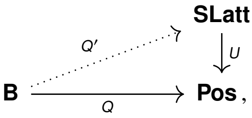
commutes, and, for each $\alpha \in Q(X)$, the map

$$1 \times Q(X \multimap Y) \xrightarrow{\alpha\times\mathtt{id}} Q(X) \times Q(X \multimap Y) \xrightarrow{\mu_{X,X\multimap Y}} Q(X \otimes X \multimap Y) \xrightarrow{Q(ev_{X,Y})} Q(Y)$$

has a right adjoint.

The beauty of SLatt

Corollary If Q factors (monoidally) as



then $\int \mathbf{Q}$ is monoidal and closed.

Corollary Q-**Set** $= \int \mathbf{Q}^{\mathbf{X}}$ is closed.
For F : **Rel** $\longrightarrow$ **Rel** comonoidal (and . . .), Q_F-**Set** $= \int \mathbf{Q}^{\mathbf{FX}}$ is closed.
nuTS $= \int \mathbf{UP}$ is monoidal closed.

Here UP : **Rel** $\longrightarrow$ **SLatt** is the "free completely distributive lattice" functor.

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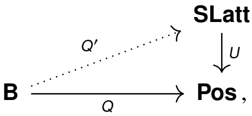

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Plan

Lifting dualizing objects

Let $X^* := X \multimap 0$. An object 0 is **dualizing** if, for each object X, the canonical map

$$j_X : X \longrightarrow X^{**}$$

is an iso.

For $\omega \in Q(0)$, let

$$\omega_X := \iota_{X,0}(\cdot, \omega) : Q(X)^{op} \longrightarrow Q(X^*)\,.$$

Proposition For an object $(0, \omega)$ of $\int \mathbf{Q}$, TFAE:

- $(0, \omega)$ is dualizing,
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From $*$-autonomous to Girard

Let **B** be $*$-autonomous, with 0 dualizing. Let

$$Q : \mathbf{B} \longrightarrow \mathbf{SLatt}$$

be monoidal (that is, let μ be natural), so $\int \mathbf{Q}$ is closed.

Remarks

- $Q(I)$ is a monoid in **SLatt**, that is, a quantale.
- If $0 = I$ and (I, ω) is a dualizing object,
  then ω is a dualizing element of the quantale $Q(I)$.

Problem

If I is a dualizing object of $\mathbf{B}$ and ω is a dualizing element of $Q(I)$,
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If I is a dualizing object of **B** *and ω is a dualizing element of $Q(I)$, is (I, ω) a dualizing object of $\int \mathbf{Q}$?*

A double negation nucleus

Recall: **B** is $*$-autonomous, $Q : \mathbf{B} \longrightarrow \mathbf{SLatt}$ is monoidal, and $\omega \in Q(0)$.

For each object X of **B**, $\alpha \in Q(X)$ and $\beta \in Q(X^*)$, let

$$\langle \alpha, \beta \rangle_X := Q(ev_{X,0})(\mu_{X,X^*}(\alpha, \beta)), \quad \text{so} \quad \omega_X(\alpha) = \bigvee \{ \beta \in X^* \mid \langle \alpha, \beta \rangle_X \leq \omega \}.$$

Define then

$$^{\perp}(\beta) := \bigvee \{ \alpha \in X \mid \langle \alpha, \beta \rangle_X \leq \omega \}.$$

Theorem

Let

$$\neg\neg^{\omega}_X(\alpha) := {}^{\perp}(\omega_X(\alpha)) \quad \textit{and} \quad Q_{\neg\neg^{\omega}}(X) := \{ \alpha \in Q(X) \mid \neg\neg^{\omega}_X(\alpha) = \alpha \}.$$

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- *$Q_{\neg\neg^{\omega}}$ is made into a monoidal functor $Q_{\neg\neg^{\omega}} : \mathbf{B} \longrightarrow \mathbf{SLatt}$,*
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Remark This generalises Hyland/Schalk focused orthogonality structures.

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A representation theorem

Phase semantics. If Q is a commutative quantale and $\omega \in Q$, then $\neg\neg^{\omega}(x) = (x \multimap \omega) \multimap \omega$ is a nucleus on Q and the quotient $Q_{\neg\neg^{\omega}}$ is a Girard quantale.

Completeness of phase semantics. If Q is a commutative Girard quantale, then we can choose $\omega \in P(Q)$, so that Q and $P(Q)_{j_\omega}$ are isomorphic quantales.

Theorem

Let $0 \in \mathbf{B}$ be dualizing and $Q : \mathbf{B} \longrightarrow \mathbf{SLatt}$ monoidal such that $\int \mathbf{Q}$ is $*$-autonomous.
Let PUQ be the functor

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Then Q is naturally isomorphic to $PUQ_{\neg\neg^{\omega}}$ for some $\omega \subseteq Q(0)$.
Thus, $\int \mathbf{Q}$ and $\int PUQ_{\neg\neg^{\omega}}$ are equivalent categories.

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Plan

Lifting coalgebras of functors

Suppose $F : \mathbf{B} \longrightarrow \mathbf{B}$ has been lifted to $\overline{F} : \int Q \longrightarrow \int Q$ by means of the lax natural $\iota_X : Q(X) \longrightarrow Q(F(X))$.

