Binomial lattice congruences and flat dihomotopy types

LambdaComb Days
Paris

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LABORATOIRE D'INFORMATIQUE & SYSTÈMES

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Introduction

Resumé

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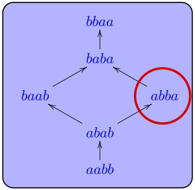
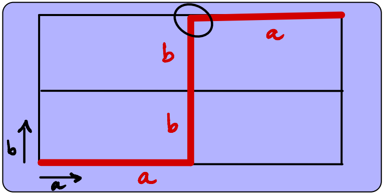


L(2,2)



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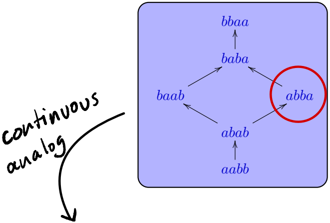

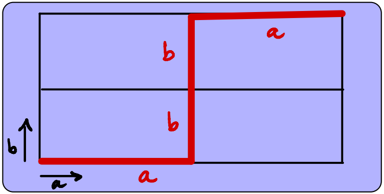

$$f : I \to I$$
$$f(\bigvee S) = \bigvee f(S)$$
form a quantale

→ Relation to dihomotopy

→ Priestley duality

→ Frame duality

Overview and context : the discrete case

- **Multinomial lattices** were introduced by Bennett & Birkhoff.
  - Study of the rewriting system associated to **commutativity** from a lattice-theoretic perspective:

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- These lattices and their congruences are strongly related to **concurrency**.
  - The word *abbaa* represents **interleaving actions** of two agents.
  - Multinomial lattice congruences give rise to certain **Parikh equivalences** central to scheduling problems in concurrency.
  - A **geometric interpretation** closely relates these lattices to a semantics of concurrent systems, namely **directed topology**.

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  - Describe the **geometric** intuition of their congruences.

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- **Directed algebraic topology**.
  - Recall the notion of directed space, and define **cubical complexes**.
  - Introduce the **binomial complex** and describe the dihomotopy types of its subcomplexes.

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  - Congruences correspond to dihomotopy types of subcomplexes.
  - The congruence lattice of a binomial lattice is isomorphic to the lattice of subcomplex dihomotopy types.

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- We will end by briefly describing ongoing work in the **continuous** setting.

Binomial lattices and their congruences

Multinomial lattices

- Given $v \in \mathbb{N}^k$, we denote by $\mathcal{L}(v)$ the set of **words** on the alphabet $\Sigma = \{a_1, \ldots, a_k\}$ such that:
  - w contains v_i occurrences of the letter a_i.

  We equip this set with the **partial order** generated by

$$w \leq w' \quad \iff \quad \exists u, v \begin{cases} w = u \cdot a_i a_j \cdot v \\ w' = u \cdot a_j a_i \cdot v \end{cases} \quad \text{and } i < j.$$

- The poset $(\mathcal{L}(v), \leq)$ has the structure of a **lattice**.

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- These structures generalize **permutations** to permutations of multisets, called **multipermutations**.
  - Indeed, for $v = (1, \ldots, 1)$, we have $\mathcal{L}(v) = S_k$.
  - The order $\leq$ generalizes the **weak Bruhat order** defining the **permutohedron**.

Binomial lattices

Today, we will focus on **binomial lattices**:

- Given $n, m \in \mathbb{N}$, we denote by $\mathcal{L}(n,m)$ the set of **words** on the alphabet $\Sigma = \{a, b\}$ such that:
  - w contains n occurrences of the letter a,
  - and m occurrences of the letter b.

  which we equip with the **partial order** generated by

$$w \leq w' \qquad \Longleftrightarrow \qquad \exists u, v \quad \begin{cases} w = u \cdot ab \cdot v, \\ w' = u \cdot ba \cdot v. \end{cases}$$

- We will henceforth denote $\mathcal{L}(n,m)$ simply by $\mathcal{L}$.

