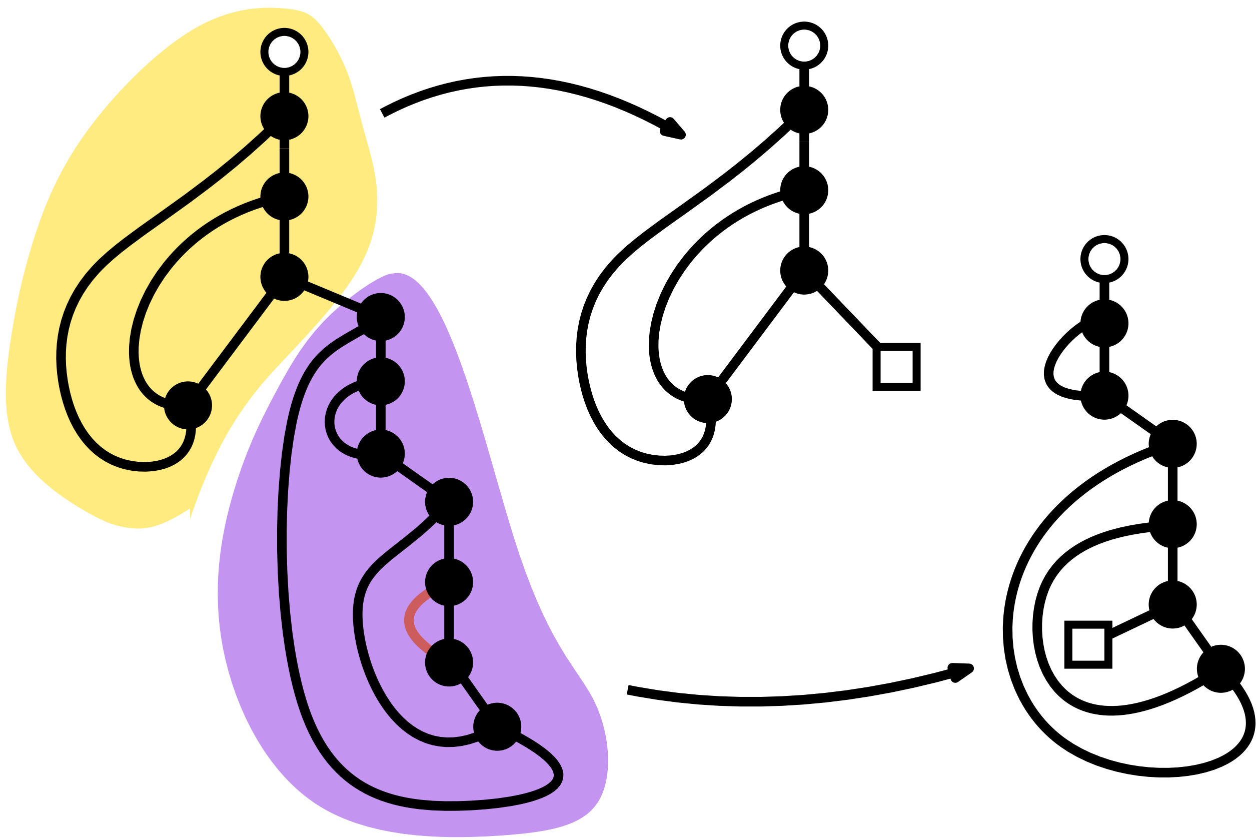
A novel interpretation of the planar Goulden-Jackson recurrence using the planar λ-calculus



Alexandros Singh (LIASD, Paris 8), Noam Zeilberger (PARTOUT, LIX)

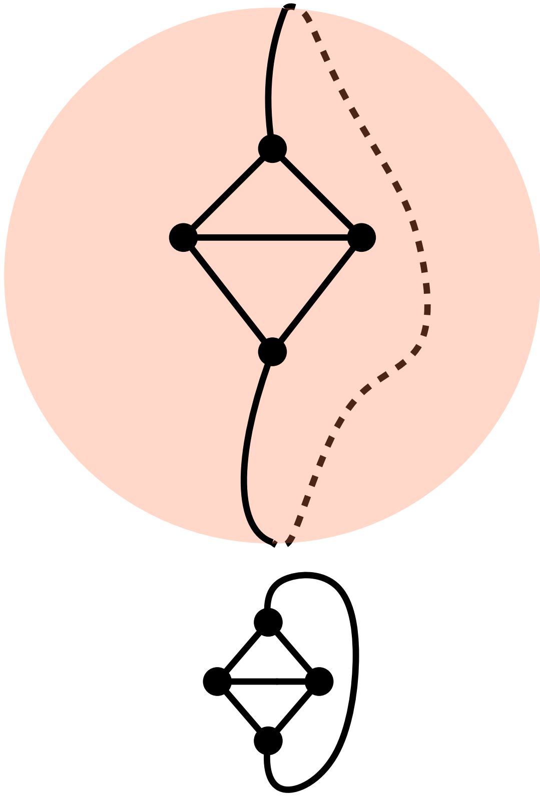
Wednesday, January 24th 2023

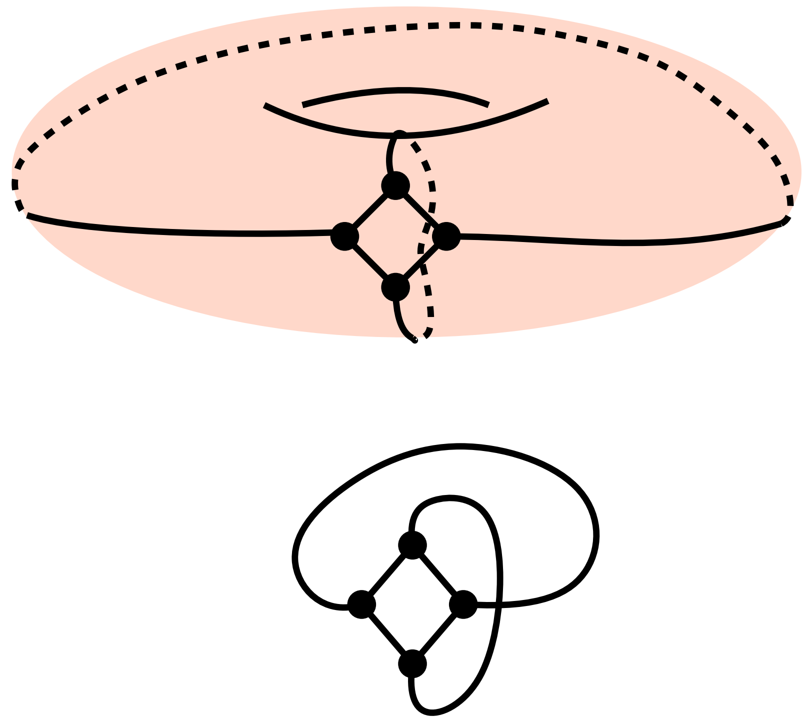
Journées LamdaComb 2024

The plan

- A brief overview of maps and the λ-calculus
- Context and related results
- The planar λ-calculus
- Goulden-Jackson recurrence for planar maps
- Closing remarks

What are maps?





