Algebraic decompositions, bijections, and universal singular exponents for equations with one catalytic parameter of order one

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based in part on join work with ENRICA DUCHI

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A crash course in generatingfunctionology

aka **symbolic and analytic combinatorics**

Enumerative combinatorics and generating functions

Let $\mathcal{A}$ be a set of combinatorial objects equipped with an integer size $|.|$ and assume that for each n the set

$$\mathcal{A}_n = \{a \in \mathcal{A} \text{ s.t. } |a| = n\}$$

is finite, and let $a_n = |\mathcal{A}_n|$ denote its cardinality.

The generating function (gf) of the class $\mathcal{A}$ w.r.t. the size is

$$A \equiv A(t) := \sum_{n \geq 0} a_n t^n = \sum_{\alpha \in \mathcal{A}} t^{|\alpha|}$$

Refined enumeration:

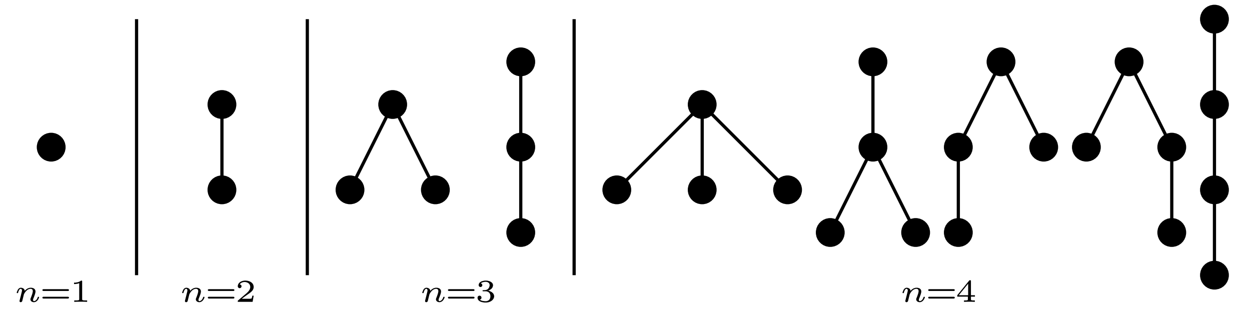
$$A(u) \equiv A(u,t) := \sum_{n,k \geq 0} a_{k,n} u^k t^n = \sum_{\alpha \in \mathcal{A}} u^{p(\alpha)} t^{|\alpha|}$$

for some parameter $p : \mathcal{A} \to \mathbb{Z}$, and $a_{k,n} = |\{a \in \mathcal{A}_n \mid p(a) = k\}$

Plane trees

Plane trees (aka ordered trees)

$\mathcal{A}_n = \{$plane trees with n vertices$\}$



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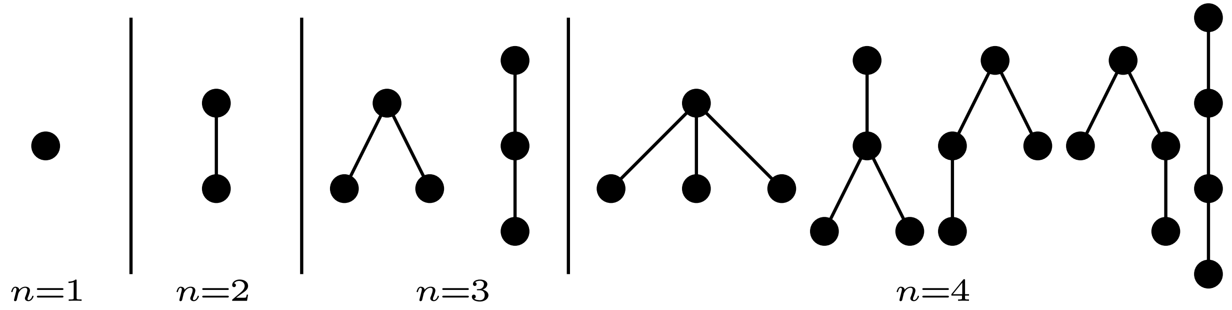
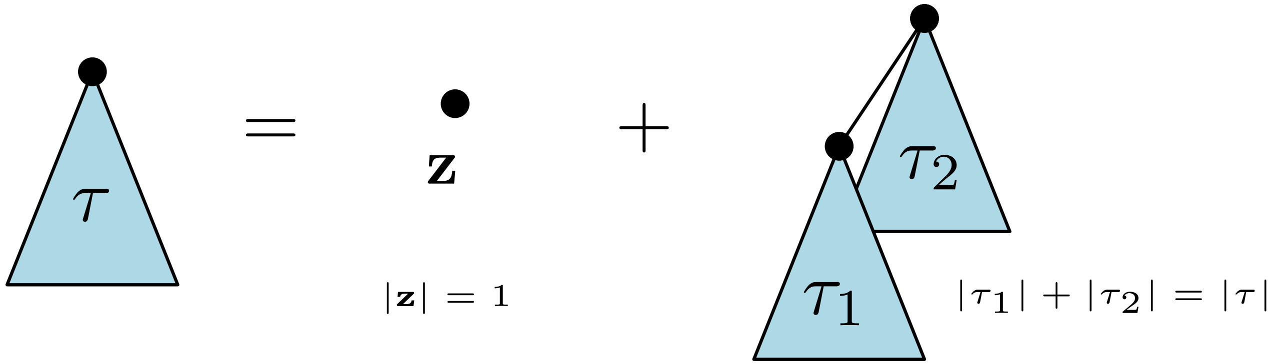


Characterized by their decomposition at root edge



$\mathcal{A} \equiv \mathbf{z} + \mathcal{A} \times \mathcal{A}$

size preserving bijection

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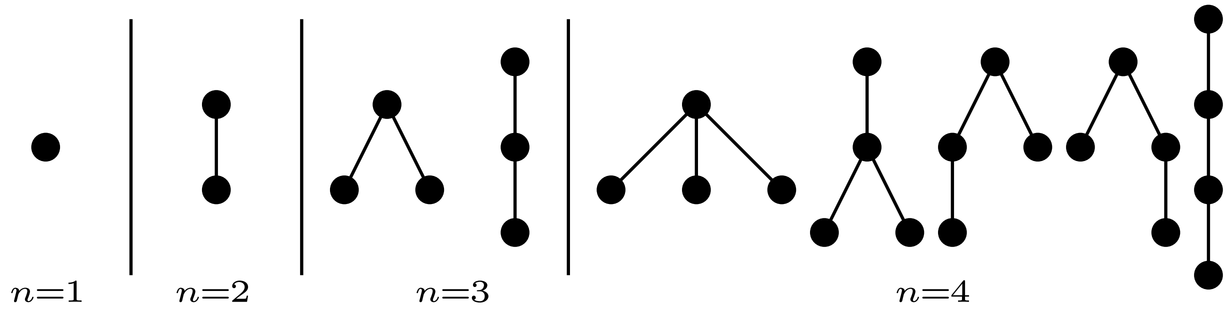

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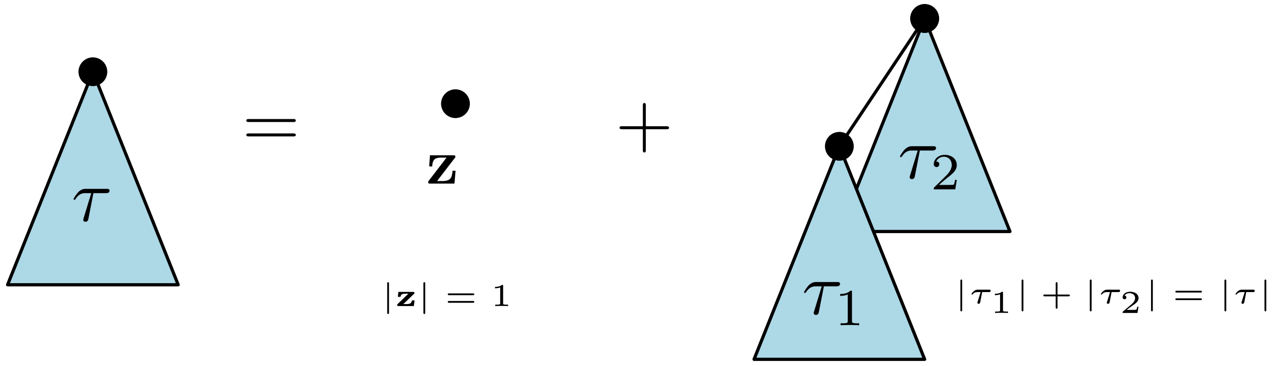

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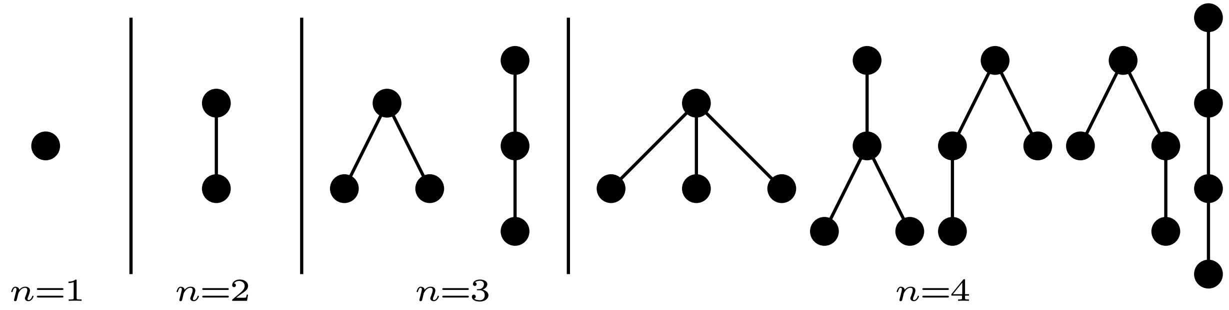

$A(t) = t + t^2 + 2t^3 + 5t^4 + O(t^5)$

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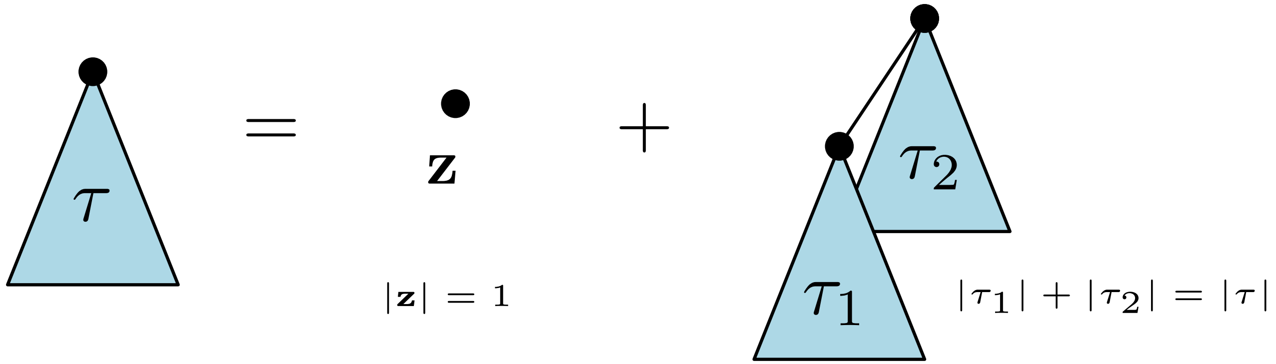

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$$A(t) = \sum_{\tau \in \mathcal{A}} t^{|\tau|} = t^1 + \sum_{(\tau_1,\tau_2) \in \mathcal{A}\times\mathcal{A}} t^{|\tau_1|+|\tau_2|} = t + \sum_{\tau_1 \in \mathcal{A}} t^{|\tau_1|} \sum_{\tau_2 \in \mathcal{A}} t^{|\tau_2|} = t + A(t)^2$$

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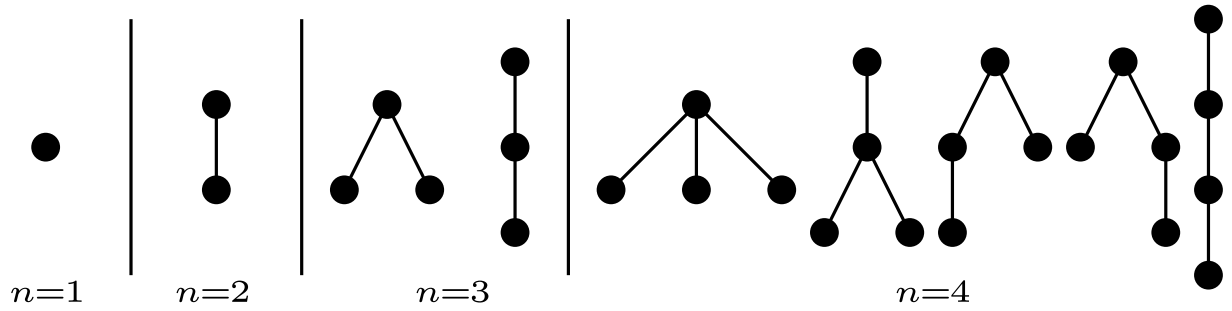

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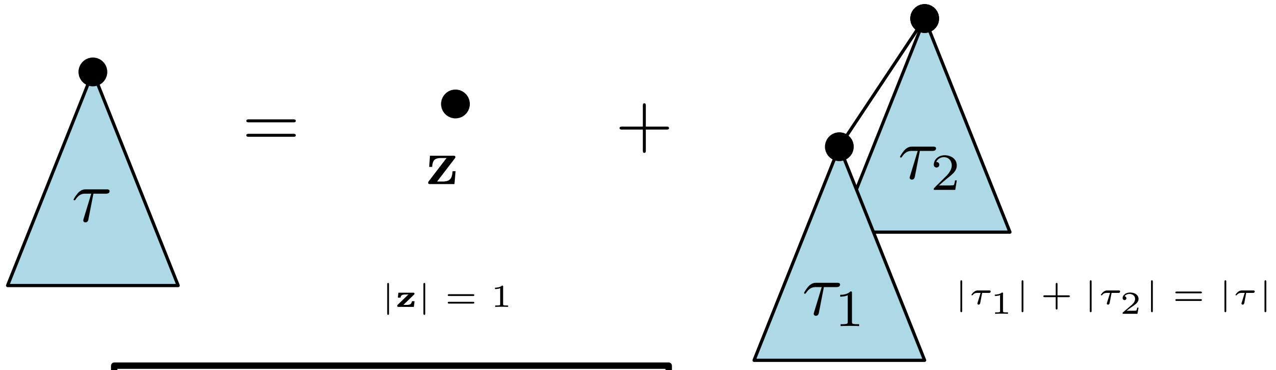

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$\Rightarrow$ **exact formulas or efficient enumeration algorithms**

$$A(t) = \frac{1-\sqrt{1-4t}}{2} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} t^{n+1} \;\Rightarrow\; a_{n+1} = \frac{1}{n+1}\binom{2n}{n} \underset{n\to+\infty}{\sim} \frac{1}{\sqrt{\pi}} \cdot 4^n\, n^{-3/2}$$

or $A = t + A^2 \;\Rightarrow\; A' = 2(2tA' - A) + 1 \;\Rightarrow\; (n+1)a_{n+1} = 2(2n-1)a_n$

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if $f(z)$ is (Δ, ρ)-analytic with dominant singular term $(1 - z/\rho)^{-\alpha}$, then $[t^n]f(t) \underset{n\to+\infty}{\sim} \frac{n^{\alpha-1}}{\Gamma(\alpha)}\rho^{-n}$.

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Here $A(z) = \frac{1}{2} - \frac{1}{2}(1-4z)^{1/2}$, so that $a_n = [t^n]A(t) \underset{n\to+\infty}{\sim} (-\frac{1}{2}) \cdot \frac{n^{-\frac{1}{2}-1}}{-2\sqrt{\pi}} (1/4)^{-n} \sim \frac{1}{4\sqrt{\pi}} \cdot 4^n\, n^{-3/2}$

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average number of leaves in trees of $\mathcal{A}_n$: $\ell_n = \frac{1}{a_n} \sum_{\tau \in A_n} \ell(\tau) = \frac{1}{a_n} \sum_{k \geq 0} k a_{k,n} = \frac{1}{a_n} [t^n] \frac{\partial A}{\partial u}(1,t)$

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$$\implies \ell_n \underset{n \to +\infty}{\sim} \frac{1}{2} n$$

Context free languages and algebraic specifications

Any well funded algebraic specification for a class $\mathcal{F}_1$

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with each $\mathcal{P}^{(i)}$ a finite combination of $+$ and $\times$ operators

Includes all languages generated by non ambiguous context free grammars

Aka multitype simply generated tree-like structures

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Applies in particular to non ambiguous context free grammars.

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Drmota Lalley Wood theorem: if spec is strongly connected and non linear then all series $F^{(i)}(z)$ have a same radius of convergence ρ and square root singular expansions in Δ-domains near ρ of the form:

$$F^{(i)}(z) = \alpha_i - \beta_i(1 - z/\rho)^{1/2} + O(1 - z/\rho)$$

with computable positive constants $\alpha_i > 0$ and $\beta_i > 0$.

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Corollary (Universal asymptotic for strongly connected algebraic specifications)

$$|\mathcal{F}_n^{(i)}| = [t^n]F^{(i)}(t) \underset{n \to +\infty}{\sim} \frac{\beta_i}{2\sqrt{\pi}} \cdot \rho^{-n} n^{-3/2}$$

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Tree counting exponent!

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Drmota Lalley Wood theorem: if spec is strongly connected and non linear then all series $F^{(i)}(z)$ have a same radius of convergence ρ and square root singular expansions in Δ-domains near ρ of the form:

$$F^{(i)}(z) = \alpha_i - \beta_i(1 - z/\rho)^{1/2} + O(1 - z/\rho)$$

with computable positive constants $\alpha_i > 0$ and $\beta_i > 0$.

Corollary (Universal asymptotic for strongly connected algebraic specifications)

$$|\mathcal{F}_n^{(i)}| = [t^n]F^{(i)}(t) \underset{n\to+\infty}{\sim} \frac{\beta_i}{2\sqrt{\pi}} \cdot \rho^{-n} n^{-3/2}$$

Tree counting exponent!