Proposition (L. Santocanale '05)

*$\mathcal{L}$ is a **distributive** lattice.*

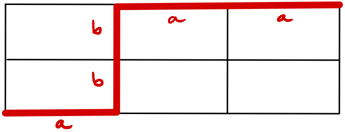
As lattices of lattice paths

- The elements of $\mathcal{L}$ are interpreted as **paths** in an n by m grid:

$$w \in \mathcal{L} \qquad \leftrightsquigarrow \qquad f_w : [n+m] \to [n] \times [m]$$

  - an occurrence of a is a step in the x-axis,
  - an occurrence of b is a step in the y-axis.

abbaa



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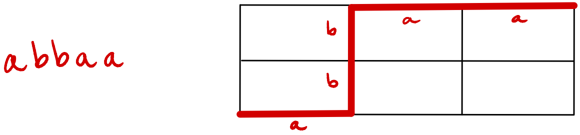
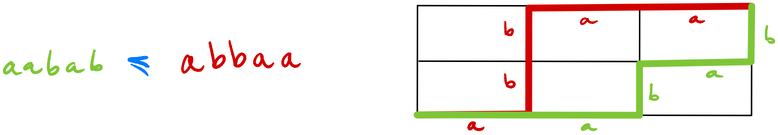


The ordering is recovered as a point-wise ordering on paths.

$$(x, y) \leq_2 (x', y') \qquad \text{iff} \qquad x' \leq x \text{ and } y \leq y',$$

Then $f \leq g$ if, and only if, $f(k) \leq_2 g(k)$ for all $k \in [n+m]$.



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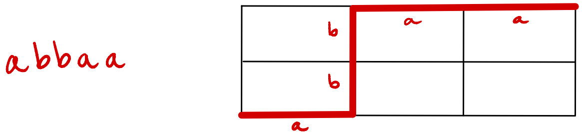

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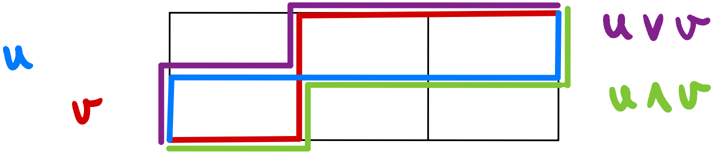
- The **join** and **meet** of $\mathcal{L}$ become point-wise maxima and minima:



- Note that these paths are **increasing** in each coordinate.

Distributive lattice congruences

Let L be a distributive lattice.

- A **congruence** on L is an equivalence relation $\theta \subseteq L \times L$ which is compatible with the lattice operations.
- In distributive lattices, congruences are given by **sets** of join-prime elements.
  - $j \in L$ is **join-prime** if

$$j = u \vee v \qquad \Rightarrow \qquad j = u \text{ or } j = v.$$

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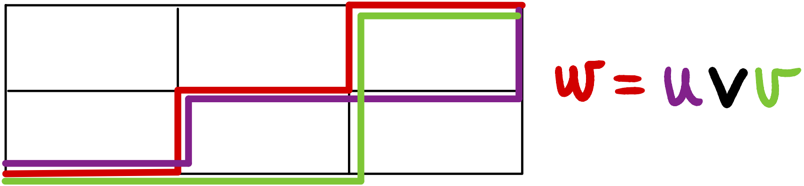
  - The set of join-prime elements of L is denoted by $\mathcal{J}$.
- Given $S \subseteq \mathcal{J}$, the congruence $\equiv_S$ is defined by:

$$u \equiv_S v \qquad \Longleftrightarrow \qquad \forall j \in S, \quad j \leq u \textit{ iff } j \leq v.$$

"u and v are above the same elements of S"

Join-prime elements of $\mathcal{L}(n, m)$

- What are the join-prime elements of $\mathcal{L}$?



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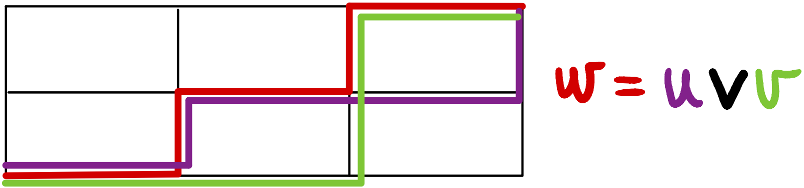
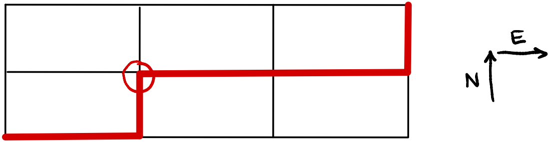