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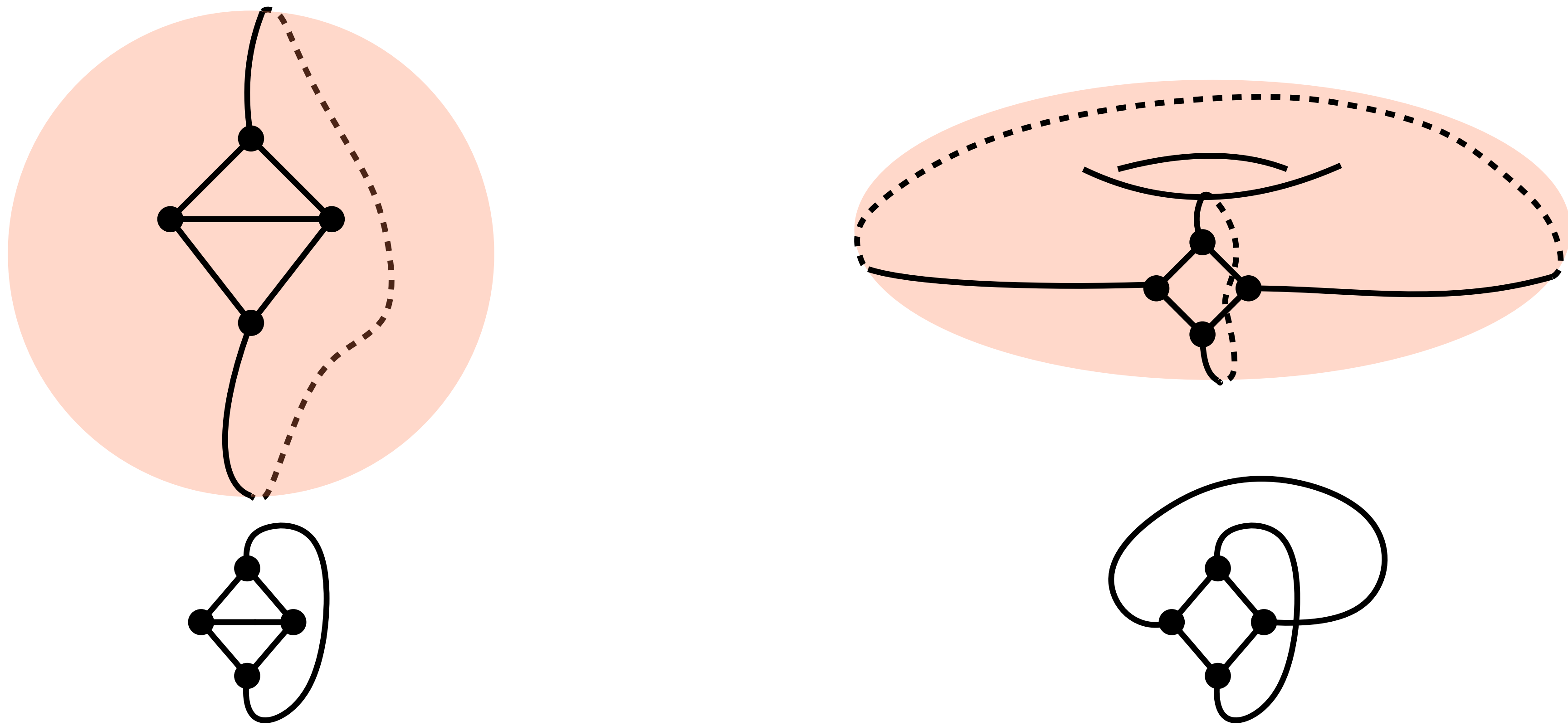

4CT...

- A central object in modern combinatorics, but not only that: probability, algebraic geometry, theoretical physics...

scaling limits... matrix integrals, Witten's conjecture, . . .

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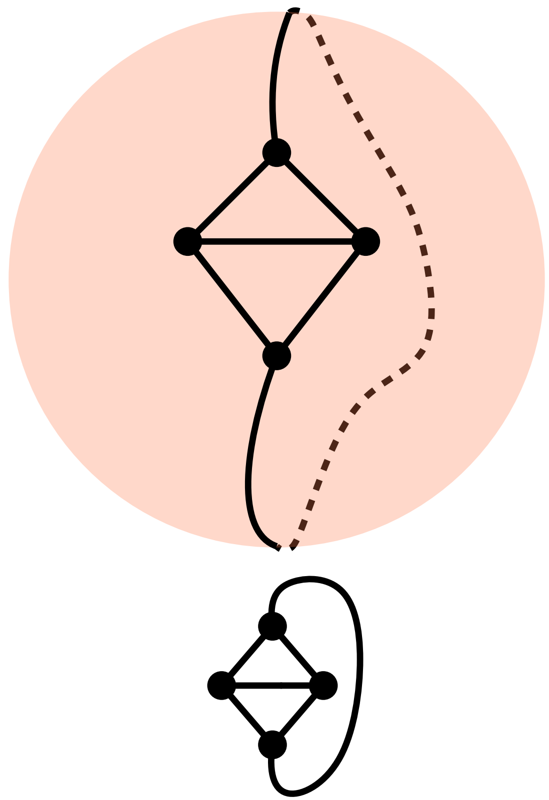

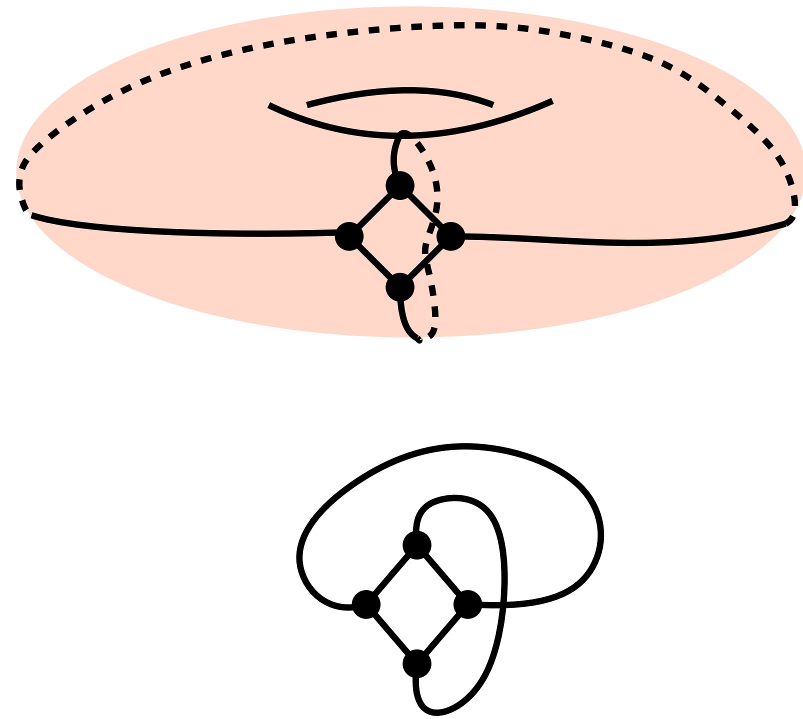

- A central object in modern combinatorics, but not only that: probability, algebraic geometry, theoretical physics...