$+$ additive auxiliary parameters have linear expectation (and Gaussian law)

A (slightly) different family of trees

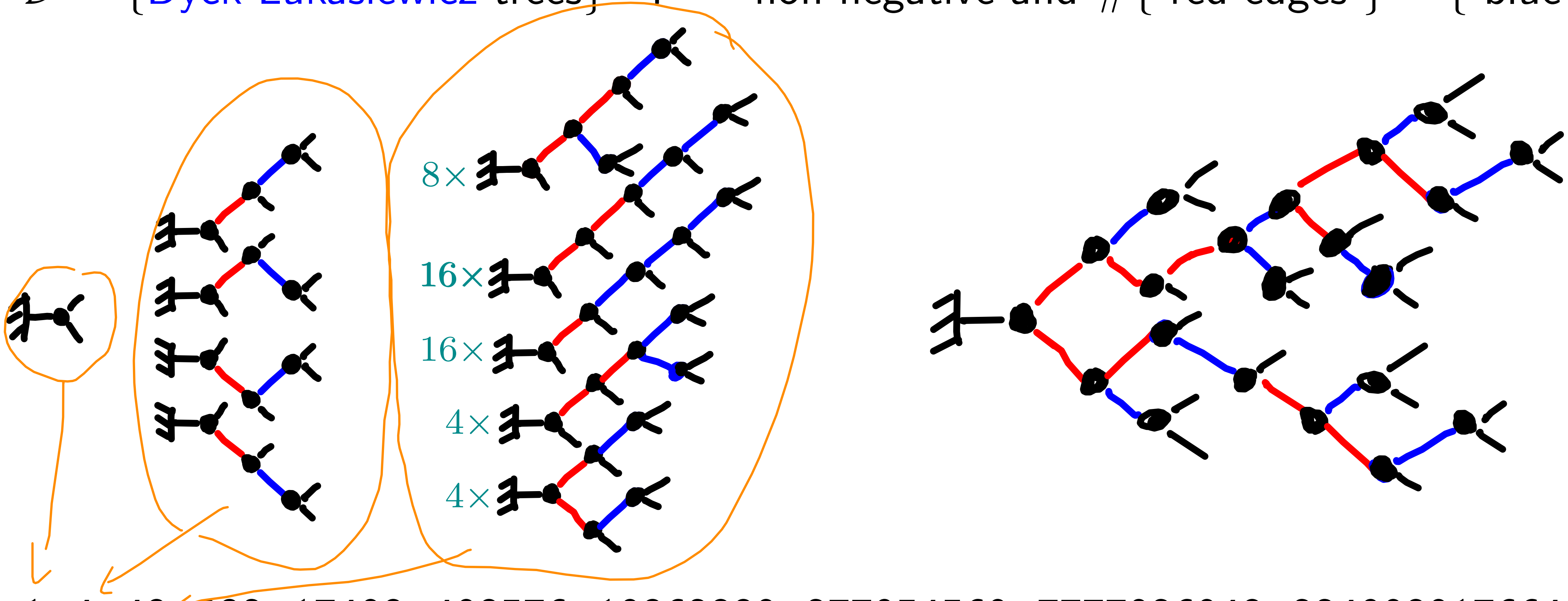
and catalytic equations of order one

Dyck-Łukasiewicz trees

$\mathcal{B} = \{$blue/red binary trees$\}$: planted binary tree with blue and red (inner) edges

$\mathcal{F} = \{$Non negative bicolored trees$\}$: no more red than blue in each planted subtree

$\mathcal{D} = \{$Dyck-Łukasiewicz trees$\}$: non negative and $\#\{$ red edges $\} = \{$ blue edges $\}$



1, 4, 48, 832, 17408, 408576, 10362880, 277954560, 7777026048, 224908017664

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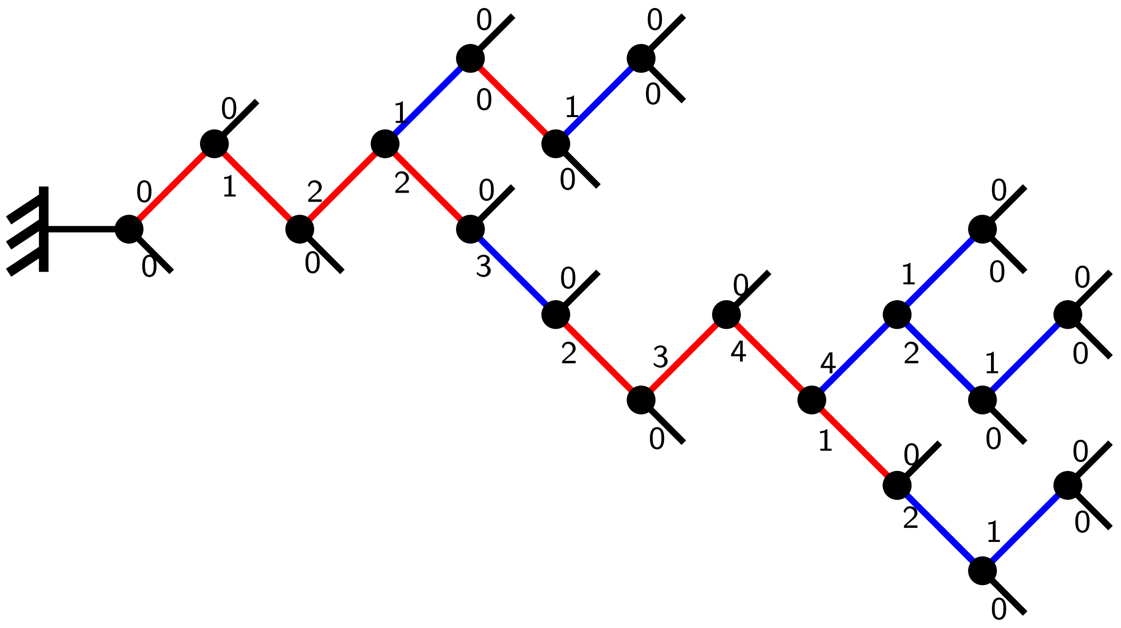
Label each inner edge or vertex with the difference between nbs of blue and red edges in corresponding subtree.



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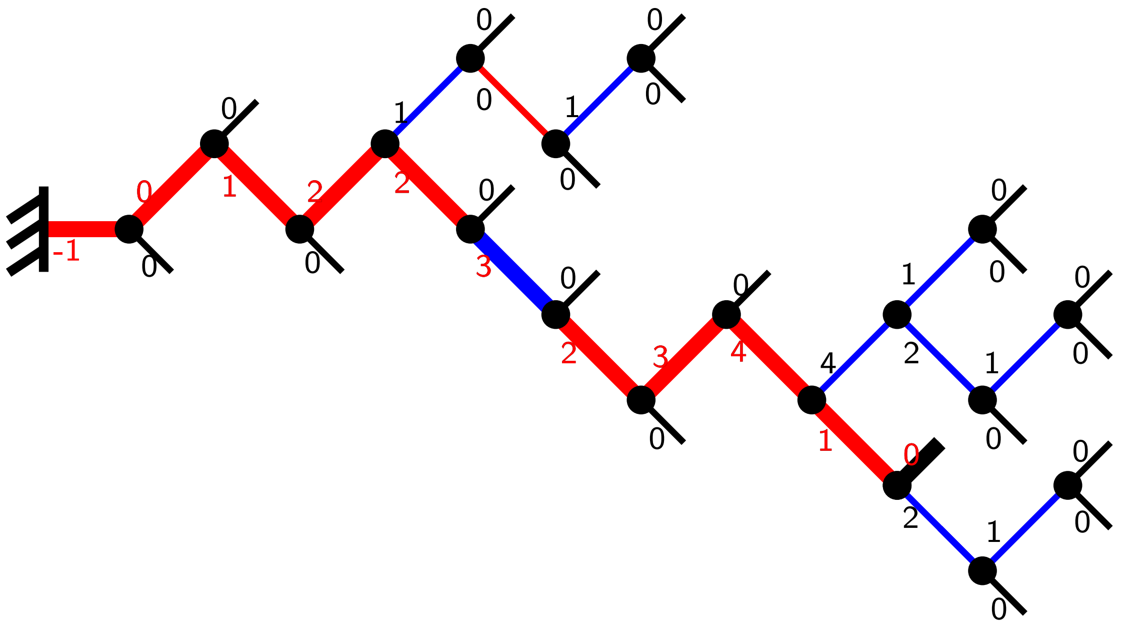
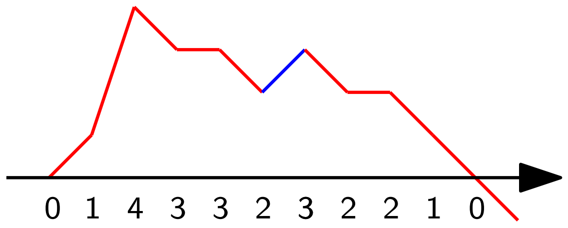


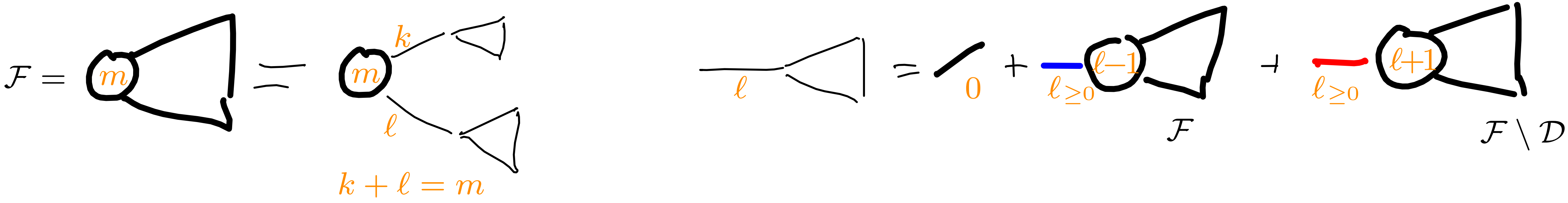
Labels from a leaf to the root form a Łukasiewicz walk.



Enumeration of Dyck-Łukasievicz trees

Proposition. The familly $\mathcal{F}$ of non negative trees admit the extended symbolic specification

$$\mathcal{F} \equiv \mathbf{z} \times \left(1 + \mathbf{a}\mathbf{u}\,\mathcal{F} + \mathbf{b}\mathbf{u}^{-1}(\mathcal{F} \setminus \mathcal{D})\right)^2$$



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The gf traduction: The bivariate gf $F(u) \equiv F(u, t)$ of non negative trees with u marking root label

satisfies $F(u) = t\,\left(1 + u\,F(u) + \dfrac{F(u) - f}{u}\right)^2$

where $f = F(0)$ is the gf of D-Ł-trees w.r.t. to the number of nodes.

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Proposition. The DŁ-tree gf is $f = V - 4V^3$ where $V = t(1 + 4V^2)^2$

In particular

$$[t^n]V = \tfrac{1}{n}[x^{n-1}](1+4x^2)^{2n} = \tfrac{4^m}{2m+1}\tbinom{4m+2}{m} \text{ with } n = 2m+1$$

$$[t^n]f = \tfrac{1}{n}[x^{n-1}](x-4x^3)'(1+4x^2)^{2n} = \tfrac{4^m}{(m+1)(2m+1)}\tbinom{4m+2}{m} \text{ with } n = 2m+1$$

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Observe. $[t^n]V \sim c_V \cdot \rho^{-n} n^{-3/2}$ with standard 3/2 tree counting exponent

but $[t^n]f \sim c_f \cdot \rho^{-n} n^{-5/2}$ with critical exponent $5/2$

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but $[t^n]f \sim c_f \cdot \rho^{-n} n^{-5/2}$ with critical exponent $5/2$

$\Rightarrow$ direct context-free specification for f cannot exist!

Other instances of catalytic equations

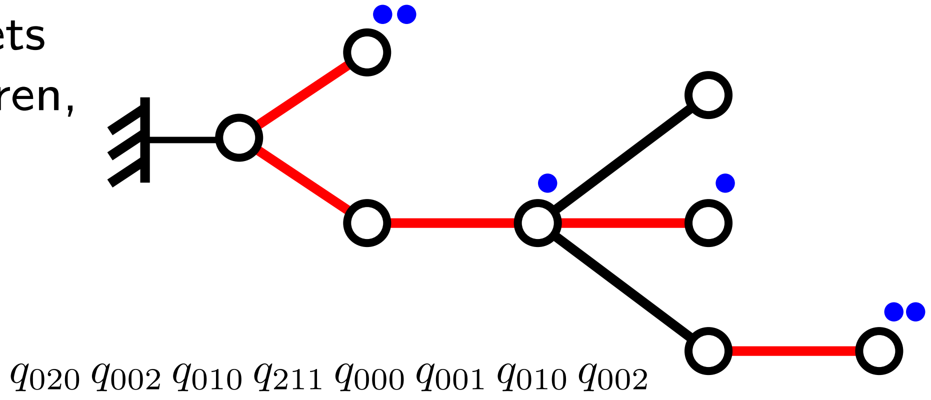
In fact, various families of combinatorial structure are known to involve such equations with the divided differences $\frac{1}{u}(F(u) - F(0))$ with respect to a *catalytic variable*.

- Various families of planar maps and triangulations (Tutte et al. 60's)
- Various families of pattern avoiding permutations (West's two-stack sortable, 90's)
- Tamari intervals (Chapoton, 2000's, Bousquet-Mélou-Chapoton 2022)
- Planar (normal) λ-terms (Zeilberger and Giorgietti, 2015)
- Duchi et al.'s fighting fish and variants (2016)
- Chen's fully parked trees (2021)

All these examples lead to the same 5/2 counting exponent.

A generic tree interpretation (Duchi-S. 2020)

A Q-tree is a plane tree with black and red edges and blue bullets on vertices, with weight q_{ijk} on vertices with i black edge children, j red edge children and k bullets.



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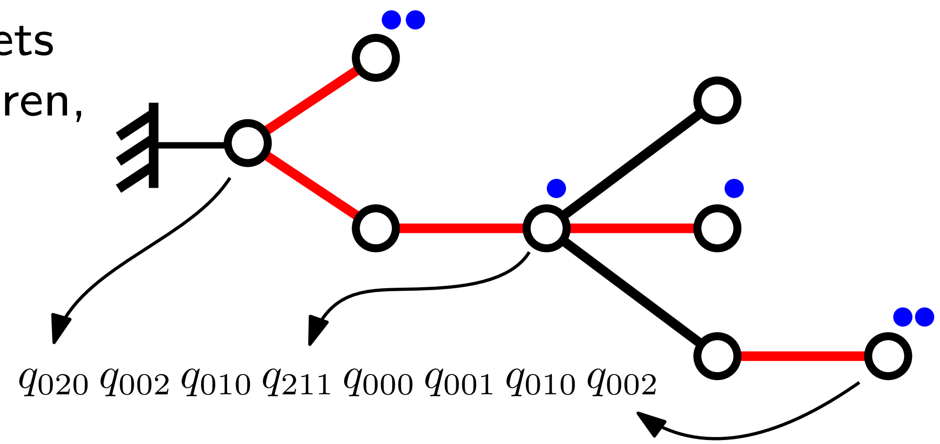

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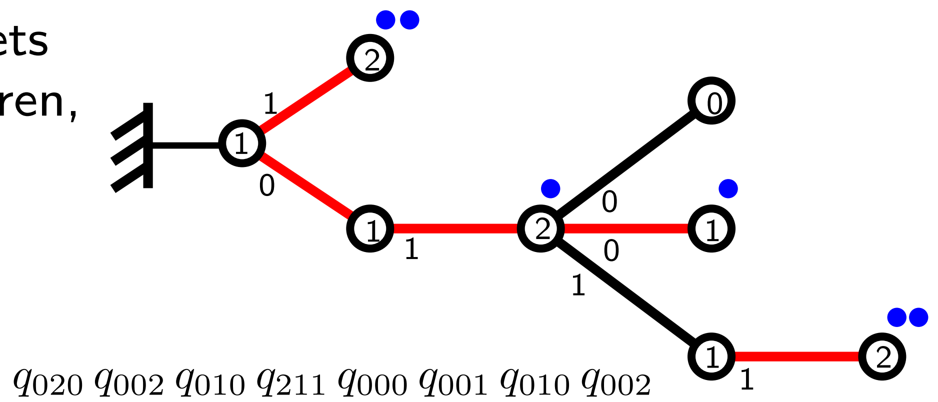
Let $\tau[x]$ denote the subtree of τ at vertex or edge x, and let $\ell_\tau(x) = \#\{\text{blue bullets in } \tau[x]\} - \#\{\text{red edges in } \tau[x]\}$

A Q-tree τ is non negative if $\ell_\tau(e) \geq 0$ for all edges e.