Proposition

$$CoAlg(\overline{F}) \simeq \int Q^{\nu} \longrightarrow CoAlg(F)$$

with $Q^{\nu} : CoAlg(F) \longrightarrow \mathbf{Pos}$ defined by

$$Q^{\nu}(\,\psi : X \longrightarrow F(X)\,) := \{\,\alpha \in Q(X) \mid Q(\psi)(\alpha) \leq \iota_X(\alpha)\,\}\,.$$

Corollary If $Q : \mathbf{B} \longrightarrow \mathbf{Pos}$, with the $Q(X)$ complete lattices, then

$$\nu_X.\overline{F}(X) = (\,(\nu.F, \xi)\,,\, \nu.\phi\,)$$

with

$$\phi := Q(\nu.F) \xrightarrow{\iota_{\nu.F}} Q(\,F(\,\nu.F\,)\,) \xrightarrow{Q(\xi^{-1})} Q(\,\nu.F\,)\,.$$

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Lifting algebras of functors

Remark
We have

$$Alg_{\mathbf{C}}(F) = CoAlg_{\mathbf{C}^{op}}(F^{op}) .$$

Considering that **SLatt** is auto dual ($*$-autonomous), we can get initial algebra lifting from the previous proposition/coroallary when $Q : \mathbf{B} \longrightarrow \mathbf{SLatt}$.

In general:
Proposition If $Q(X)$ is a complete lattice (for all objects X), then define $Q^{\mu} : Alg(F) \longrightarrow \mathbf{Pos}$ by

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Plan

1. Background
2. Lifting closed structure
3. Dualizing objects, star-autonomy
4. Coalgebras and algebras of a functor
5. **Ongoing and future work**

TODO list

- Other kind of liftings:
  - limits/colimits,
  - monads, comonads,
  - algebras of a functor,
  - linearly distributive structures, ...
- Understand various monoidal categories of the form $\int \mathbf{Q}$ w.r.t. the theory just developed. In particular:
  - finite dimensional Banach (normed) spaces and contracting linear maps.
- Understand the categorical structure of various categories of fuzzy relations, as generalization of Q-**Set**, by replacing **Rel** by **Rel**(Q).

TODO list: an interesting conjecture

All the previous computations as if we had a typed quantale.

Conjecture Let $\mathbf{B}$ be $*$-autonomous and let $Q : \mathbf{B} \longrightarrow \mathbf{SLatt}$ be monoidal. Then $\int \mathbf{Q}$ is $*$-autonomous if and only if Q is a Girard monoid in the monoidal category $[\mathbf{B}, \mathbf{SLatt}]$ (with convolution as tensor).

Remarks

- If $\mathbf{B}$ is $*$-autonomous, then $[\mathbf{B}, \mathbf{SLatt}]$ is $*$-autonomous as well (Egger 2008).
- The conjecture yields a test ground for the results in (De Lacroix and S., CSL 2023).

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Thanks!

Some relevant (and incomplete) literature

A. Carboni and R. F. C. Walters.
Cartesian bicategories. I.
J. Pure Appl. Algebra, 49:11–32, 1987.

C. de Lacroix and L. Santocanale.
Unitless Frobenius quantales.
working paper or preprint, Apr. 2022.

C. de Lacroix and L. Santocanale.
Frobenius structures in star-autonomous categories.
In B. Klin and E. Pimentel, editors, *CSL 2023*, volume 252 of *LIPIcs*, pages 18:1–18:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2023.

J. Egger.
Star-autonomous functor categories.
Theory Appl. Categ., 20:307–333, 2008.

M. Fiore, Z. Galal, and F. Jafarrahmani.
Fixpoint constructions in focused orthogonality models of linear logic.
To appear in the proceeedings of the conference MFPS 2023, Electronic Notes in Theoretical Informatics and Computer Science, 2023.

J. Harding, C. Walker, and E. Walker.
Categories with fuzzy sets and relations.
Fuzzy Sets Syst., 256:149–165, 2014.

I. Hasuo, K. Cho, T. Kataoka, and B. Jacobs.
Coinductive predicates and final sequences in a fibration.
Electronic Notes in Theoretical Computer Science, 298:197–214, 2013.
Proceedings of MFPS XXIX.

C. Hermida.
On fibred adjunctions and completeness for fibred categories.
In *Recent trends in data type specification. 9th workshop on specification of abstract data types joint with the 4th COMPASS workshop, Caldes de Malavella, Spain, October 26–30, 1992. Selected papers*, pages 235–251. Berlin: Springer, 1994.

M. Hyland and A. Schalk.
Glueing and orthogonality for models of linear logic.
Theor. Comput. Sci., 294(1-2):183–231, 2003.

F. Jafarrahmani.
Fixpoints of Types in Linear Logic from a Curry-Howard-Lambek Perspective.
PhD thesis, Université Paris Cité, 2023.

J. Moeller and C. Vasilakopoulou.
Monoidal Grothendieck construction.
Theory Appl. Categ., 35:1159–1207, 2020.

A. Schalk and V. de Paiva.
Poset-valued sets or how to build models for linear logics.
Theor. Comput. Sci., 315(1):83–107, 2004.

M. Shulman.
Framed bicategories and monoidal fibrations.
Theory and Applications of Categories, 20(18):650–738, 2008.