- Their enumeration was pioneered by Tutte in the 60s, as part of his approach to the four colour theorem.

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- In its typed form: functional programming, proof theory,...

Linking terms and maps

- In 2013, Bodini, Gardy, Jacquot, describe a series of bijections:

  rooted trivalent maps $\leftrightarrow$ closed linear terms
  rooted (2,3)-valent maps $\leftrightarrow$ closed affine terms

 In the same year, together with Gittenberger, they study:

 $\mathrm{BCI}(\mathrm{p})$ terms (each bound variable appears p times)

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- In 2015, Zeilberger advocates for

"linear lambda terms as invariants of rooted trivalent maps"

Some results

●=w. Bodini, Zeilberger ● = ● + Gittenberger, Wallner

Parameters on maps and terms of arbitrary genus (number of):

- Loops in trivalent maps and identity-subterms in closed linear terms

  Limit law: $\mathrm{Poisson}(1)$

- Bridges in trivalent maps and closed subterms in closed linear terms

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- Vertices of degree 1 in (1,3)-valent maps and free variables in open linear terms

  Limit law: $\mathcal{N}((2n)^{1/3}, (2n)^{1/3})$

- Patterns in trivalent maps and redices in closed linear terms

  Asymptotic mean and variance: $\frac{n}{24}$

- Steps to reach normal form for closed linear terms

  Asymptotic mean bound below by: $\frac{11n}{240}$

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Similar results for planar maps/terms, plus: a new interpretation of a recurrence of Goulden and Jackson.

This talk!

The planar λ-calculus - formally

Inductive definition (keeping track of variables not “captured” by a λ):

$$\dfrac{}{x \vdash x}\ \text{var}$$

free

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free

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disjoint

$$\frac{}{x \vdash x}\ \text{var} \qquad \frac{\Gamma, x \vdash t}{\Gamma \vdash \lambda x.t}\ \text{abs} \qquad \frac{\Gamma \vdash f \quad \Delta \vdash t}{\Gamma, \Delta \vdash (f\ t)}\ \text{app}$$

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Decomposing planar trivalent maps

(with a boundary)

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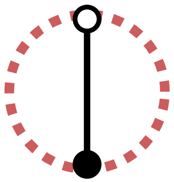

edges

non-root boundary vertices

$$P(z, u) = uz$$

Decomposing planar trivalent maps

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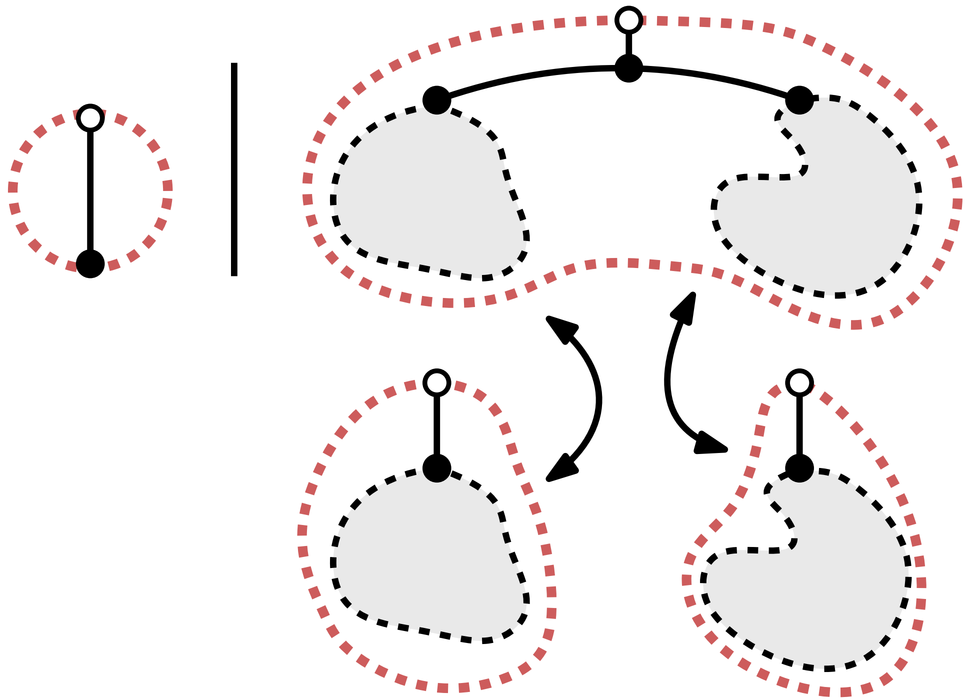

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$$P(z,u) = uz + zP(z,u)^2$$

Decomposing planar trivalent maps

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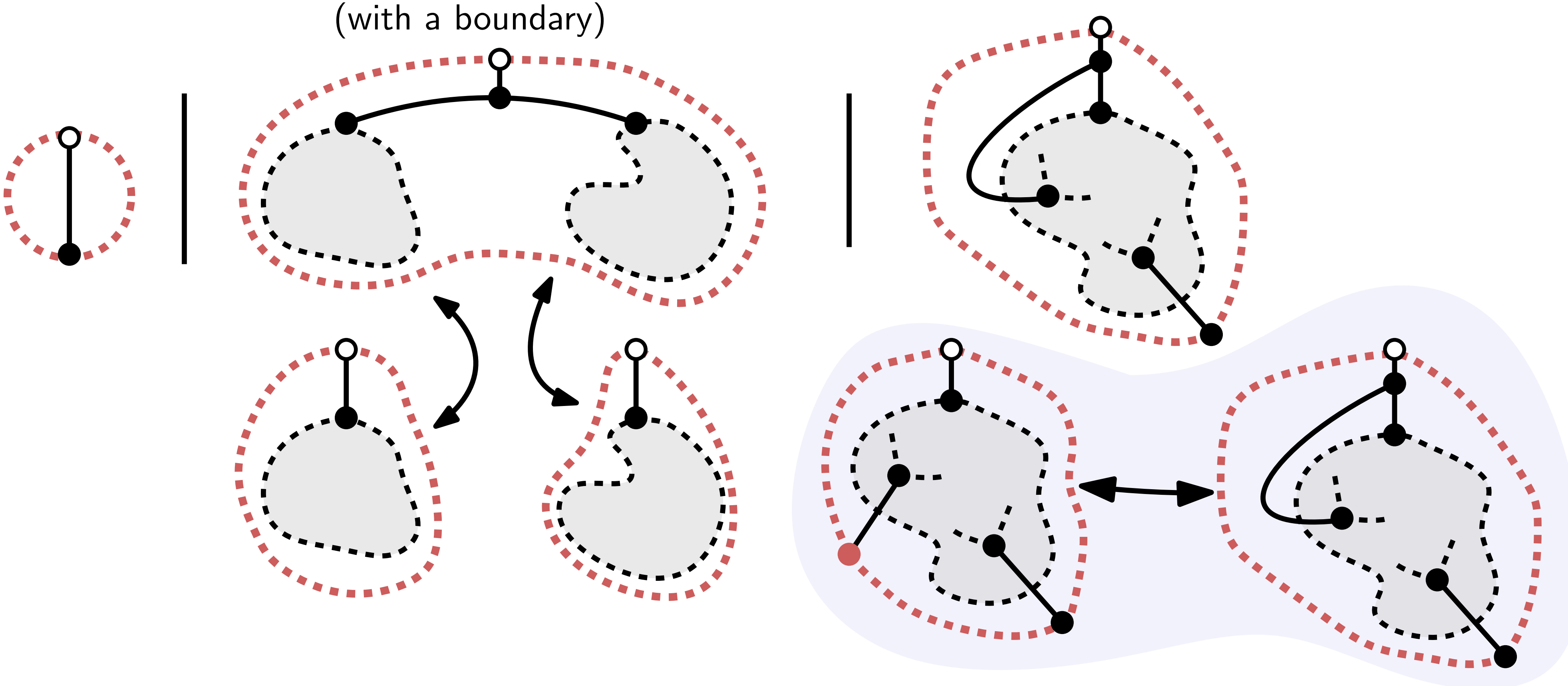