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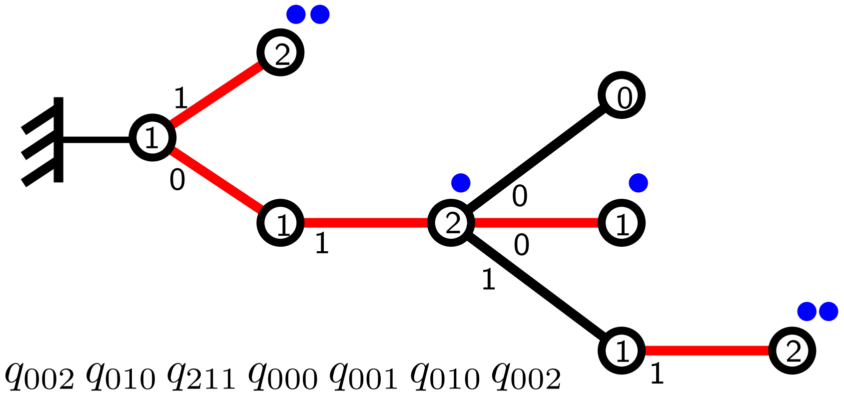

$q_{020}\, q_{002}\, q_{010}\, q_{211}\, q_{000}\, q_{001}\, q_{010}\, q_{002}$

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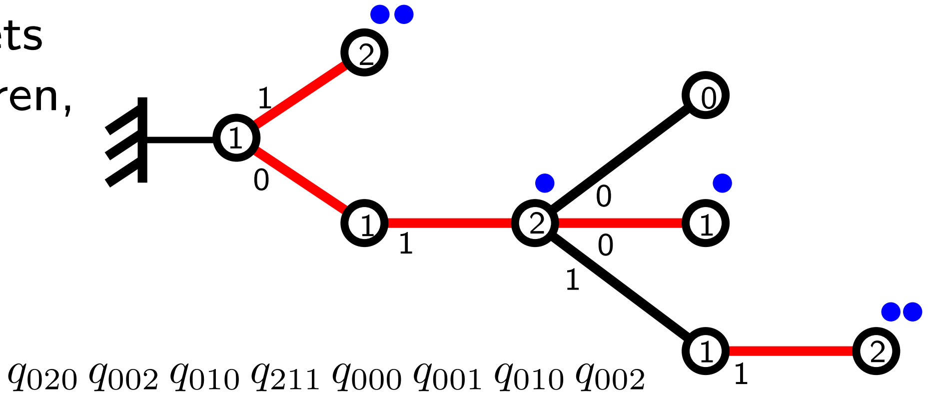

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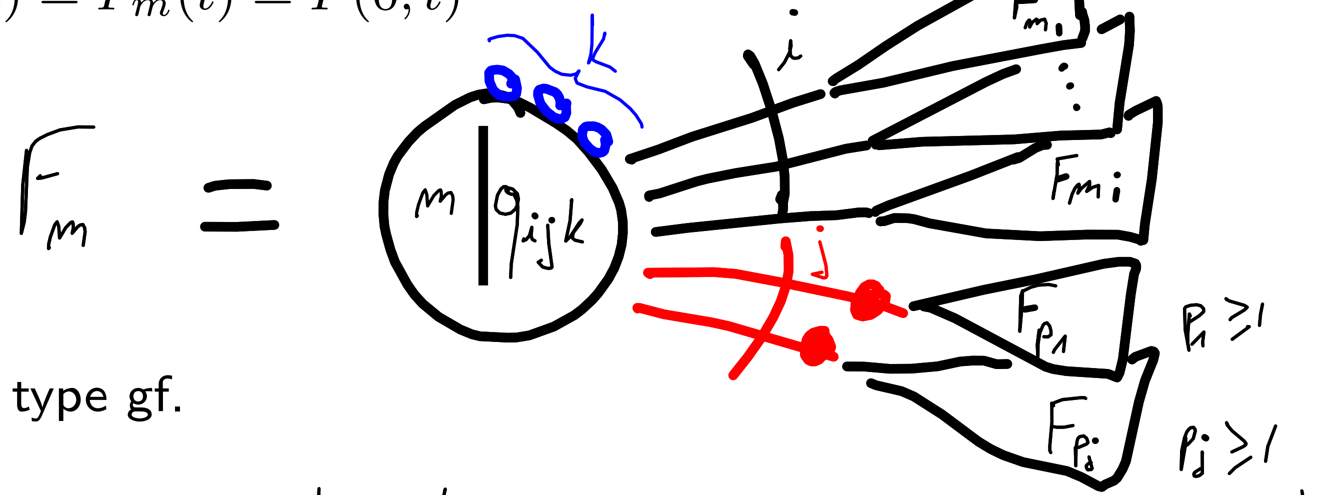
Let $F_m(t)$ be the weighted gf of non negative Q-trees τ with $\ell(\tau) = m$ and $F(u) \equiv F(u,t) = \sum_{m\geq 0} F_m(t)u^k$ and $f \equiv f(t) = F_m(t) = F(0,t)$ be the gf of DŁ-Q-trees.

Proposition. $F(u)$ is* the unique fps solution of

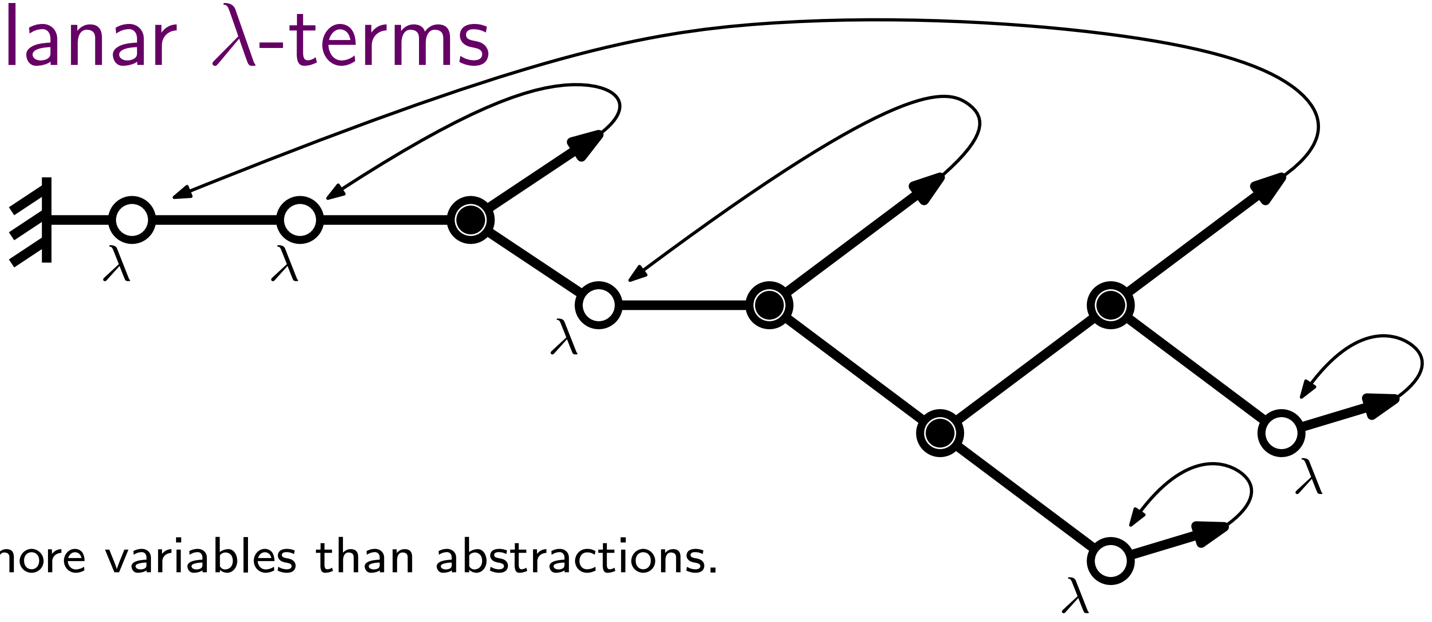
$$F(u) = t\,Q\Big(F(u), \tfrac{1}{u}(F(u) - f), u\Big)$$

where $Q(v,w,u) = \sum_{i,j,k\geq 0} q_{ijk} v^i w^j u^k$ is the vertex type gf.



$$m = k + (m_1 + \dots + m_i) + ((p_1 - 1) + \dots + (p_j - 1))$$

Open and closed planar λ-terms



The skeleton trees have

- applications: binary nodes
- abstractions: unary nodes
- variables: leaves, represented as arrow.

with condition that in each subterm there are more variables than abstractions.

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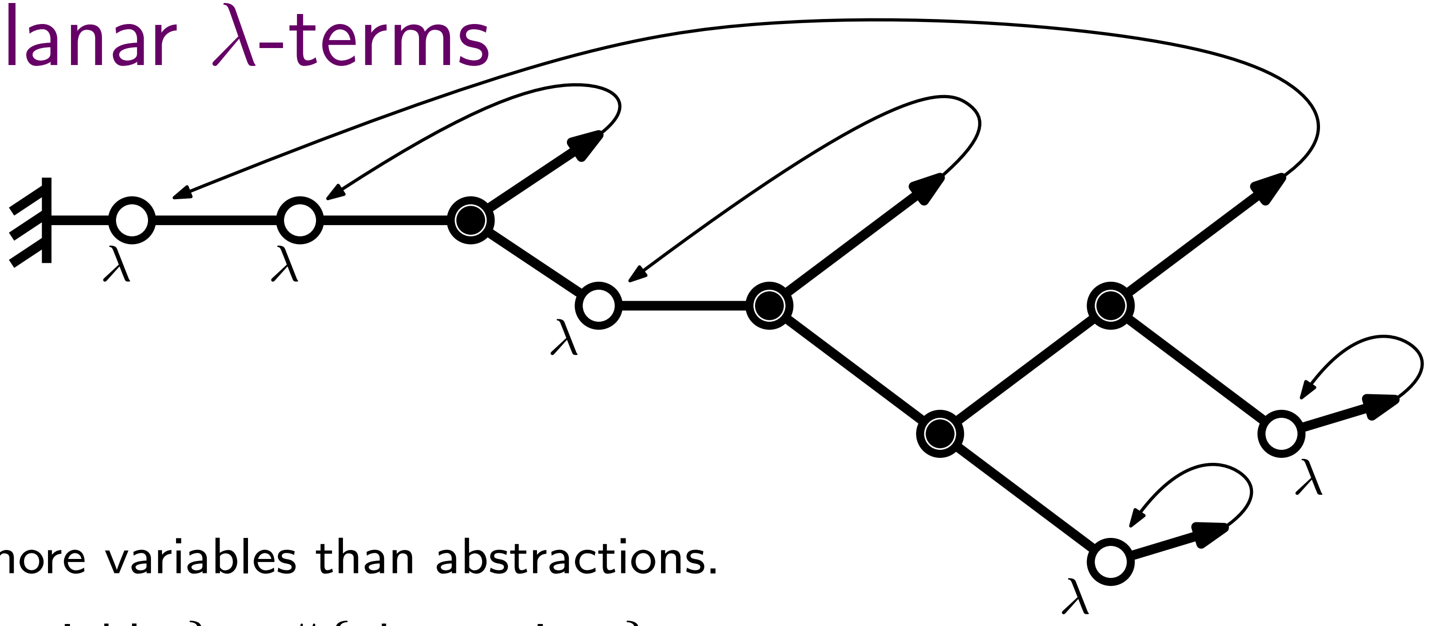

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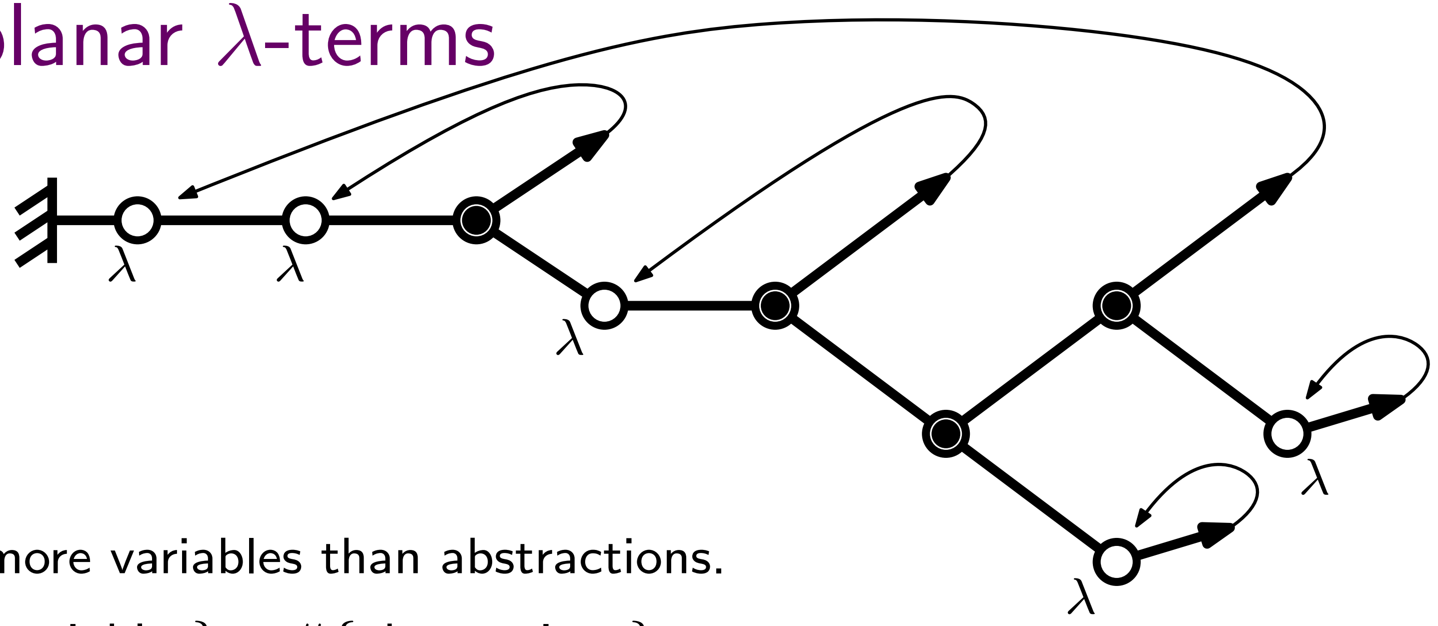
Let the *catalytic parameter* be $\text{excess}(\tau) = \#\{\text{variables}\} - \#\{\text{abstractions}\}$

Then the catalytic equation is

$$P(u) = tu + tP(u)^2 + \frac{t}{u}(P(u) - p)$$

cf Zeilberger-Giorgietti 15, Singh 22 Eq (3.5)

Open and closed planar λ-terms



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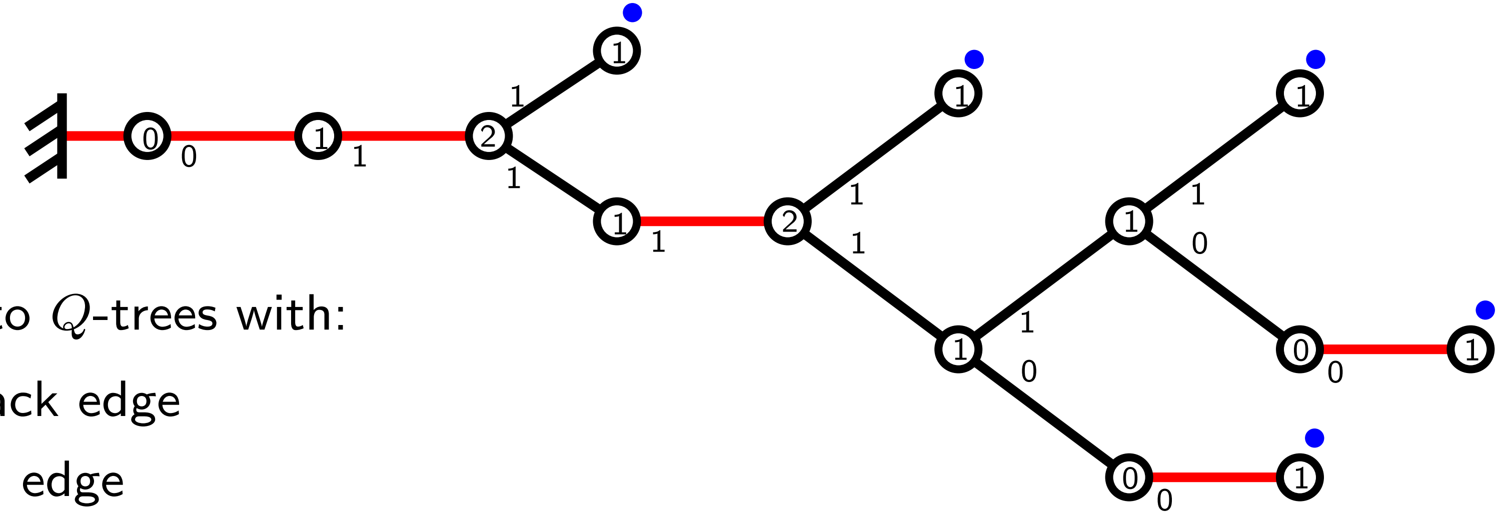

Open planar λ-terms immediately correspond to Q-trees with:

- applications: binary nodes carrying two black edge
- abstractions: unary nodes carrying one red edge
- variables: leaves with unit catalytic increment.

Then closed planar λ-terms correspond to DŁ-Q-trees.

Analytic combinatorics for catalytic equations

and the universality of counting exponents

Drmota-Noy-Yu theorem

Theorem (Drmota, Noy, Yu, 2020): Let $F(u) \equiv F(u,t)$ be the unique power series solution of the equation
$$F(u) = t\,Q\left(F(u), \frac{1}{u}(F(u)-f),\, u\right)$$
where $Q(v,w,u)$ is a non linear* polynomial with non negative coefficients. Then $f \equiv f(t) = F(0)$ has a dominant singularity $\rho > 0$ with singular expansion
$$f(z) = \alpha_f - \gamma_f(1-z/\rho) - \delta_f(1-z/\rho)^{3/2} + O((1-z/\rho)^2)$$
with computable positive constants $\alpha_f > 0$, γ_f and $\delta_f > 0$.

Under standard technical aperiodicity conditions, transfer theorems then imply
$$[t^n]f(t) \sim \frac{\delta_f}{\Gamma(-3/2)} \cdot \rho^{-n} n^{-5/2}.$$

Corollary (Drmota, Noy, Yu, 2020): The critical counting exponent $5/2$ is generic for combinatorial classes governed by a non negative equation with one catalytic variable of order one.

$+$ additive auxiliary parameters have linear expectation (and Gaussian law)

Proof technics: Bousquet-Mélou–Jehanne's method

$\frac{\partial}{\partial u}$ applied to $$F(u) = t\,Q\left(F(u), \frac{1}{u}(F(u) - f), u\right)$$

yields $$F'_u(u) = F'_u(u) \cdot t\,\left(Q'_v(\ldots) + \tfrac{1}{u} Q'_w(\ldots)\right) - t\,\tfrac{1}{u}\,\tfrac{F(u)-f}{u} Q'_w(\ldots) + t\,Q'_u(\ldots)$$

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The unique fps $U \equiv U(t)$ satisfying $U = t\,U\,Q'_v\left(F(U), \frac{F(U)-f}{U}, U\right) + t\,Q'_w\left(F(U), \frac{F(U)-f}{U}, U\right)$ cancels the left term $(*)$.