- They are the paths that have exactly one **north-east turn**:



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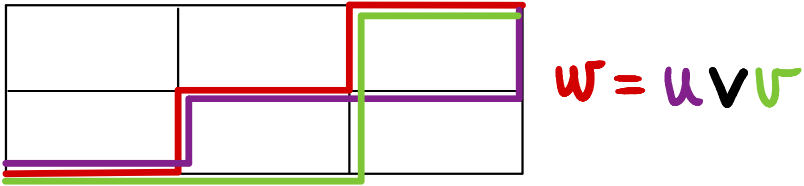

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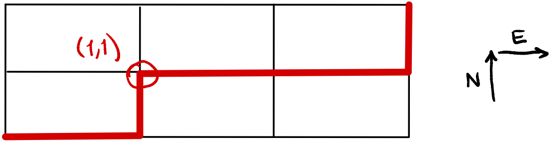


- As **words**, these are of the form

$$a^k b^l a^{n-k} b^{m-l}$$

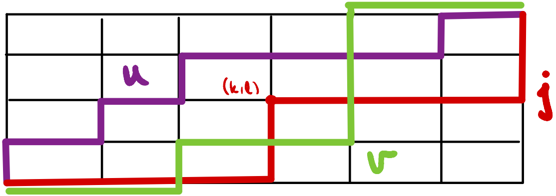
They are thus characterized by (k, l), with $\begin{cases} 0 \leq k < n \\ 0 < l \leq m \end{cases}$

Geometric interpretation of congruences

- Let us look at the particular case when $S = \{j\}$.
- Recall that

$$w \equiv_S w' \iff j \leq w \text{ iff } j \leq w'.$$

- Let (k, l) be the coordinate of the NE turn of j.
  - $j \leq u$ means f_u passes "above" (k, l),
  - $j \not\leq v$ means f_v passes "below" (k, l).



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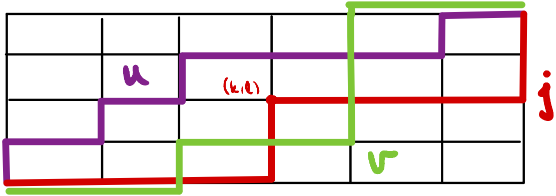

- The same holds for arbitrary $S \subseteq \mathcal{J}$.
- So, lattice congruences of $\mathcal{L}$ correspond to separating directed paths by squares. This reminds us of **directed homotopy**. . .

Directed homotopy and binomial complexes

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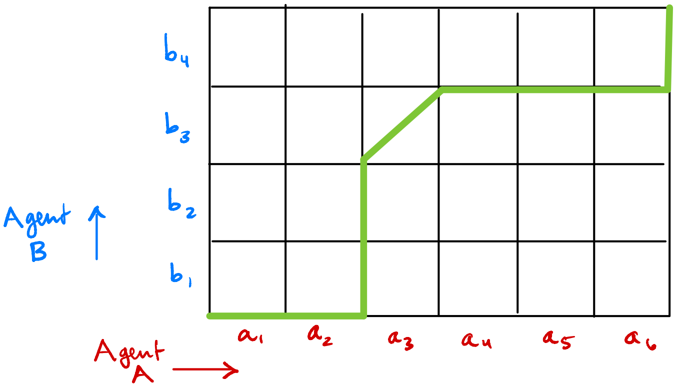
  - A topological space X,
  - A set of **directed paths** dX.
- We interpret **directed paths** as **executions**.



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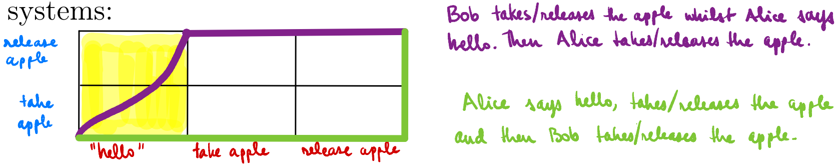
- We interpret **directed paths** as **executions**.
- Today, we focus on a particular class of directed spaces, namely **cubical complexes**. In two dimensions, these consist of:
  - **vertices**, which may be related by...
  - **edges**, which may form the border of...
  - **squares**.
- Such two-dimensional complexes model two-agent concurrent systems:



- Directed paths are those which increase in each coordinate.