Boundary contains at least one non-root vertex
&
Consume first according to contour

edges

non-root boundary vertices

$$P(z,u) = uz + zP(z,u)^2 + z\frac{P(z,u) - P(z,0)}{u}$$

Decomposing planar trivalent maps

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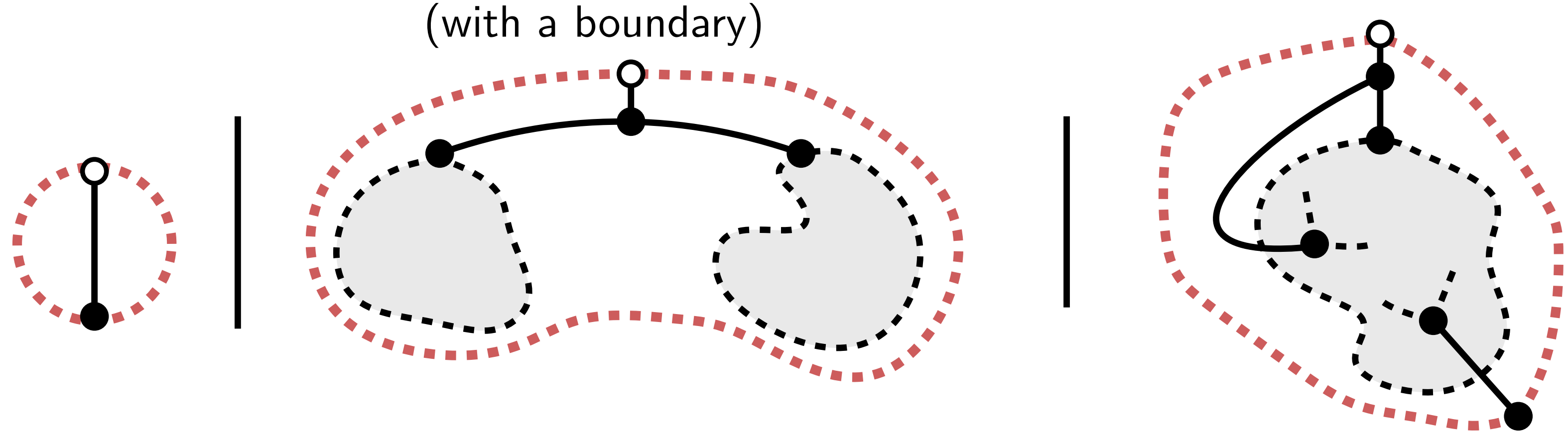


$$\frac{}{x \vdash x}\,\text{var} \quad \Big| \quad \frac{\Gamma \vdash f \qquad \Delta \vdash t}{\Gamma, \Delta \vdash (f\ t)}\,\text{app} \quad \Big| \quad \frac{\Gamma, x \vdash t}{\Gamma \vdash \lambda x.t}\,\text{abs}$$

At least 1 free var
&
Consume rightmost one

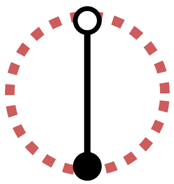
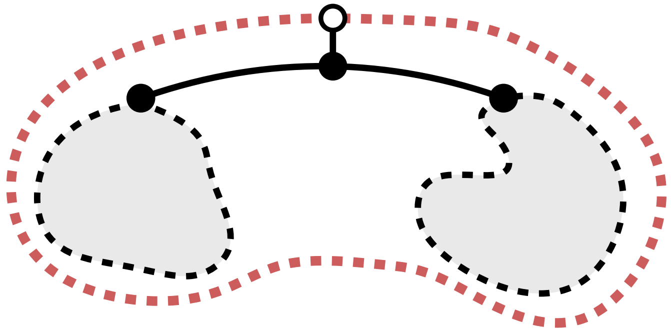
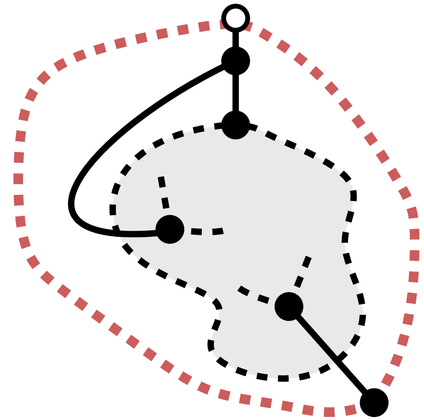
edges

non-root boundary vertices

$$P(z,u) = uz + zP(z,u)^2 + z\frac{P(z,u) - P(z,0)}{u}$$

Decomposing planar trivalent maps and open planar terms!

(with a boundary)

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$$P(z,u) = uz + zP(z,u)^2 + z\frac{P(z,u)-P(z,0)}{u}$$

free vars.

subterms

Decomposing planar trivalent maps and open planar terms!

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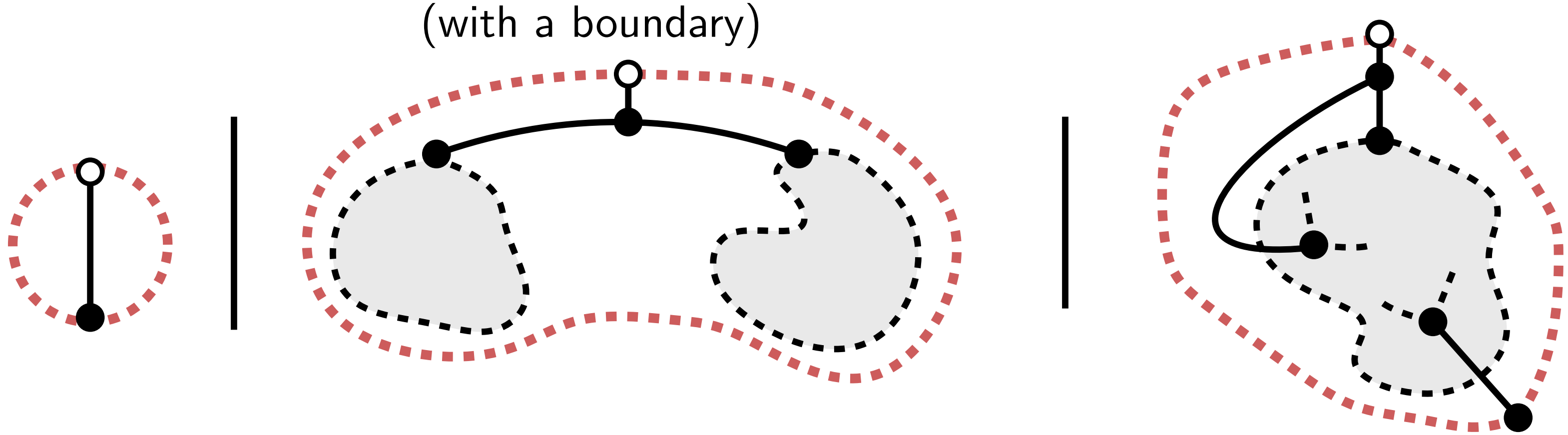