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Then U, $V = F(U)$, $W = \frac{F(U)-f}{U}$ and f satisfy a polynomial system

$$\begin{cases} U &= t\,U\,Q'_v(V,W,U) + t\,Q'_w(V,W,U) \\ V &= t\,Q(V,W,U) \\ 0 &= -t\,\frac{1}{U}\,W\,Q'_w(V,W,U) + t\,Q'_u(V,W,U) \\ f &= V - U\,W \end{cases}$$

This system shows that f is algebraic but it is not non negative in general

$\Longrightarrow$ Drmota-Lalley-Wood does not apply (except if $Q'_w = 1$, cf Chapuy 2006)

Proof technics: Drmota, Noy, Yu's trick and tour de force

U, V, W and f satisfy

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Use Line 1 to replace Q'_w by Q'_v in Line 3:

$$\Rightarrow \begin{cases} U &= t\,U\,Q'_v(V,W,U) + t\,Q'_w(V,W,U) \\ V &= t\,Q(V,W,U) \\ W &= t\,W\,Q'_v(V,W,U) + t\,Q'_u(V,W,U) \\ f &= V - U\,W \end{cases}$$

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The power series U, V and W are now defined by a non negative strongly connected system, and DLW theorem immediately implies

$$\begin{cases} U &= \alpha_U - \beta_U(1-z/\rho)^{1/2} + O(1-z/\rho) \\ V &= \alpha_V - \beta_V(1-z/\rho)^{1/2} + O(1-z/\rho) \\ W &= \alpha_W - \beta_W(1-z/\rho)^{1/2} + O(1-z/\rho) \end{cases}$$

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DNY then show that there is a systematic cancellation in $f = V - UW$:

$$f = \alpha_f - \underbrace{(\beta_V - \alpha_U\beta_W - \alpha_W\beta_U)}_{=0}(1-z/\rho)^{1/2} + O(1-z/\rho)$$

Proof technics: Drmota, Noy, Yu's trick and tour de force

U, V, W and f satisfy

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$$\Rightarrow \left\{\begin{array}{lcl} U & = & t\,U\,Q'_v(V,W,U) + t\,Q'_w(V,W,U) \\ V & = & t\,Q(V,W,U) \\ W & = & t\,W\,Q'_v(V,W,U) + t\,Q'_u(V,W,U) \\ f & = & V - U\,W \end{array}\right.$$

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$$\left\{\begin{array}{lcl} U & = & \alpha_U - \beta_U(1-z/\rho)^{1/2} + \gamma_U(1-z/\rho) - \delta_U(1-z/\rho)^{3/2} + O((1-z/\rho)^2) \\ V & = & \alpha_V - \beta_V(1-z/\rho)^{1/2} + \gamma_V(1-z/\rho) - \delta_V(1-z/\rho)^{3/2} + O((1-z/\rho)^2) \\ W & = & \alpha_W - \beta_W(1-z/\rho)^{1/2} + \gamma_W(1-z/\rho) - \delta_W(1-z/\rho)^{3/2} + O((1-z/\rho)^2) \end{array}\right.$$

DNY then show that there is a systematic cancellation in $f = V - UW$:

$$f = \alpha_f - \underbrace{(\beta_V - \alpha_U\beta_W - \alpha_W\beta_U)}_{=0}(1-z/\rho)^{1/2} + \gamma_f(1-z/\rho) - \delta_f(1-z/\rho)^{3/2} + O((1-z/\rho)^2)$$

Proof technics: Drmota, Noy, Yu's trick and tour de force

U, V, W and f satisfy

Use Line 1 to replace Q'_w by Q'_v in Line 3:

$$\left\{\begin{array}{lcl} U &=& t\,U\,Q'_v(V,W,U) + t\,Q'_w(V,W,U) \\ V &=& t\,Q(V,W,U) \\ 0 &=& -t\,\frac{1}{U}\,W\,Q'_w(V,W,U) + t\,Q'_u(V,W,U) \\ f &=& V - U\,W \end{array}\right. \quad \Rightarrow \quad \left\{\begin{array}{lcl} U &=& t\,U\,Q'_v(V,W,U) + t\,Q'_w(V,W,U) \\ V &=& t\,Q(V,W,U) \\ W &=& t\,W\,Q'_v(V,W,U) + t\,Q'_u(V,W,U) \\ f &=& V - U\,W \end{array}\right.$$

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and a quite delicate analysis allows to check that $\delta_f \neq 0 \quad \Rightarrow \quad [t^n]f(t) \sim \frac{\delta_f}{\Gamma(-3/2)} \cdot \rho^{-n} n^{-5/2}$.

Proof technics: Drmota, Noy, Yu's trick and tour de force

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The power series U, V and W are now defined by a non negative strongly connected system, and DLW theorem immediately implies

$$\left\{\begin{array}{lcl} U &=& \alpha_U - \beta_U\,(1-z/\rho)^{1/2} + \gamma_U\,(1-z/\rho) - \delta_U\,(1-z/\rho)^{3/2} + O((1-z/\rho)^2) \\ V &=& \alpha_V - \beta_V\,(1-z/\rho)^{1/2} + \gamma_V\,(1-z/\rho) - \delta_V\,(1-z/\rho)^{3/2} + O((1-z/\rho)^2) \\ W &=& \alpha_W - \beta_W\,(1-z/\rho)^{1/2} + \gamma_W\,(1-z/\rho) - \delta_W\,(1-z/\rho)^{3/2} + O((1-z/\rho)^2) \end{array}\right.$$

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and a quite delicate analysis allows to check that $\boxed{\delta_f \neq 0}$ $\Rightarrow$ $[t^n]f(t) \sim \frac{\delta_f}{\Gamma(-3/2)} \cdot \rho^{-n} n^{-5/2}$.

A new proof: singular behavior via marking

$$F(u) = t\,Q\left(F(u), \frac{1}{u}(F(u)-f), u\right)$$

$\frac{\partial}{\partial u}$: $\underbrace{F'_u(u) = F'_u(u)\cdot t\,\left(Q'_v(\ldots) + \tfrac{1}{u}Q'_w(\ldots)\right)}_{\text{canceled by } u=U} - t\,\frac{1}{u}\,\frac{F(u)-f}{u}Q'_w(\ldots) + t\,Q'_u(\ldots)$

A new proof: singular behavior via marking

$$F(u) = t\,Q\left(F(u), \frac{1}{u}(F(u) - f), u\right)$$

$$\frac{\partial}{\partial u}: \quad \underbrace{F'_u(u) = F'_u(u) \cdot t\,\left(Q'_v(\ldots) + \tfrac{1}{u} Q'_w(\ldots)\right)}_{\text{canceled by } u = U} - t\,\tfrac{1}{u}\,\tfrac{F(u)-f}{u} Q'_w(\ldots) + t\,Q'_u(\ldots)$$

$$\frac{\partial}{\partial t}: \quad \underbrace{F'_t(u) = F'_t(u) \cdot t\,\left(Q'_v(\ldots) + \tfrac{1}{u} Q'_w(\ldots)\right)}_{u = U \qquad 0} - t\,\tfrac{1}{u} f'_t\, Q'_w(\ldots) + Q(\ldots)$$

$$\Rightarrow \boxed{t f'_t = U\,\frac{Q(\ldots)}{Q'_w(\ldots)}}$$

A new proof: singular behavior via marking

$$F(u) = t\,Q\left(F(u), \frac{1}{u}(F(u)-f), u\right)$$

$\frac{\partial}{\partial u}$: $F'_u(u) = F'_u(u)\cdot t\,\left(Q'_v(\dots) + \frac{1}{u}Q'_w(\dots)\right) - t\,\frac{1}{u}\,\frac{F(u)-f}{u}\,Q'_w(\dots) + t\,Q'_u(\dots)$

canceled by $u = U$

$\frac{\partial}{\partial t}$: $F'_t(u) = F'_t(u)\cdot t\,\left(Q'_v(\dots) + \frac{1}{u}Q'_w(\dots)\right) - t\,\frac{1}{u}\,f'_t\,Q'_w(\dots) + Q(\dots)$

$u = U$ $\quad 0$

$$\Rightarrow \boxed{t f'_t = U\,\frac{Q(\dots)}{Q'_w(\dots)}}$$

$$\Rightarrow \boxed{t f'_t = \frac{t\,Q(\dots)}{1 - t\,Q'_v(\dots)} = \frac{V}{1 - t\,Q'_v(V,W,U)}}$$

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$$F(u) = t\,Q\left(F(u), \frac{1}{u}(F(u) - f), u\right)$$

Then U, V, W and f are the unique fps satisfying the system

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This system for V, R, U and W is* strongly connected, non linear and non negative and tf'_t is a positive combination of R and V.

Drmota-Lalley-Wood then immediately implies that tf'_f has generic square root singularity

$$tf'_t = (1+\alpha_R)\alpha_V - (\alpha_V\beta_R + (1+\alpha_R)\beta_V)(1-z/\rho)^{1/2} + O(1-z/\rho)$$

$$> 0$$

A new proof: singular behavior via marking

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Then U, V, W and f are the unique fps satisfying the system

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$$tf'_t = (1+\alpha_R)\alpha_V - \underbrace{(\alpha_V\beta_R + (1+\alpha_R)\beta_V)}_{>0}(1-z/\rho)^{1/2} + O(1-z/\rho)$$

Corollary (Drmota-Noy-Yu 2020):

$$[t^n]f(t) = \frac{1}{n}[t^n]tf'(t) \underset{n\to\infty}{\sim} \frac{\alpha_V\beta_R+(1+\alpha_R)\beta_V}{2\sqrt{\pi}}\cdot\rho^{-n}n^{-5/2}.$$

Universal exponents for typical label and depth

Theorem (Duchi-S. 2020) The series $V = F(U)$ is the gf of DŁ-Q-trees with a marked red edge.

Theorem (S.23). The series $\Lambda = F'_u(U)$ is the gf for of DŁ-Q-trees with a red marked edge counted by the value of the label of the marked red edge. Then*

$$\Lambda = \alpha_\Lambda - \beta'_\Lambda(1 - z/\rho)^{1/4} + O((1 - z/\rho)^{1/2})$$

Corollary (S. 23) The average label value in DŁ-Q-trees of size n is

$$\frac{[t^n]\Lambda(t)}{[t^n]V(t)} \underset{n\to\infty}{\sim} cte \cdot \frac{n^{-5/4}}{n^{-3/2}} \sim cte \cdot n^{1/4}.$$

Theorem (S. 23). The series $\Delta = F'_t(U)$ is the gf of DŁ-Q-trees with a red marked edge with a marked vertex in its subtree. Equivalently Δ is the gf of DŁ-Q-trees with a red marked vertex counted by the red-depth of the marked vertex. Then*

$$\Delta = \frac{\beta''_\Delta}{(1-z/\rho)^{1/4}} + O((1 - z/\rho)^0)$$

Corollary (S. 23). The average vertex red-depth in DŁ-Q-trees of size n is

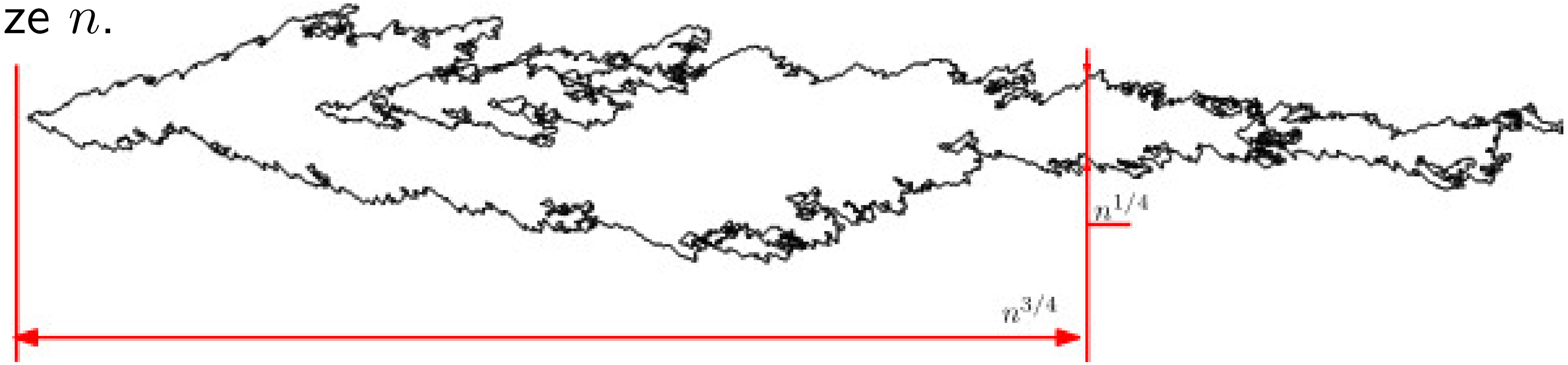
$$\frac{[t^n]\Delta(t)}{[t^n]tf'(t)} \underset{n\to\infty}{\sim} cte \cdot \frac{n^{-3/4}}{n^{-3/2}} \sim cte \cdot n^{3/4}.$$

Applications

- The average value λ_n of node labels and δ_n of red edges on path to the root in a random Q-tree of size n.

$$\mathbb{E}(\lambda_n) \underset{n\to\infty}{\sim} cte \cdot n^{1/4}. \qquad \mathbb{E}(\delta_n) \underset{n\to\infty}{\sim} cte \cdot n^{3/4}.$$

- The width λ_n and depth δ_n of a random cut in a uniform random fighting fish of size n.



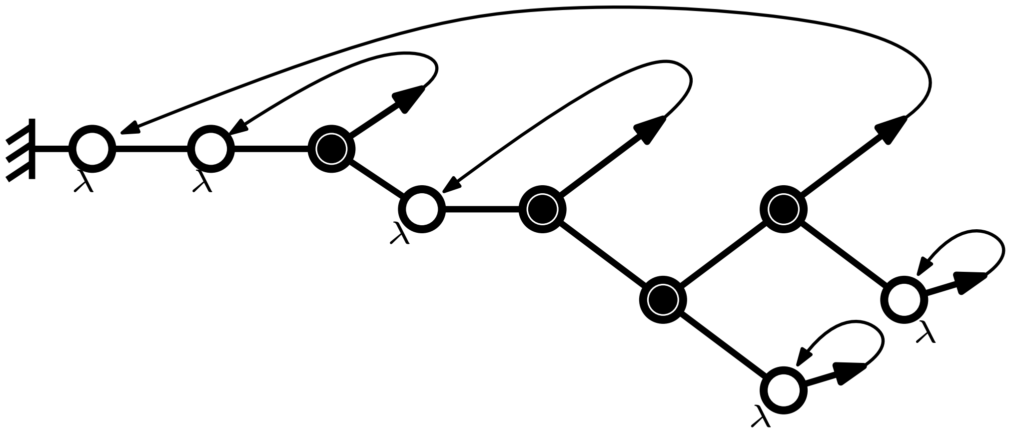
- The average length λ_n of backward edges and recursion stack size δ_n during the leftmost depth first search traversal of a uniform random planar map with n edges.
- The average flow λ_n of cars at a random vertex and its depth δ_n in a random fully parked parking tree of size n.

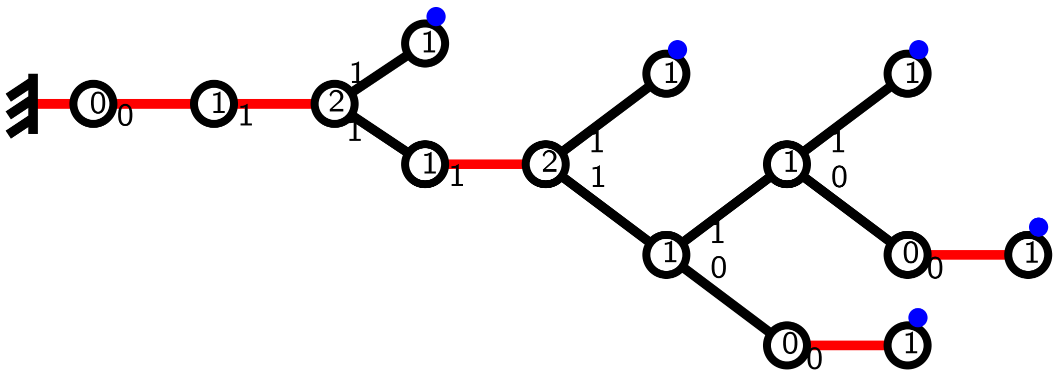
Applications

- The average value λ_n of node labels and δ_n of red edges on path to the root in a random Q-tree of size n.

$$\mathbb{E}(\lambda_n) \underset{n\to\infty}{\sim} cte \cdot n^{1/4}. \qquad \mathbb{E}(\delta_n) \underset{n\to\infty}{\sim} cte \cdot n^{3/4}.$$

- The average excess of nodes λ_n and average number of abstractions above node δ_n in a closed planar λ-term of size n.



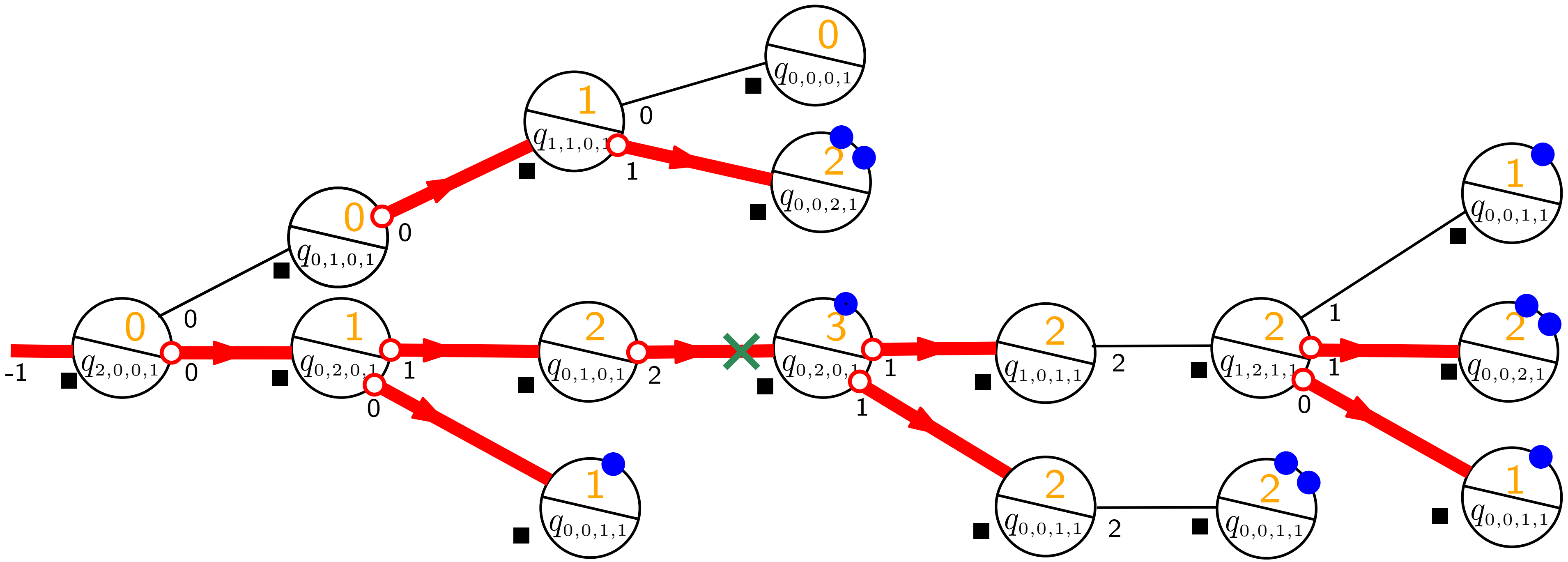


And now for something different...