Binomial complexes

- In particular, for $n, m \in \mathbb{N}$, we consider the **binomial complex** C:
  - $C_0 := \{v_{(i,j)} \mid 0 \leq i \leq n \text{ and } 0 \leq j \leq m\}$,
  - $C_1 := \{e_{(i_1,j_1),(i_2,j_2)} \mid i_2 = i_1 + 1 \text{ exor } j_2 = j_1 + 1\}$,
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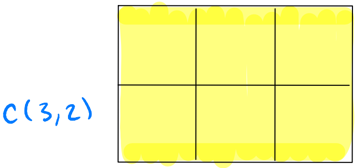
- This cubical complex corresponds to the n by m grid, with **all** "holes" filled by squares.



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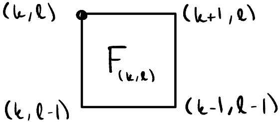

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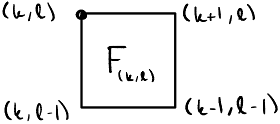
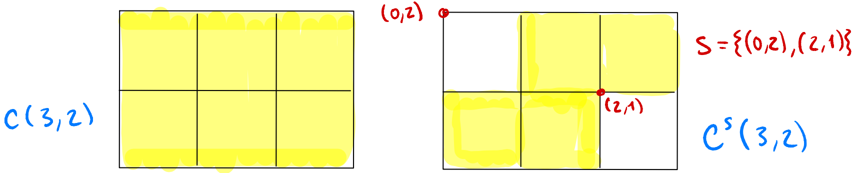
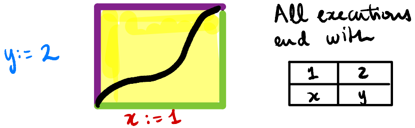


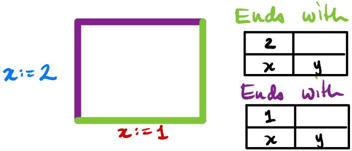
- Given $S \subseteq C_2$, we denote by C^S the cubical complex with the same set of vertices and edges, but in which $C_2^S := C_2 \setminus S$.



Cubical homotopy

- Given a concurrent system, which executions produce the same output?





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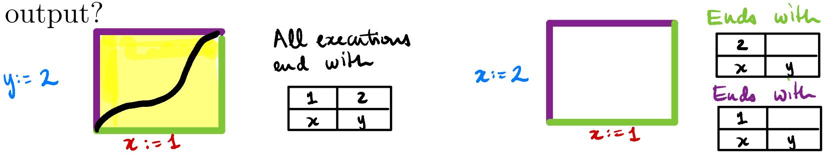
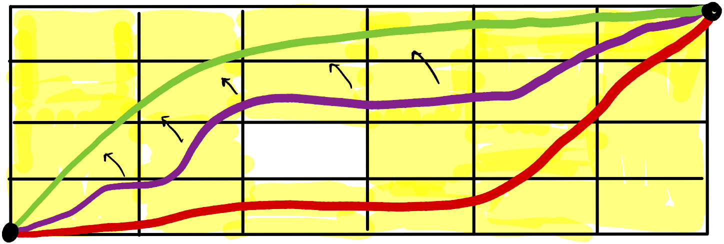


- We say that paths are **dihomotopic** if we can "slide" one onto the other through a sequence of directed paths, and if they start and end at the same point.



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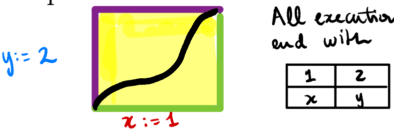

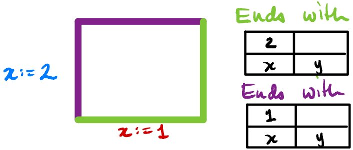

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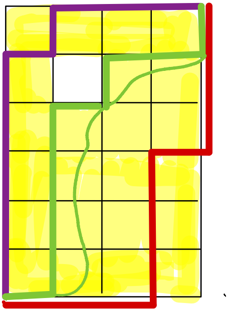

- In a cubical complex Γ, it suffices to consider
  - **combinatorial dipaths**, *i.e.* those which are contained in the set of edges Γ_1,
  - **combinatorial homotopy**, *i.e.* dipaths are equivalent when the space between them is filled by squares in Γ_2.