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For arbitrary genus replace $z\frac{F(z,u)-F(z,0)}{u}$ by $z\partial_u F(z,u)$!

edges

non-root boundary vertices

$$P(z,u) = uz + zP(z,u)^2 + z\frac{P(z,u)-P(z,0)}{u}$$

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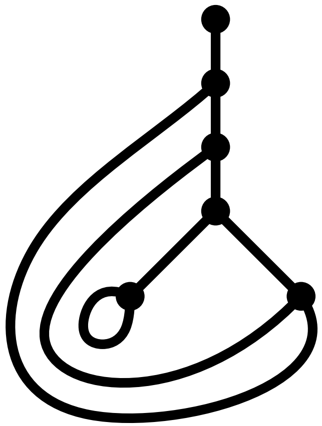
Closed planar terms and contexts

- Restricting the previous bijection we have:

  closed planar terms $\Leftrightarrow$ rooted trivalent planar maps

$$\lambda x.\lambda y.((x\ y)\ (\lambda z.z)) \qquad \leftrightarrow$$



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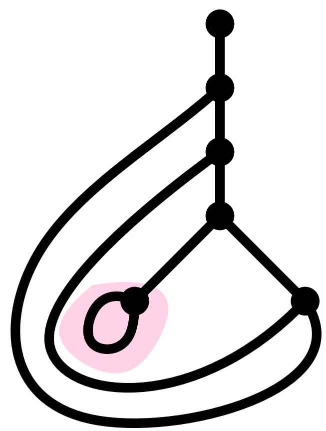
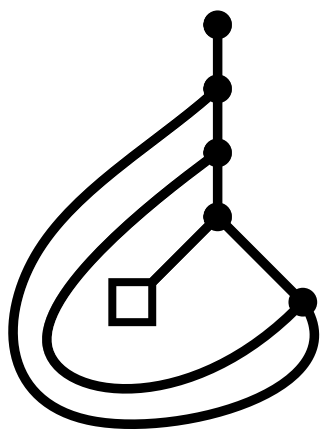


- We can also consider contexts:

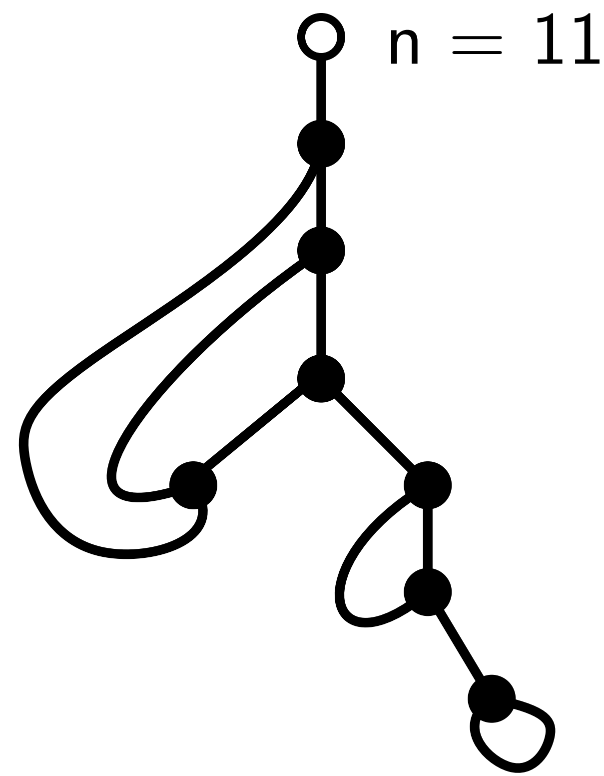
$$\lambda x.\lambda y.((x\ y)\ \Box) \quad \leftrightarrow$$



Closed planar terms and contexts

Lemma

A closed planar term with $n = 3k + 2, k \in \mathbb{N}$, subterms has:



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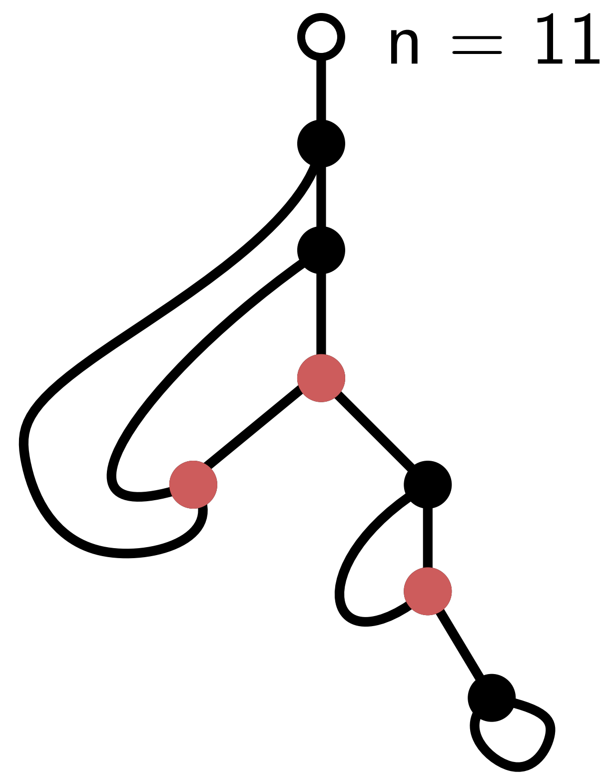