Bijections!

From catalytic to context free specs, bijectively

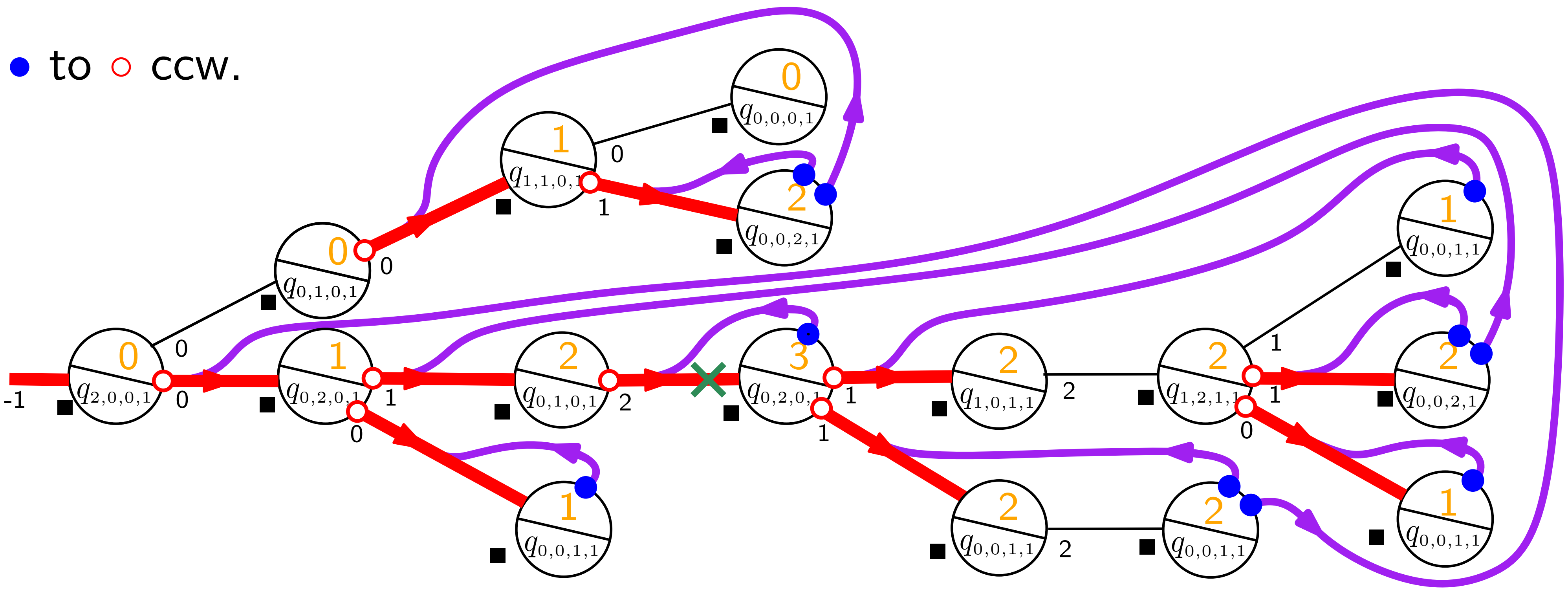
$$F(u) = t\,Q\left(F(u), \frac{1}{u}(F(u) - f), u\right)$$



From catalytic to context free specs, bijectively

$$F(u) = t\,Q\left(F(u), \frac{1}{u}(F(u) - f), u\right)$$

Ccw-closure: from ● to ○ ccw.

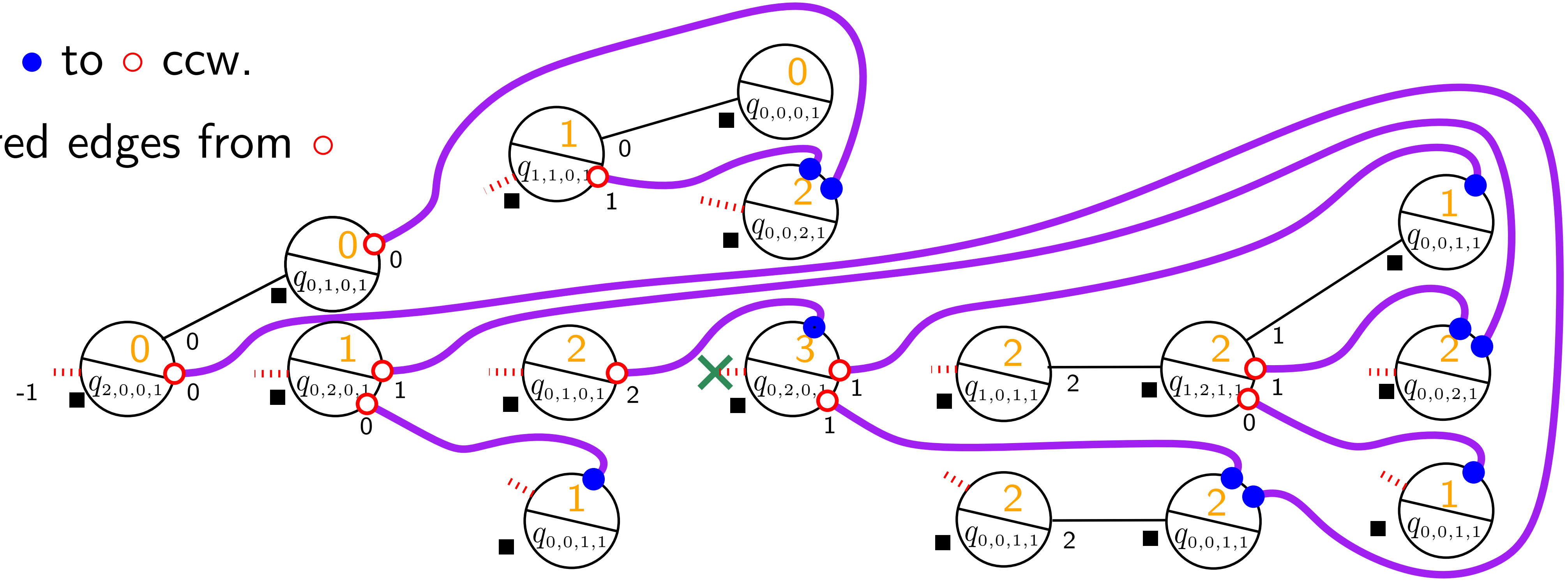


From catalytic to context free specs, bijectively

$$F(u) = t\,Q\left(F(u), \frac{1}{u}(F(u) - f), u\right)$$

Ccw-closure: from • to ○ ccw.

Rewiring: detach red edges from ○



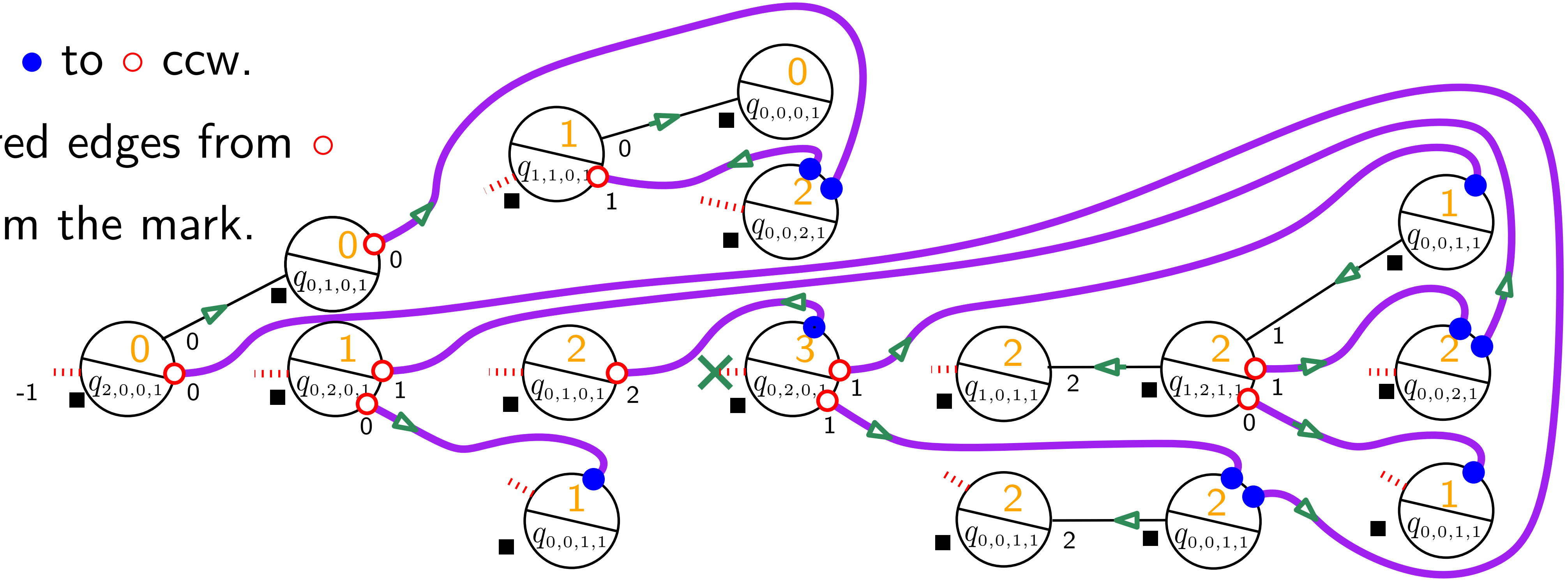
From catalytic to context free specs, bijectively

$$F(u) = t\,Q\left(F(u), \frac{1}{u}(F(u) - f), u\right)$$

Ccw-closure: from • to ○ ccw.

Rewiring: detach red edges from ○

Reorient: away from the mark.



From catalytic to context free specs, bijectively

$$F(u) = t\,Q\left(F(u), \frac{1}{u}(F(u) - f), u\right)$$

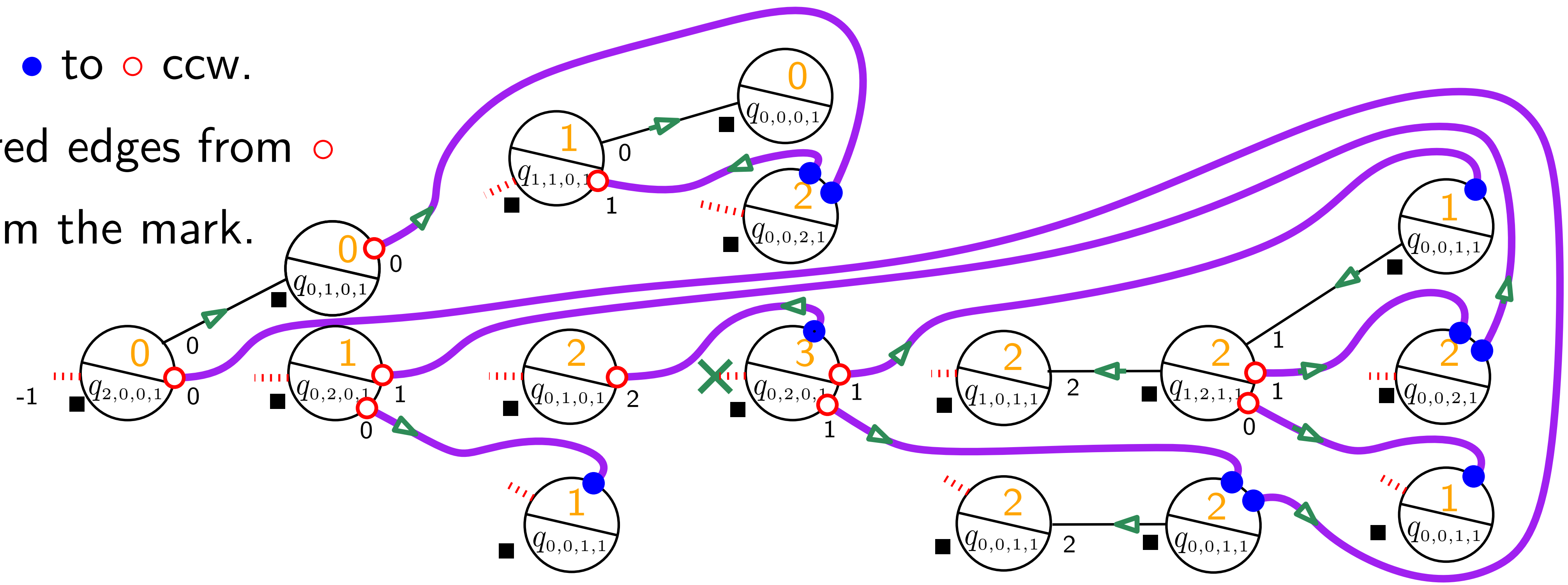
Lemma (Duchi-S. 23). The ccw-closure and rewiring of a DŁ-Q-tree is a tree.

Proposition (Duchi-S. 23).
Ccw-closure and rewiring of DŁ-Q-trees are injective mappings, and their inverse are cw-closure followed by rewiring.

Ccw-closure: from • to ○ ccw.

Rewiring: detach red edges from ○

Reorient: away from the mark.



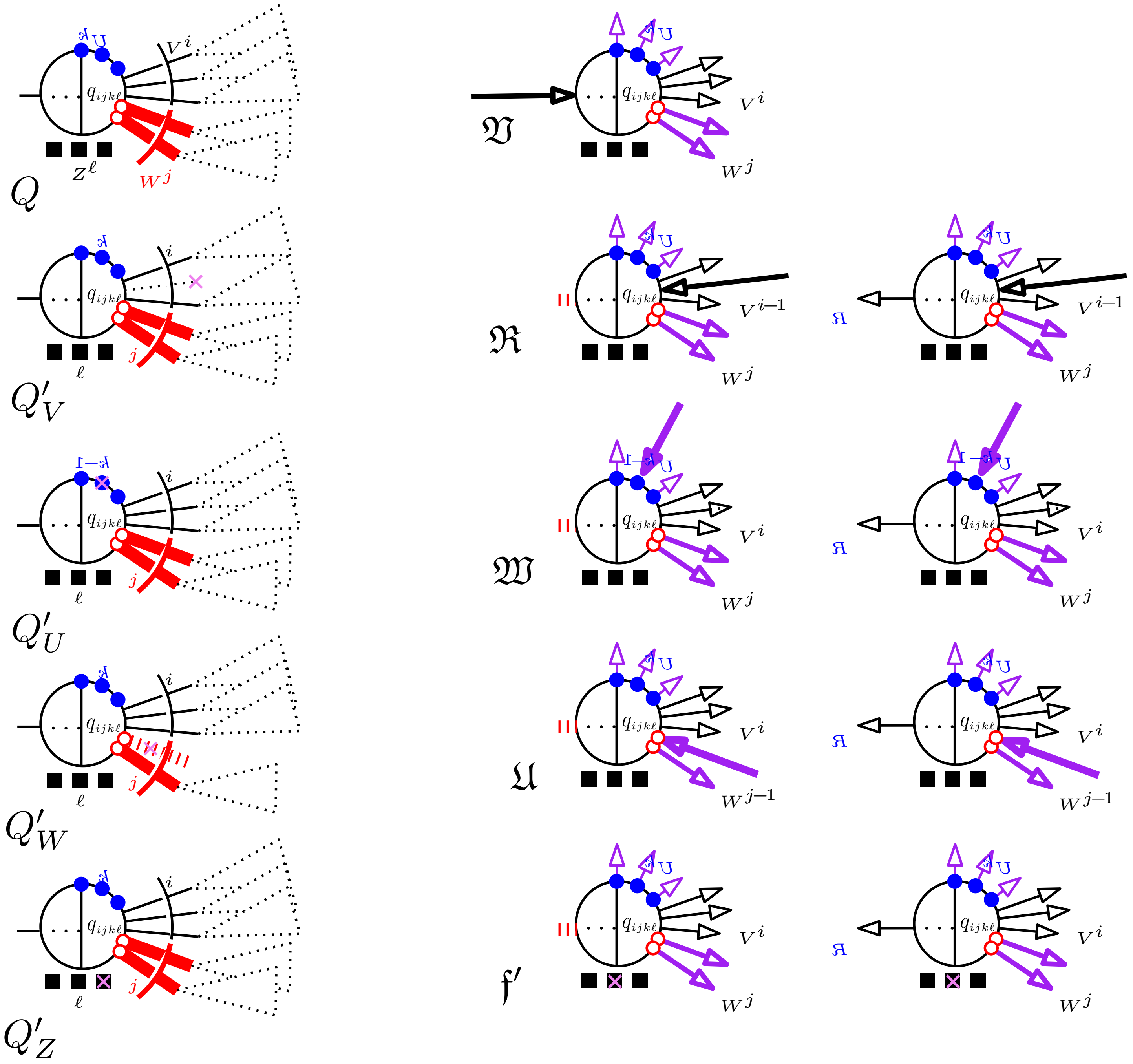
From catalytic to context-free specs, bijectively

$$F(u) = t\,Q\left(F(u), \frac{1}{u}(F(u)-f), u\right)$$

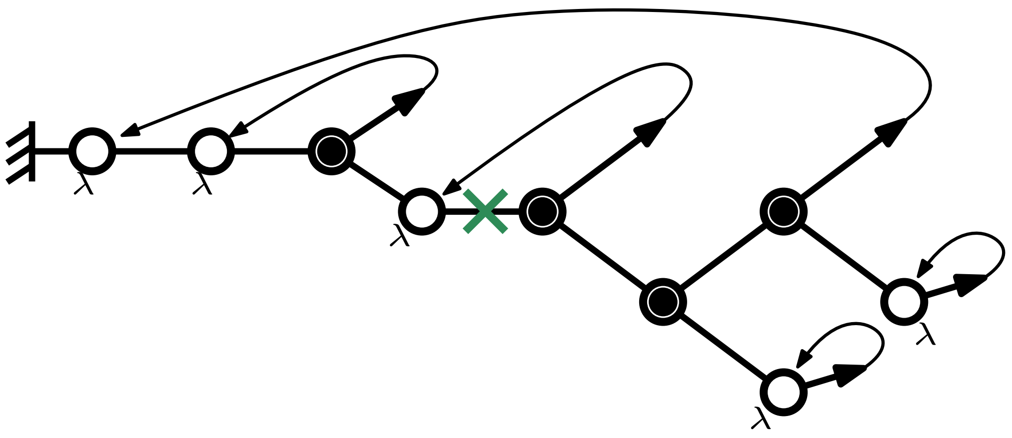
$$\Rightarrow \begin{cases} V &= t\cdot Q(V,W,U) \\ R &= t\cdot(1+R)\cdot Q'_v(V,W,U) \\ U &= t\cdot(1+R)\cdot Q'_w(V,W,U) \\ W &= t\cdot(1+R)\cdot Q'_u(V,W,U) \\ tf'_t &= (1+R)\cdot Q(V,W,U) \end{cases}$$

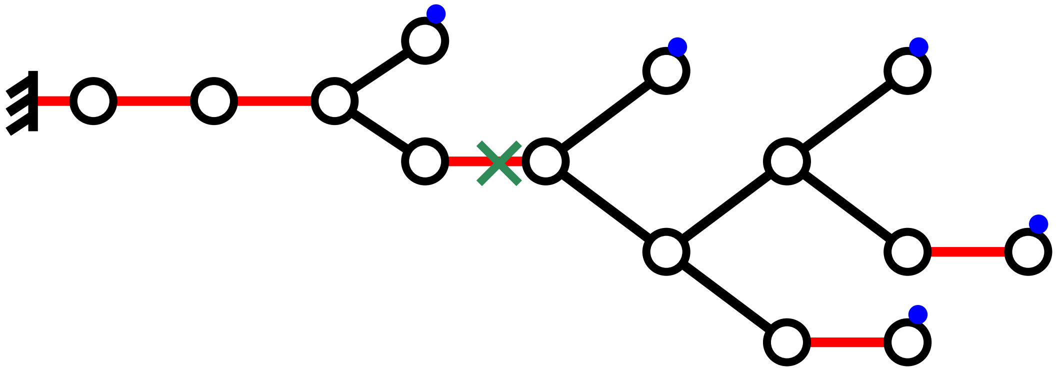
Theorem (Duchi-S. 23).
The node gf Q and its derivatives induce the node gfs of a family of multitype trees governed by the companion algebraic system.