Dihomotopy quotients

- The combinatorial dihomotopy relation, denoted by $\overset{*}{\leftsquigarrow\!\!\rightsquigarrow}$, is an **equivalence relation** on combinatorial dipaths with the same beginning- and end-points.

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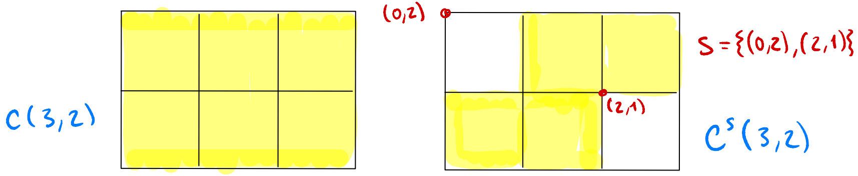
- Given the binomial complex C, we denote by $\overrightarrow{\mathbb{P}}(C)$ the set of combinatorial dipaths from $(0,0)$ to (n,m).
- Note that for any $S \subseteq C_2$, we have $\overrightarrow{\mathbb{P}}(C) = \overrightarrow{\mathbb{P}}(C^S)$.



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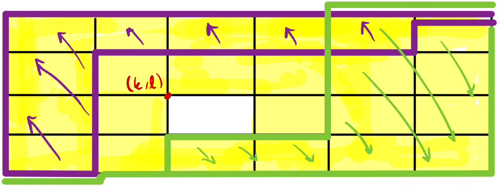
- We are interested in the **quotient** by combinatorial dihomotopy:

$$\overrightarrow{\mathbb{P}}(C^S) \,/\, \overset{*}{\leftrightsquigarrow} .$$

- In the particular case in which $S = \{F_{(k,l)}\}$...



Paths going above F are all identified.

Paths going under F are all identified.

The correspondence

Correspondences

- Elements of $\mathcal{L}$.
- Elements of $\overrightarrow{\mathbb{P}}(C^S)$.

Lattice paths

Correspondences

- Elements of $\mathcal{L}$.
- Elements of $\overrightarrow{\mathbb{P}}(C^S)$.

Lattice paths

- Join prime elements of $\mathcal{L}$.
- Squares in C.

$$\mathcal{J} \simeq \{(\mathbf{k},\mathbf{l}) \mid \mathbf{0} \leq \mathbf{k} < \mathbf{n} \textbf{ and } \mathbf{0} < \mathbf{l} \leq \mathbf{m} \} \simeq \mathbf{C_2}$$

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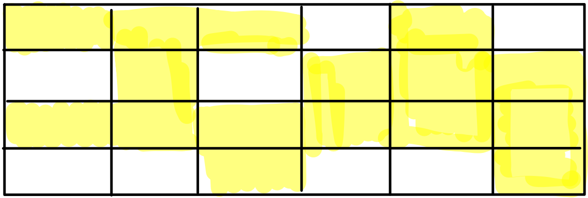
- Congruences $\equiv_S$ of $\mathcal{L}(n,m)$
- Subcomplexes $C^S(n,m)$.



Results

- Using the point-wise order induced on paths in C, we have that $\mathcal{L} \simeq \overrightarrow{\mathbb{P}}(C^S)$ as lattices.

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For any $S \subseteq \mathcal{J} \simeq C_2$, we have the lattice isomorphism

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- Moreover, the maps induced by inclusions $S' \subseteq S$ on each side correspond, *i.e.* the following maps coincide:

$$q_{S',S} : \overrightarrow{\mathbb{P}}(C^{S'}) \,/_{\overset{*}{\leftrightsquigarrow}} \longrightarrow \overrightarrow{\mathbb{P}}(C^S) \,/_{\overset{*}{\leftrightsquigarrow}}$$
$$p_{S',S} : \mathcal{L}(n,m) \,/_{S'} \longrightarrow \mathcal{L}(n,m) \,/_{S}.$$

Ongoing work

Multinomial lattice quotients

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- In this direction, we are studying the **higher dimensional automata** associated to the multinomial complexes.