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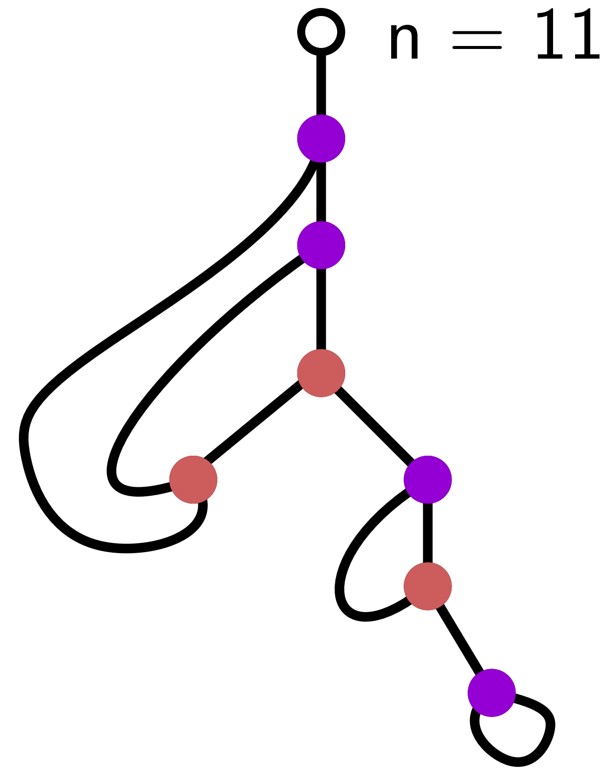

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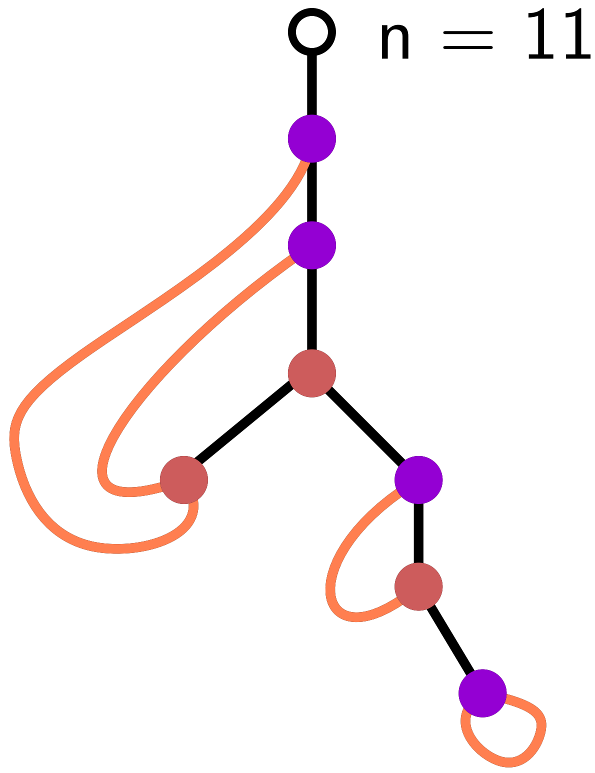

The planar Goulden-Jackson recurrence

In [GJ08], Goulden and Jackson give the following recurrence for $F(k, g) = \#$ of rooted triangulations of k faces and genus g:

$$F(k, g) = \frac{f(k,g)}{3k+2}, \text{ for } (k, g) \in S \setminus \{(-1, 0), (0, 0)\},$$

where $S = \{(k, g) \in \mathbb{Z}^2 \mid k \geqslant -1, 0 \leqslant g \leqslant \frac{k+1}{2}\}$ and $f(k, g)$ is

$$f(-1, 0) = \frac{1}{2}$$

$$f(k, g) = 0, \text{ for } (k, g) \notin S.$$

$$f(k, g) = \frac{4(3k+2)}{k+1} \left(k(3k-2)f(k-2, g-1) + \sum f(i, h)f(j, \ell)\right),$$

with the sum being taken over all pairs $(i, h) \in S$, $(j, \ell) \in S$ such that $i + j = k - 2$ and $h + \ell = g$.

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using the KP hierarchy!

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Open problem: give a combinatorial interpretation of the above. Planar case resolved by Baptiste Louf [B19].

The planar Goulden-Jackson recurrence

Reparameterising and setting $g = 0$, we have:

$$u(0) = 1$$

$$u(k+1) = 2(3k+2)p(k)$$

$$(k+1)p(k) = \sum_{i=0}^{n} u(i)u(n-i)$$

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duality

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To keep in mind:
$3k + 2$ edges $\leftrightarrow$ $2k$ vertices

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- rooted planar trivalent maps with $2k$ vertices
- closed planar terms with k applications

duality

bijection

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$3k + 2$ subterms $\leftrightarrow$ k applications

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k applications $\Rightarrow 3k+2$ subterms

2 ways to introduce a new application

$$\lambda x.\lambda y.(x\ y) \Leftrightarrow \begin{array}{c} \lambda x.\lambda y.(\square\ (x\ y)) \\ \text{or} \\ \lambda x.\lambda y.((x\ y)\ \square) \end{array}$$

How to (re)prove the Planar G&J Recurrence

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2k vertices $\Rightarrow$ $3k + 2$ edges

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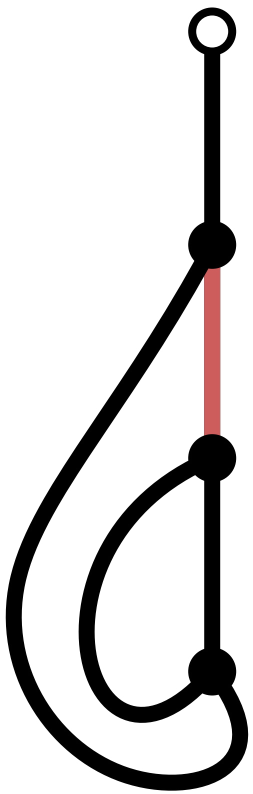

How to (re)prove the Planar G&J Recurrence

- Step 1:

2k vertices $\Rightarrow$ $3k+2$ edges

$$u(0) = 1$$

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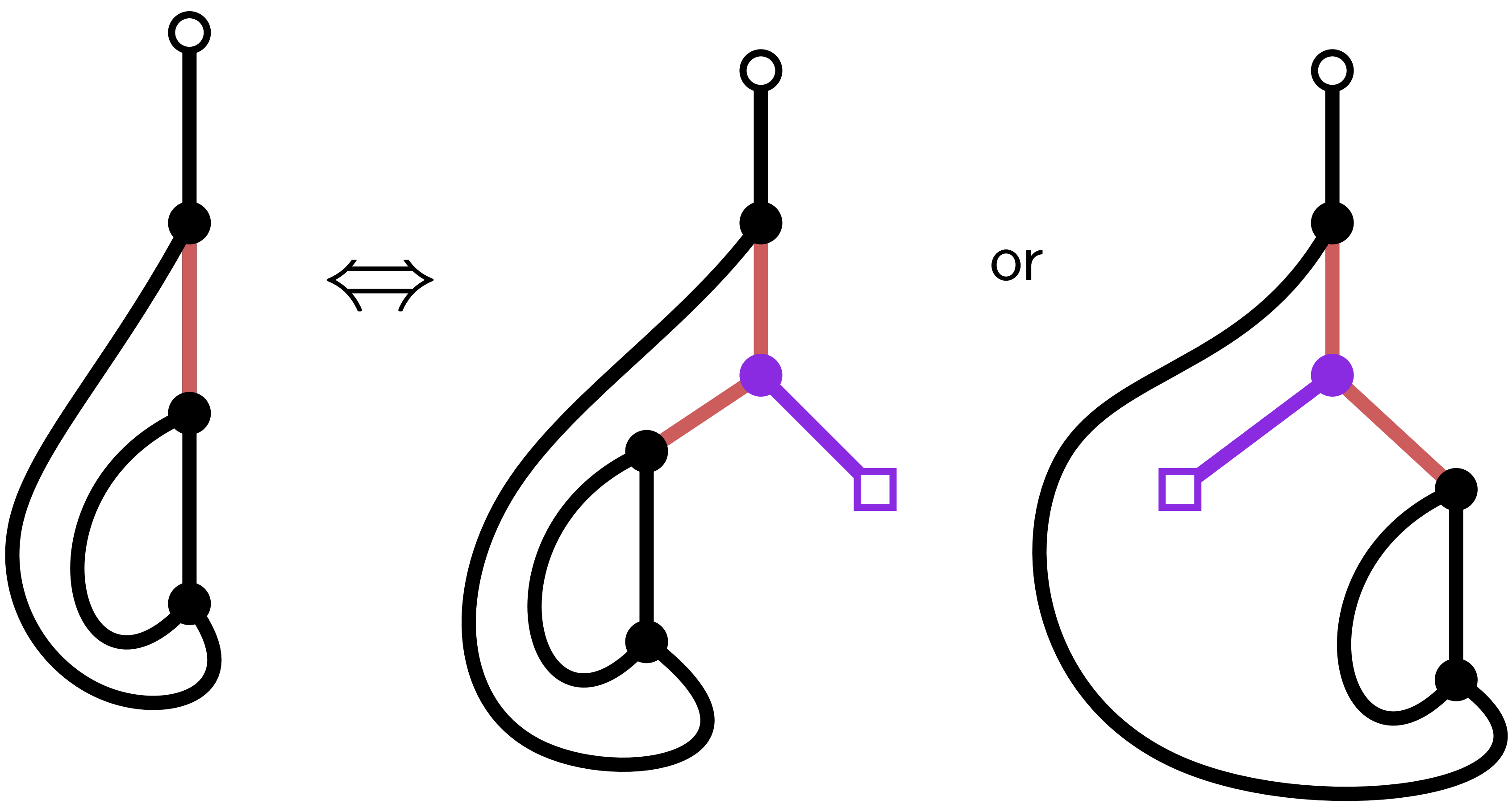
2 ways to introduce a new box vertex



How to (re)prove the Planar G&J Recurrence

- Step 2:

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How to (re)prove the Planar G&J Recurrence

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k applications $\Rightarrow (k+1)$ variables

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$$\lambda x.\lambda y.(\lambda z.\lambda w.(\lambda u.\lambda v.z\ u\ v)\ w)(x\ y)$$

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split var-pointed term into two contexts

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split var-pointed term into two contexts

minimal closed subterm that contains v

(always starts with λ!)

$$\lambda x.\lambda y.(\lambda z.\lambda w.(\lambda u.\lambda v.z\ u\ v)\ w)(x\ y)$$

$$\lambda x.\lambda y.\square(x\ y)$$

$$\lambda z.\lambda w.(\lambda u.\lambda v.z\ u\ v)\ w$$

How to (re)prove the Planar G&J Recurrence

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How to (re)prove the Planar G&J Recurrence

- Step 2:

k applications $\Rightarrow (k+1)$ variables

$$(k+1)p(k) = \sum_{i=0}^{n} u(i)u(n-i)$$

split var-pointed term into two contexts

minimal closed subterm that contains v

(always starts with λ!)

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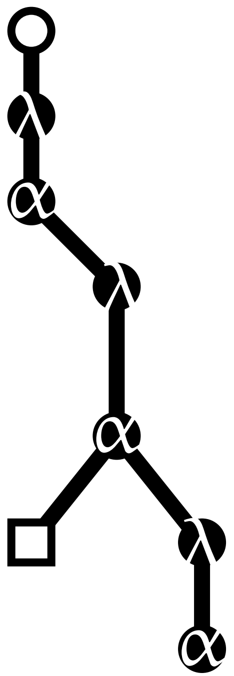

$$\lambda\_.(\lambda\_.((\lambda\_.\_\_)\square))\_$$

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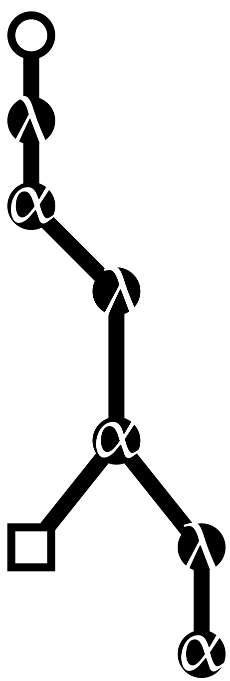
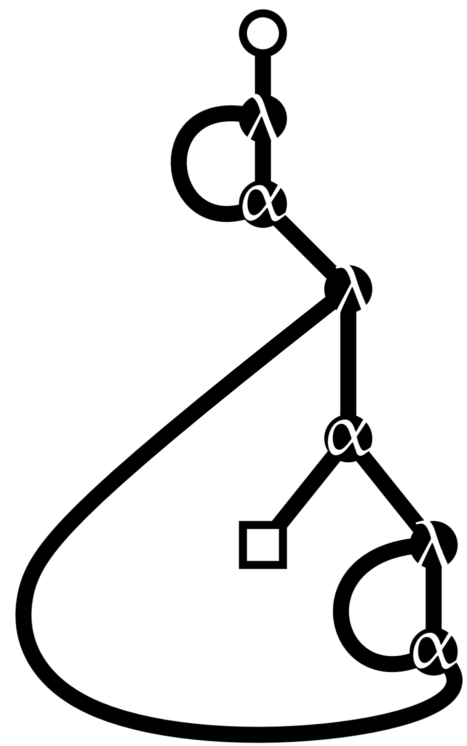


$\leftrightarrow$



$\lambda\textvisiblespace.(\lambda\textvisiblespace.((\lambda\textvisiblespace.\textvisiblespace\textvisiblespace)\Box))\textvisiblespace$

$\lambda x.(\lambda y.((\lambda z.yz)\Box))x$

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$$\longrightarrow \lambda\text{\textvisiblespace}.(\lambda\text{\textvisiblespace}.\lambda\text{\textvisiblespace}.\text{\textvisiblespace}\ \text{\textvisiblespace}\ \square)\ \text{\textvisiblespace}$$