Moreover these multitype trees are exactly the images of marked DŁ-Q-trees by closure and rewiring!

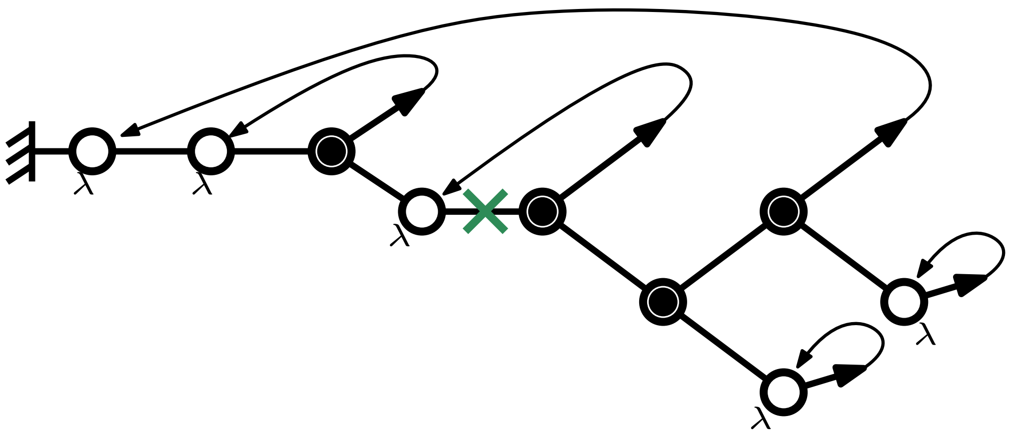


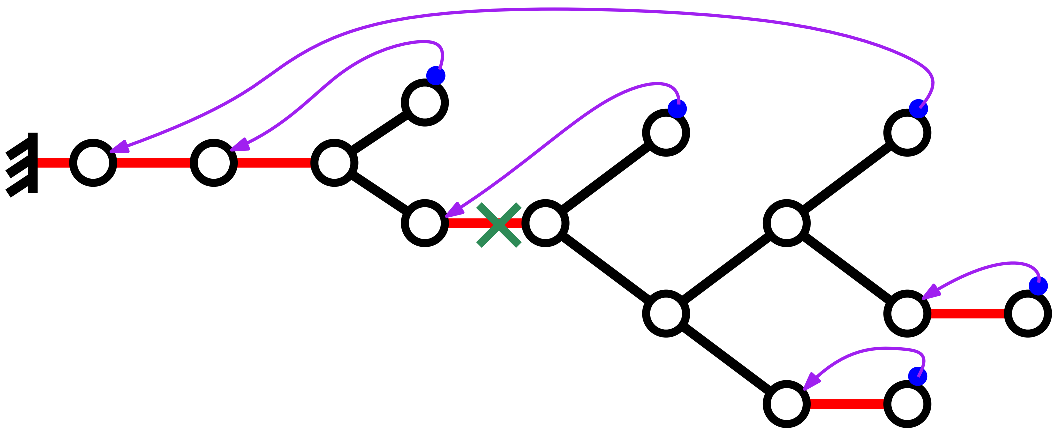
Planar λ-terms, closure and rewiring

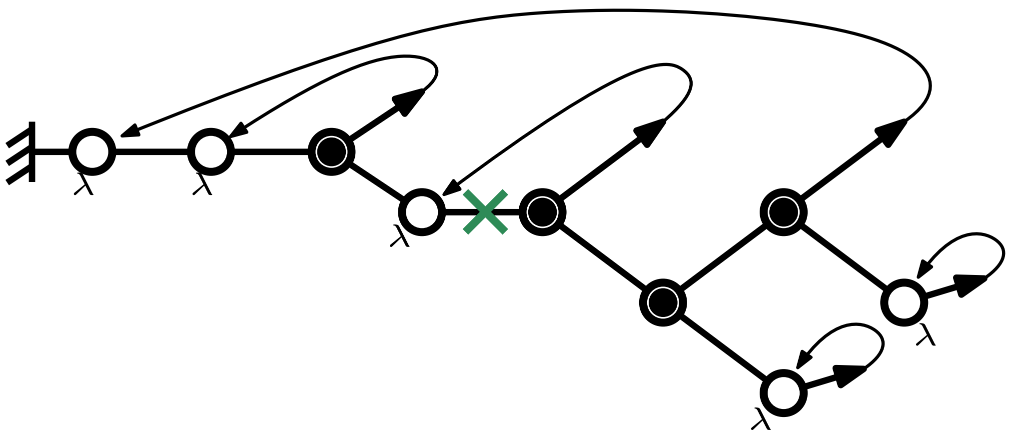


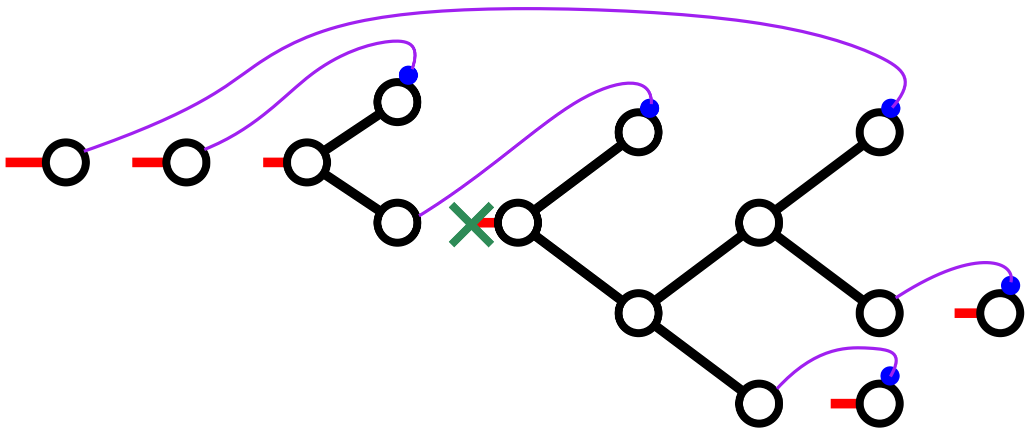


Planar λ-terms, closure and rewiring



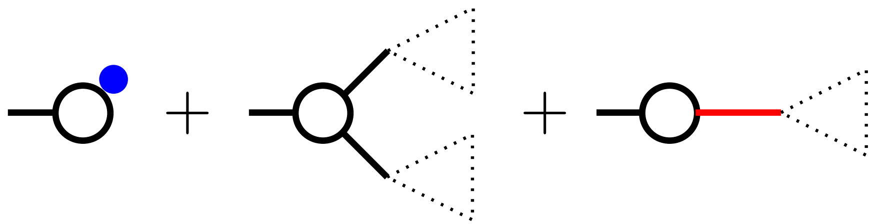






Planar λ-terms, companion context-free spec

$$P(u) = tu + tP(u)^2 + \frac{t}{u}(P(u) - p) = Q(P(u), \frac{1}{u}(P(u) - p), u, t)$$

$Q(v, w, u, t) =$ 

Planar λ-terms, companion context-free spec

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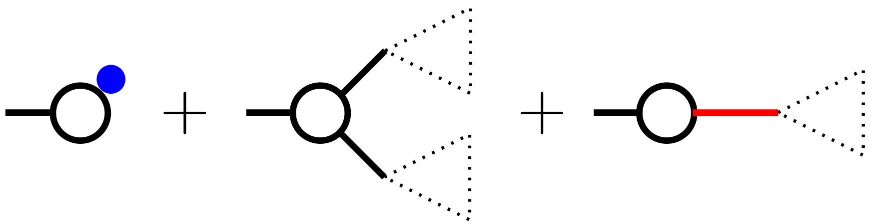
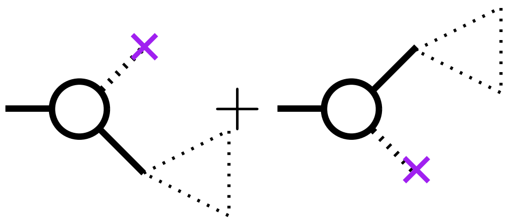


$Q'_v = \emptyset +$



$+ \emptyset$

Planar λ-terms, companion context-free spec

$$P(u) = tu + tP(u)^2 + \frac{t}{u}(P(u) - p) = Q(P(u), \frac{1}{u}(P(u) - p), u, t)$$

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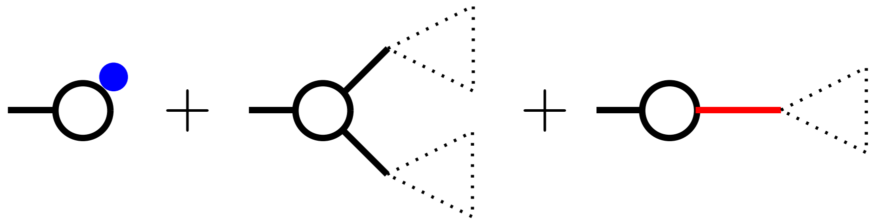
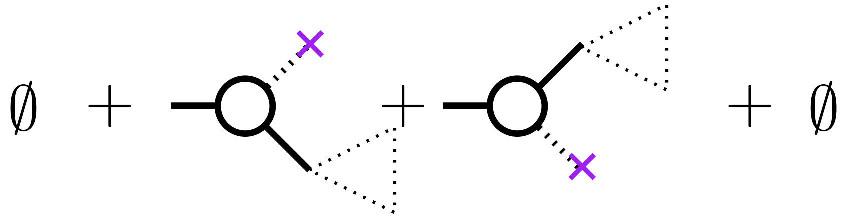


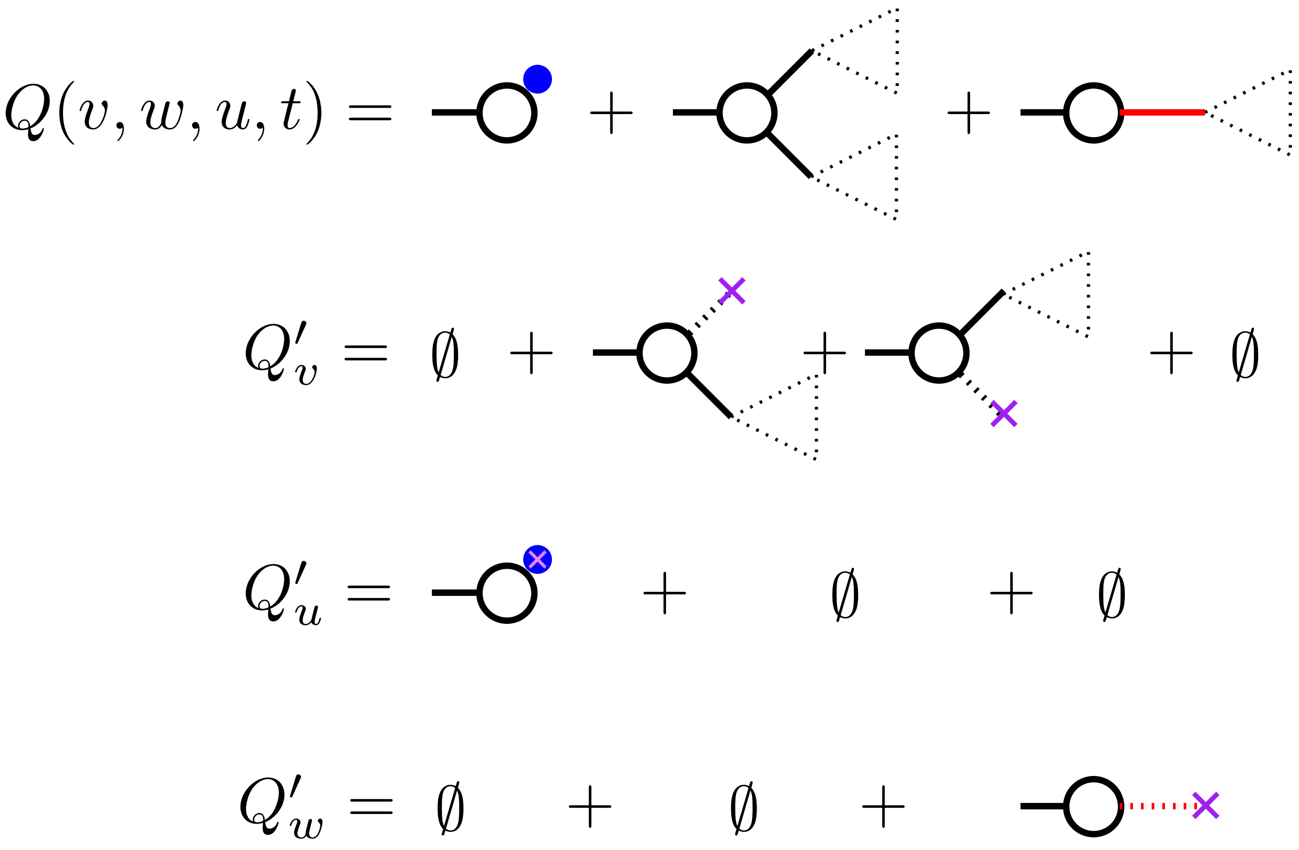
$Q'_v =$



$Q'_u =$

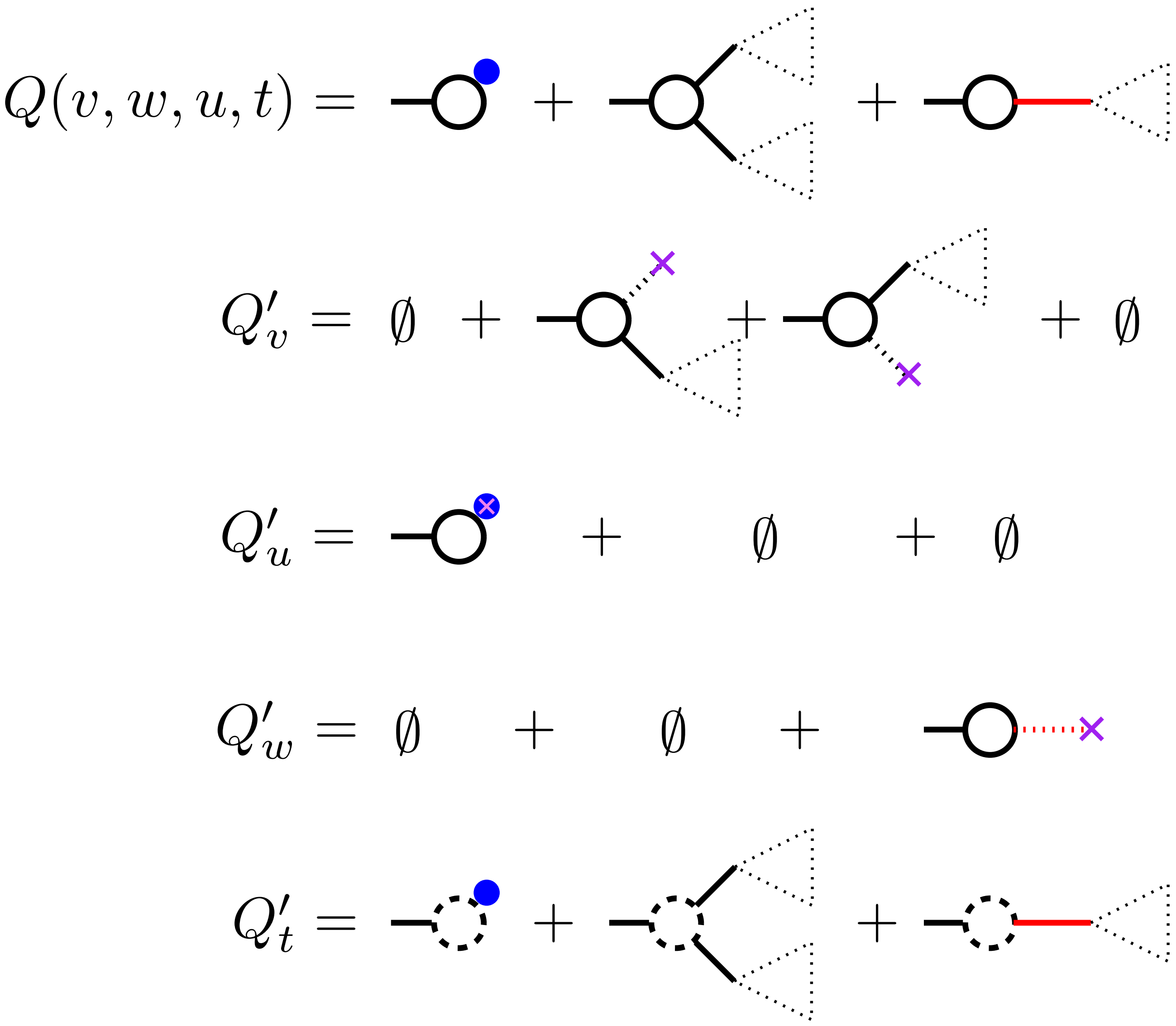
Planar λ-terms, companion context-free spec

$$P(u) = tu + tP(u)^2 + \frac{t}{u}(P(u) - p) = Q(P(u), \frac{1}{u}(P(u) - p), u, t)$$