The continuous case

- Let $Q_\vee(I)$ denote the set of order preserving maps

$$f : I \to I \qquad \text{s.t.} \qquad f(\bigvee X) = \bigvee f(X),$$

equipped with the point-wise ordering $\leq$.

Proposition (M.J. Gouveia, L. Santocanale '18)

- *The structure $(Q_\vee(I), \leq)$ is a completely distributive lattice.*
- *With composition $\circ$, the lattice $Q_\vee(I)$ is a $\star$-autonomous quantale which moreover satisfies the **mix rule**.*

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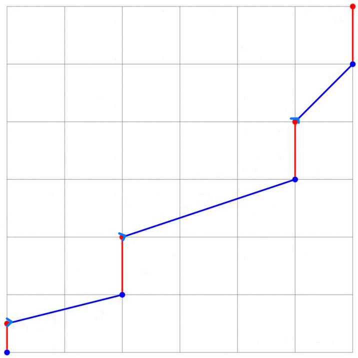
f is left-continuous

Segments in red are discontinuity points



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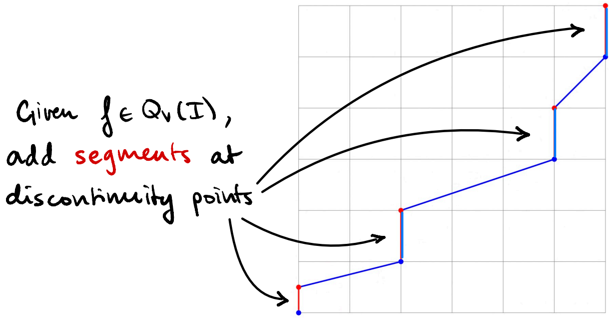

$f : I \to I$ $f(\bigvee X) = \bigvee f(X)$	$\simeq$	$C \subseteq I^2$ complete, dense, totally ordered	$\twoheadleftarrow$	$p : I \to I^2$ continuous, monotone paths

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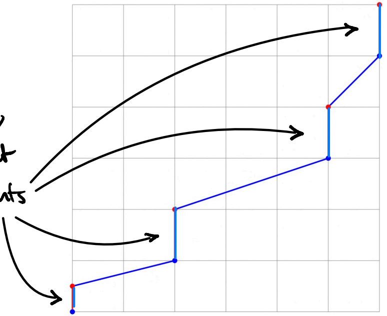
Given $f \in Q_\vee(I)$, add **segments** at discontinuity points



mod out by para-metrisation

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Dihomotopy and the continuous order

- In the discrete case, paths $f[n+m] \to [n] \times [m]$ are parametrised by **arc-length**.
- We can recover the ordering on $\mathcal{L}$ in two ways:
  - As the **point-wise order** inherited from

$$(x, y) \leq_2 (x', y') \qquad \text{iff} \qquad x' \leq x \text{ and } y \leq y',$$

  - or as that generated by the **elementary cubical homotopy** relation $\rightsquigarrow$:

$\gamma_1 \rightsquigarrow \gamma_2 \iff$ $\exists F \in C_2$

Dihomotopy and the continuous order

- In the continuous case, **parametrisation** is an obstruction to this characterisation.

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Proposition (C.C, L. Santocanale)

Let $f, g \in Q_\vee(I)$ such that $f \leq g$. There exist parametrisations π_f, π_g of f and g such that:

- $\pi_f(t) \leq_2 \pi_g(t)$ *for all* $t \in I$,
- *there exists an* ***increasing*** *homotopy* $\psi_{f,g} : \pi_f \Rightarrow \pi_g$.

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- A characterisation of **all** congruences of $Q_\vee(I)$ akin to that obtained for $\mathcal{L}(n, m)$ via dihomotopy types is not possible…

Dualities

- **Priestley duality** relates bounded, distributive lattices to topological spaces:
  - Given a lattice L, construct a space X whose points are **prime filters** of L.
  - There is a Galois connection

fixed points are lattice congruences

$$[\![-]\!] : \mathcal{P}(L^2) \rightleftharpoons \mathcal{P}(X) : \theta$$

fixed points are closed sets

  - We have **identified** the topology on $X_J \subset X$, the set of **principal** prime filters, as a directed-suprema closure topology on I^2.

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  - Which congruences are **complete**?

Thank you