Planar λ-terms, companion context-free spec

$$P(u) = tu + tP(u)^2 + \frac{t}{u}(P(u) - p) = Q(P(u), \tfrac{1}{u}(P(u) - p), u, t)$$



Planar λ-terms, companion context-free spec

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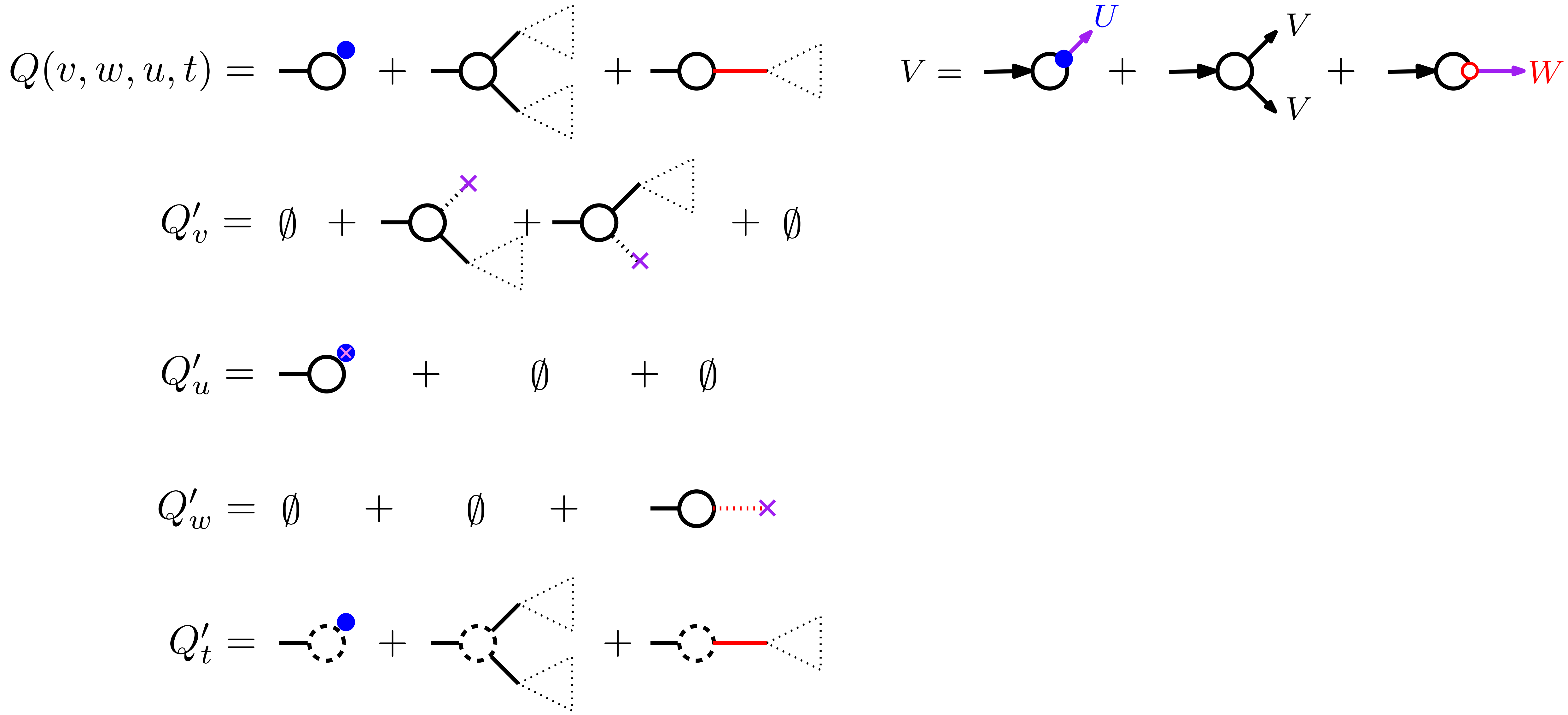
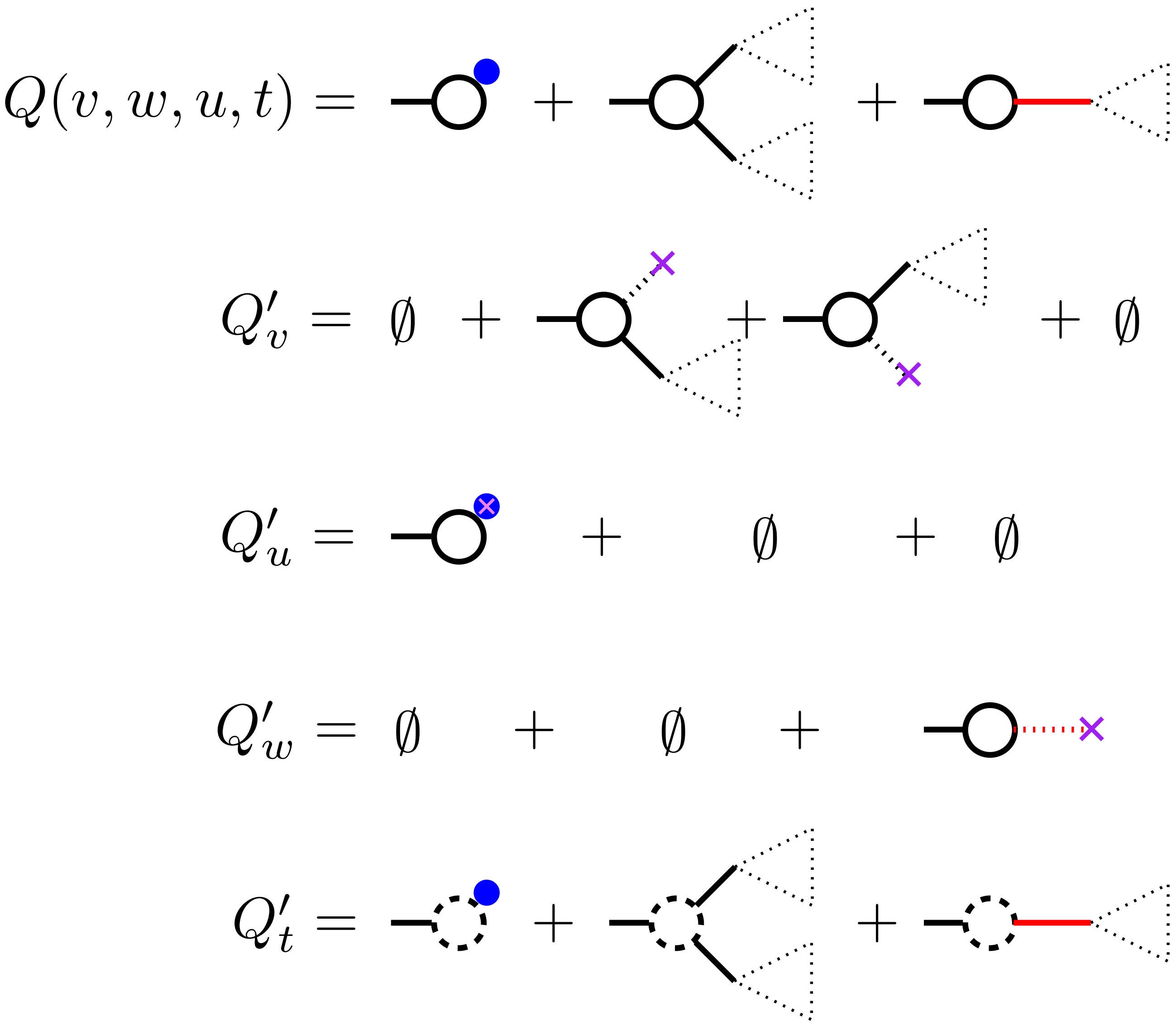
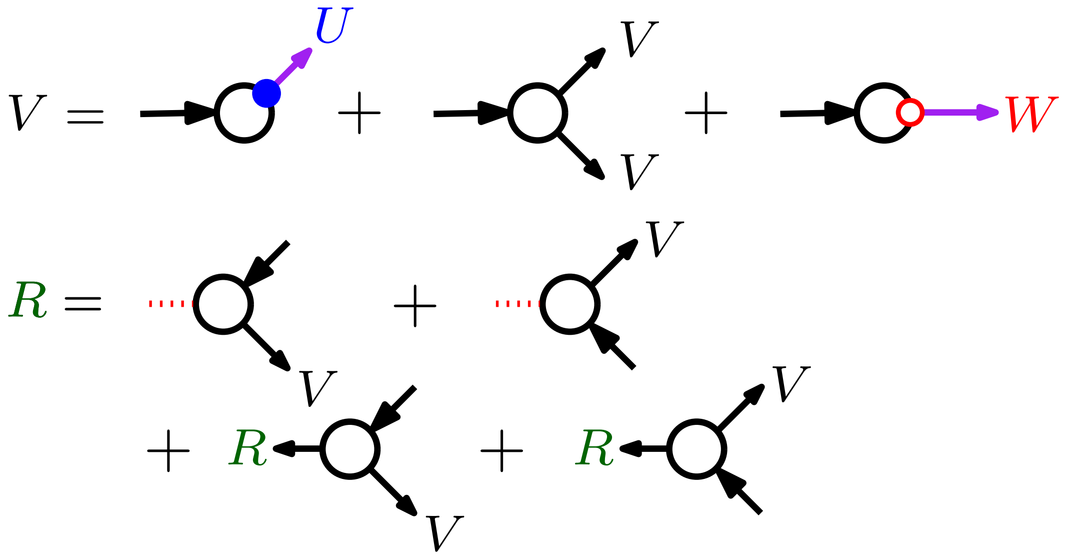


Planar λ-terms, companion context-free spec

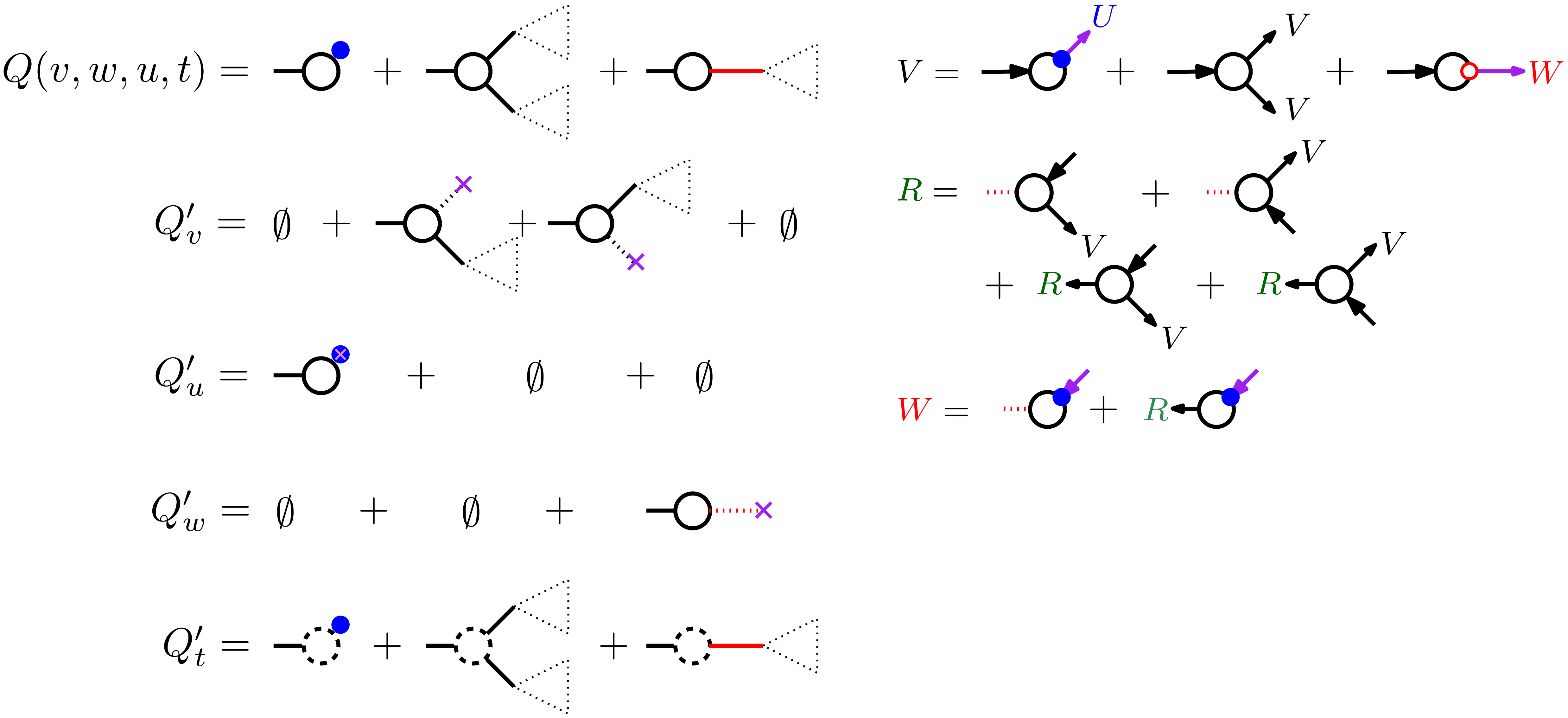
$$P(u) = tu + tP(u)^2 + \tfrac{t}{u}(P(u)-p) = Q(P(u), \tfrac{1}{u}(P(u)-p), u, t)$$





Planar λ-terms, companion context-free spec

$$P(u) = tu + tP(u)^2 + \frac{t}{u}(P(u) - p) = Q(P(u), \frac{1}{u}(P(u) - p), u, t)$$



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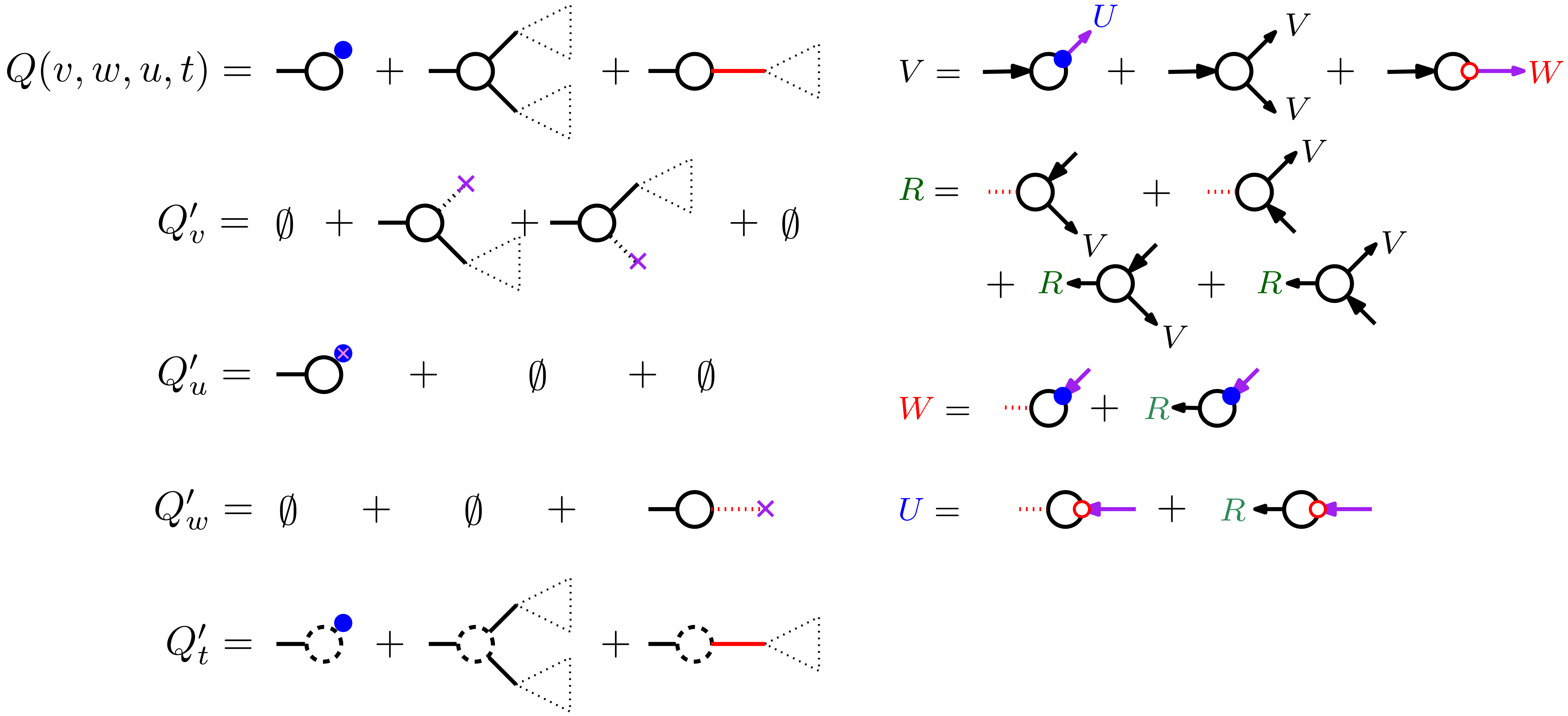

Planar λ-terms, companion context-free spec

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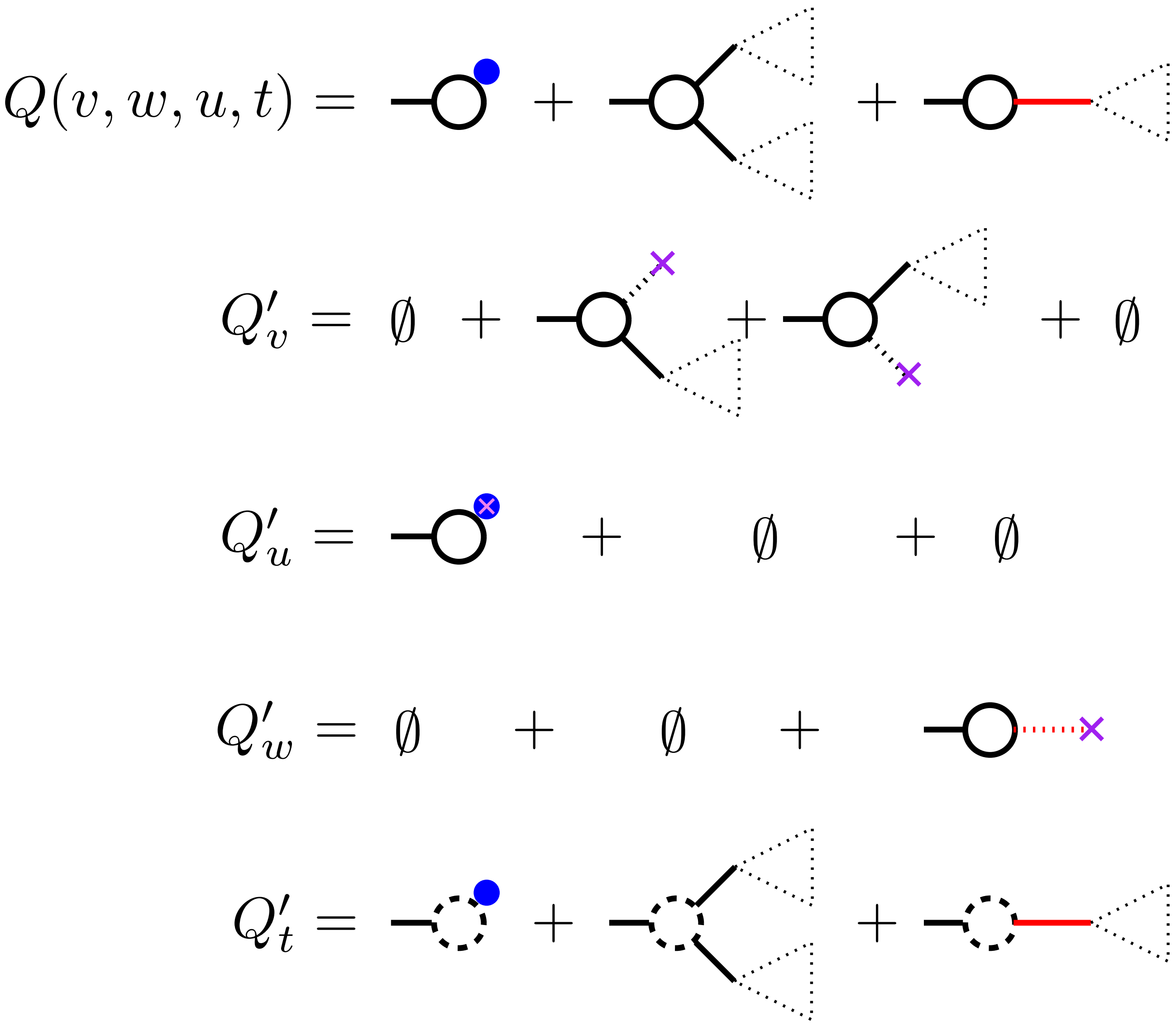
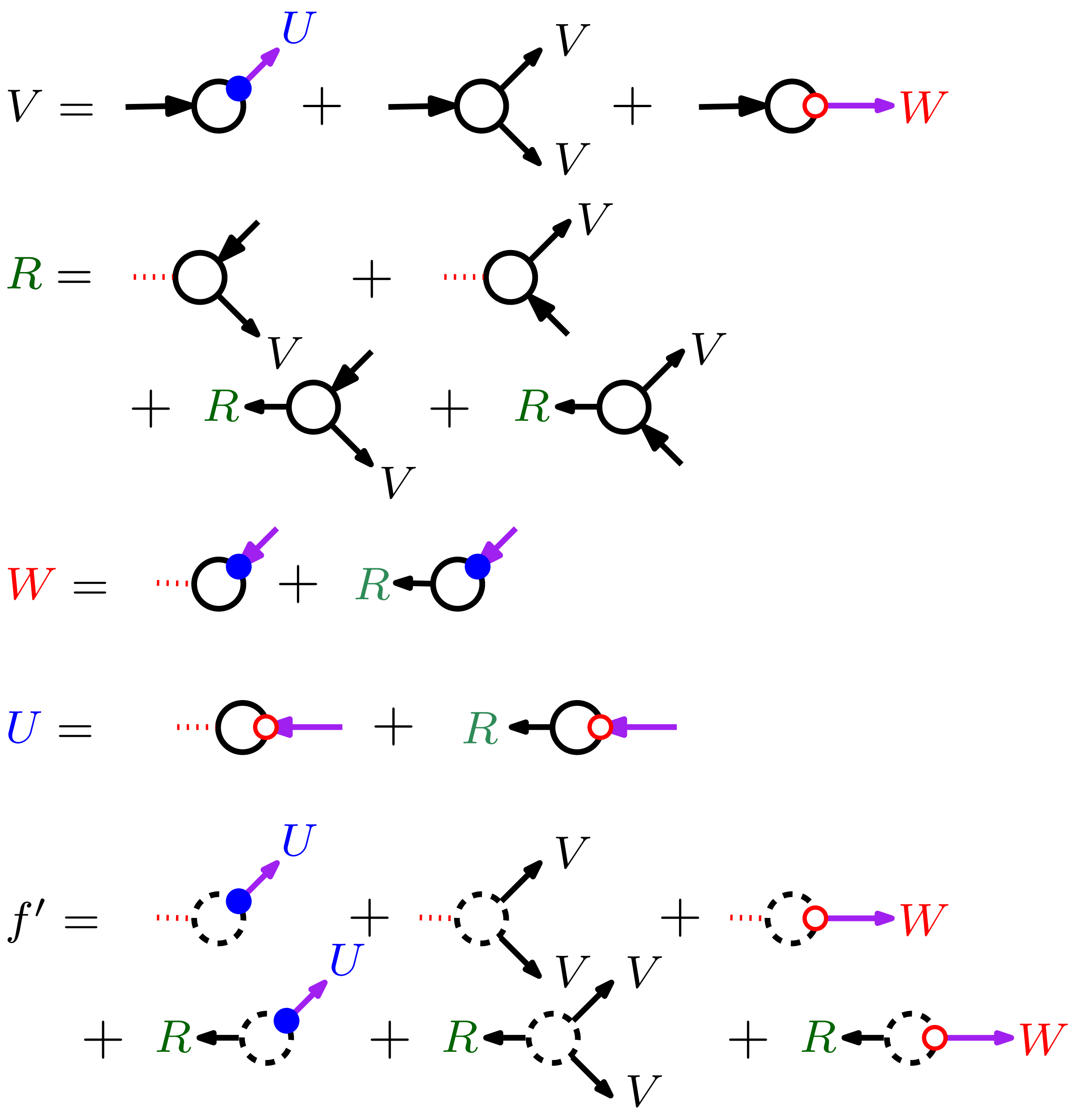
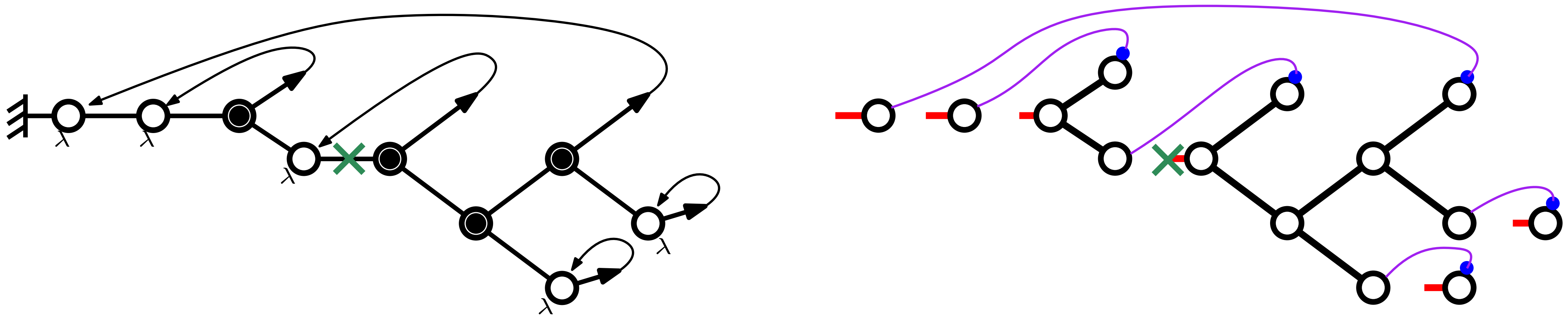
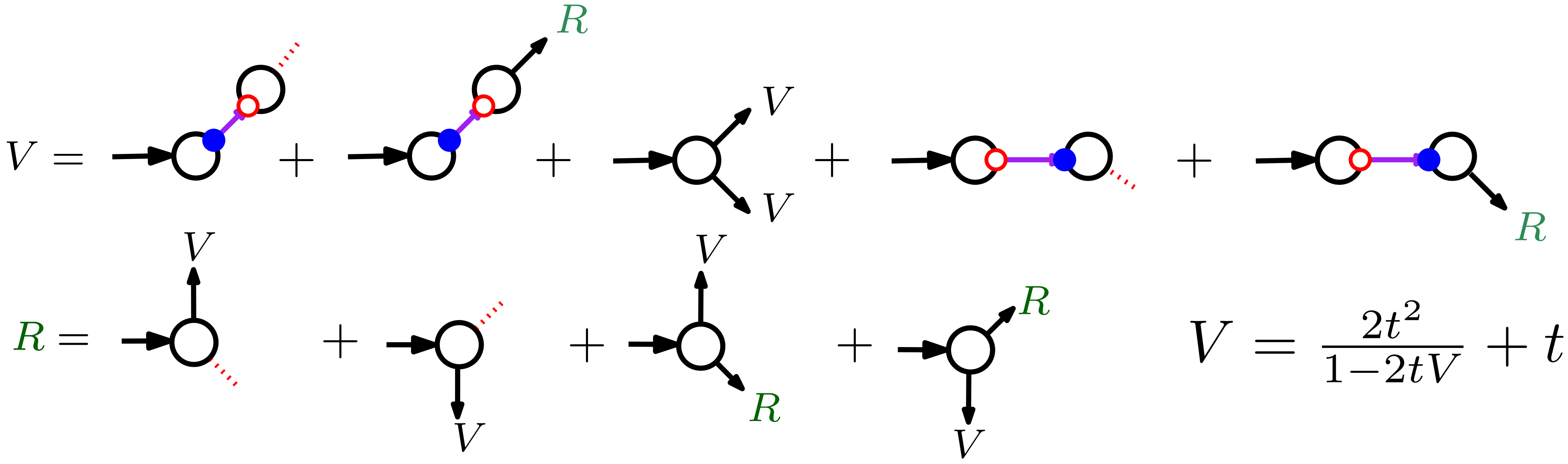




Planar λ-terms, closure and rewiring



Corollary.
Rewiring yields a size-preserving bijection between marked planar λ-terms and multitype trees with context-free spec:



$$V = \frac{2t^2}{1-2tV} + tV^2$$

Conclusion

The method is systematic:

catalytic equation of order one
$\Rightarrow$ bijection via ccw closure and rewiring
$\Rightarrow$ multitype trees with companion context free spec

Proofs based on context free decomposition of marked DŁ-Q-trees (Duchi-S. 22)

However it often does not gives directly the simplest context free spec.

Moreover in general the bijection starts from the *derivation trees* of the catalytic decomposition...

Thank you!

The general case: further useful observations!

$$F(u) = t\,Q\left(F(u), \frac{b}{u}(F(u)-f), a\,u\right)$$

$$\frac{\partial}{\partial u}: \quad F'_u(u) = F'_u(u)\cdot t\left(Q'_v(\ldots) + \frac{b}{u}Q'_w(\ldots)\right) - t\,\frac{b}{u}\,\frac{F(u)-f}{u}Q'_w(\ldots) + t\,aQ'_u(\ldots)$$

canceled by $u = U$

$$\frac{\partial}{\partial t}: \quad F'_t(u) = F'_t(u)\cdot t\left(Q'_v(\ldots) + \frac{b}{u}Q'_w(\ldots)\right) - t\,\frac{b}{u}f'_t\,Q'_w(\ldots) + Q(\ldots)$$

$$\frac{\partial}{\partial b}: \quad F'_b(u) = F'_b(u)\cdot t\left(Q'_v(\ldots) + \frac{b}{u}Q'_w(\ldots)\right) + t\left(\frac{F(u)-f}{u} - \frac{b}{u}f'_b\right)Q'_w(\ldots)$$

The general case: further useful observations!

$$F(u) = t\,Q\left(F(u), \frac{b}{u}(F(u)-f), a\,u\right)$$

$$\frac{\partial}{\partial u}:\quad \underbrace{F'_u(u) = F'_u(u)\cdot t\left(Q'_v(\dots) + \tfrac{b}{u} Q'_w(\dots)\right)}_{\text{canceled by } u=U} - t\,\tfrac{b}{u}\,\tfrac{F(u)-f}{u} Q'_w(\dots) + t\,a Q'_u(\dots)$$

$$\frac{\partial}{\partial t}:\quad F'_t(u) = F'_t(u)\cdot t\left(Q'_v(\dots) + \tfrac{b}{u} Q'_w(\dots)\right) - t\,\tfrac{b}{u} f'_t\, Q'_w(\dots) + Q(\dots)$$

$$\frac{\partial}{\partial b}:\quad \underbrace{F'_b(u) = F'_b(u)\cdot t\left(Q'_v(\dots) + \tfrac{b}{u} Q'_w(\dots)\right)}_{u=U \qquad 0} + t\left(\tfrac{F(u)-f}{u} - \tfrac{b}{u} f'_b\right) Q'_w(\dots)$$

$$\Rightarrow \boxed{WU = b f'_b = a f'_a \quad \text{and} \quad V = UW + f = b(bf)'_b}$$

The general case: further useful observations!

$$F(u) = t\,Q\left(F(u), \frac{b}{u}(F(u)-f), a\,u\right)$$

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$$\Rightarrow \boxed{WU = bf'_b = af'_a \quad \text{and} \quad V = UW + f = b(bf)'_b}$$

Systematic combinatorial interpretation of V as gf of DŁ-trees with a marked red edge!

Derivation of the generic $1/4$ exponent

Returning to the original system:

$$P(F(u), f, u, t) = 0$$

where $P(v, f, u, t) = -v + tQ(v, \frac{1}{u}(v - f), u)$

Upon derivating w.r.t. u:

$$P'_v(F(u), f, u, t)\frac{\partial}{\partial u}F(u) + P'_u(F(u), f, u, t) = 0$$

so that the system of equations defining $V \equiv V(t)$, $U \equiv U(t)$ and $f \equiv f(t)$ reads

$$(\mathcal{S}_t) \begin{cases} P(V, f, U, t) = 0 \\ P'_v(V, f, U, t) = 0 \\ P'_u(V, f, U, t) = 0 \end{cases}$$

In particular the dominant singularity ρ is the unique solution of $(\mathcal{S}_\rho)$ and

$$\det\begin{pmatrix} P_v(\alpha_V, \alpha_f, \alpha_U, \rho) & P_f(\alpha_V, \alpha_f, \alpha_U, \rho) & P_u(\alpha_V, \alpha_f, \alpha_U, \rho) \\ P''_{vv}(\alpha_V, \alpha_f, \alpha_U, \rho) & P''_{vf}(\alpha_V, \alpha_f \alpha_U, \rho) & P''_{vu}(\alpha_V, \alpha_f, \alpha_U, \rho) \\ P''_{vu}(\alpha_V, \alpha_f, \alpha_U, t) & P''_{uf}(\alpha_V, \alpha_f, \alpha_U, \rho) & P''_{uu}(\alpha_V, \alpha_f, \alpha_U, \rho) \end{pmatrix}$$

$$= -P'_f(\ldots) \cdot (P''_{vv}(\ldots)P''_{uu}(\ldots) - P''_{vu}(\ldots)^2) = 0$$

(Drmota, Noy, Yu 2020)

Derivation of the generic $1/4$ exponent

Restarting from $P'_v(F(u), f, u, t)\frac{\partial}{\partial u}F(u) + P'_u(F(u), f, u, t) = 0$

and $\left\{\begin{array}{l} P(V(t), f(t), U(t), t) = 0 \\ P'_v(V(t), f(t), U(t), t) = 0 \\ P'_u(V(t), f(t), U(t), t) = 0 \end{array}\right.$ with dominant ρ s.t. $P''_{uu} \cdot P''_{vv} - {P''_{vu}}^2 = 0$

upon derivating again

$$P'_v(F(u), f, u, t)\frac{\partial^2}{\partial u^2}F(u)$$
$$+P''_{vv}(\ldots)\left(\frac{\partial}{\partial u}F(u)\right)^2 + 2P''_{vu}(\ldots)\frac{\partial}{\partial u}F(u) + P''_{uu}(\ldots) = 0$$

So that V_λ satisfies the quadratic equation

$$P''_{vv}(V, f, U, t) \cdot V_\lambda^2 + 2P''_{vu}(V, f, U, t) \cdot V_\lambda + P''_{uu}(V, f, U, t) = 0$$

with reduced discrimimant

$$\Delta \equiv \Delta(t) = P''_{vu}(V, f, U, t)^2 - P''_{vv}(V, f, U, t)P''_{uu}(V, f, U, t)$$

which cancels at $t = \rho$: $\Delta(t) = \beta_\Delta(1 - t/\rho)^{1/2} + O(1 - t/\rho)$

Derivation of the generic $1/4$ exponent

V_λ satisfies the quadratic equation

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with reduced discrimimant

$$\Delta = P''_{vu}(V,f,U,t)^2 - P''_{vv}(V,f,U,t)P''_{uu}(V,f,U,t)$$

which cancels at $t=\rho$: $\Delta(t) = \beta_\Delta(1-t/\rho)^{1/2} + O(1-t/\rho)$

Hence:

$$V_\lambda(t) = \frac{-P''_{vu} - \sqrt{\Delta(t)}}{P''_{vv}} = \alpha_\lambda - \sqrt{\beta_\Delta}(1-t/\rho)^{1/4} + O(\sqrt{1-t/\rho})$$

and by transfert theorem: $[t^n]V_\lambda(t) \sim \frac{\sqrt{\beta_\Delta}}{4\Gamma(\frac{3}{4})}\cdot\frac{\rho^{-n}}{n^{5/4}}$

so that $\mathbb{E}(\lambda_n) = \frac{[t^n]V_\lambda(t)}{[t^n]V(t)} \sim (\frac{\sqrt{\beta_\Delta}}{4\Gamma(\frac{3}{4})}\cdot\frac{\rho^{-n}}{n^{5/4}})/(\frac{\beta_V}{2\sqrt{\pi}}\cdot\frac{\rho^{-n}}{n^{3/2}}) \sim \frac{\sqrt{\beta_\Delta}}{\beta_V}\frac{\sqrt{\pi}}{2\Gamma(\frac{3}{4})}\cdot n^{1/4}$