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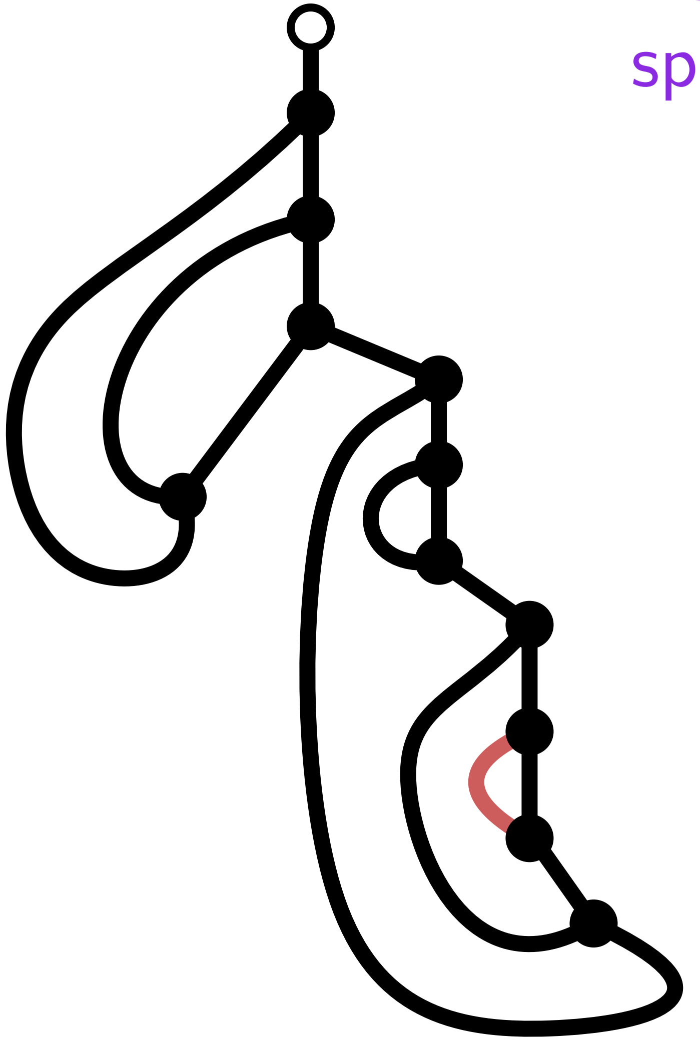

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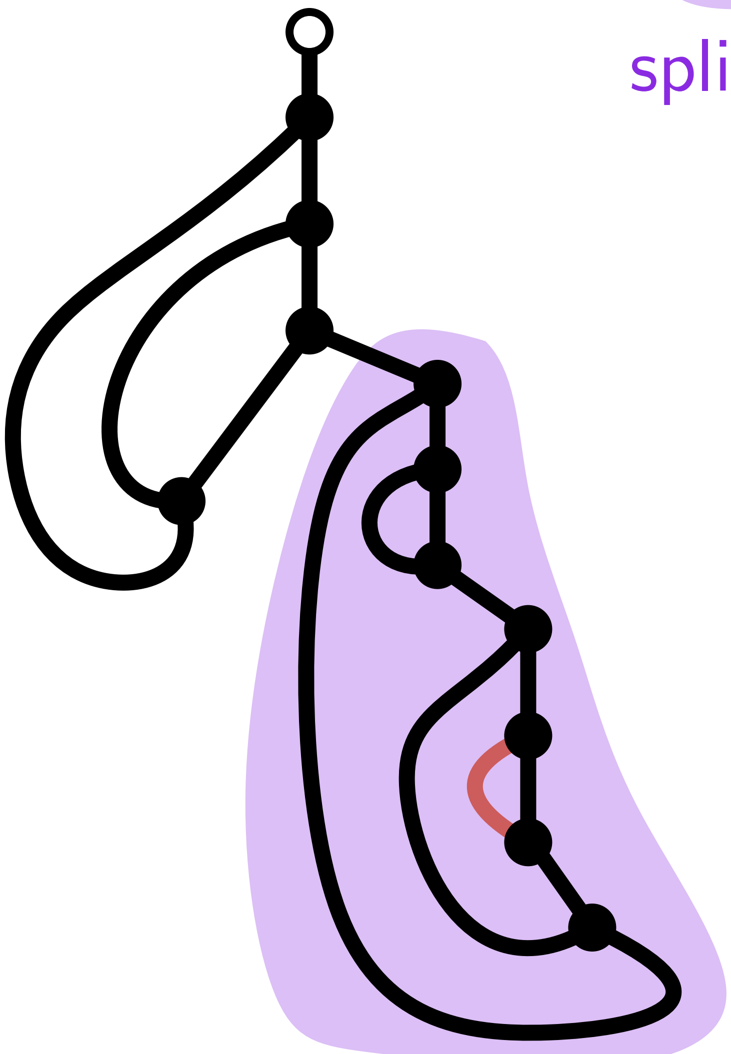

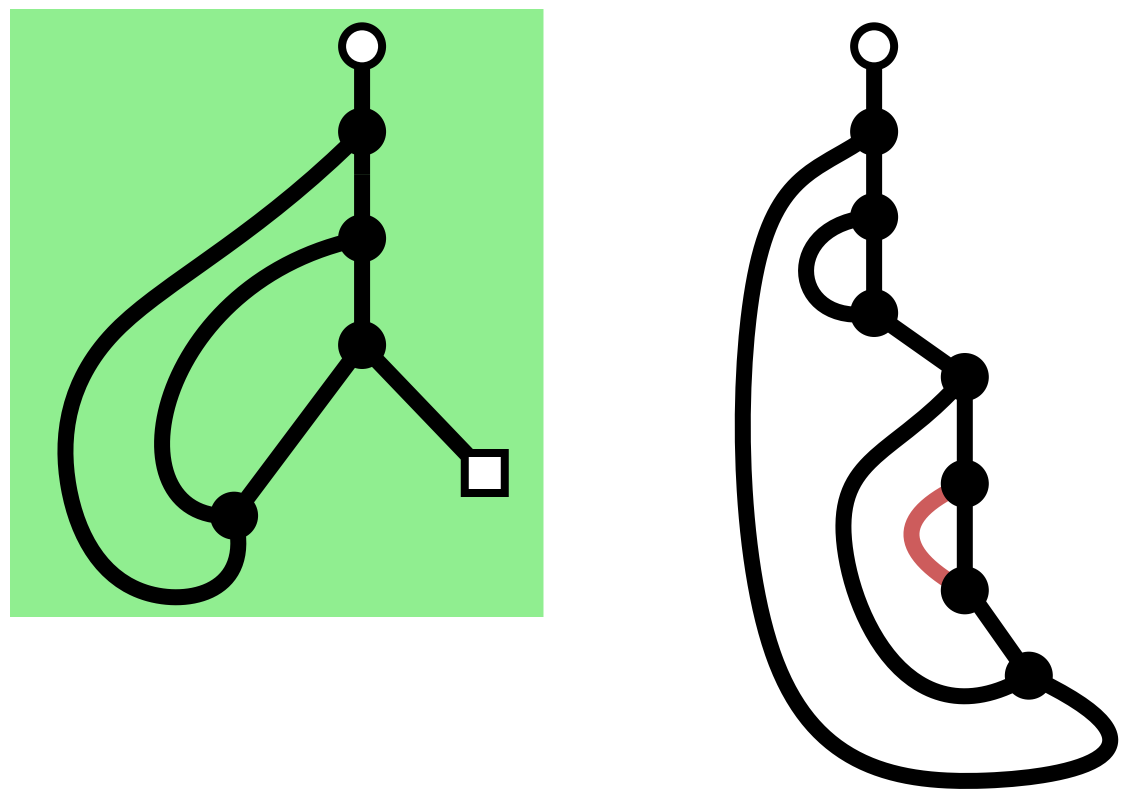

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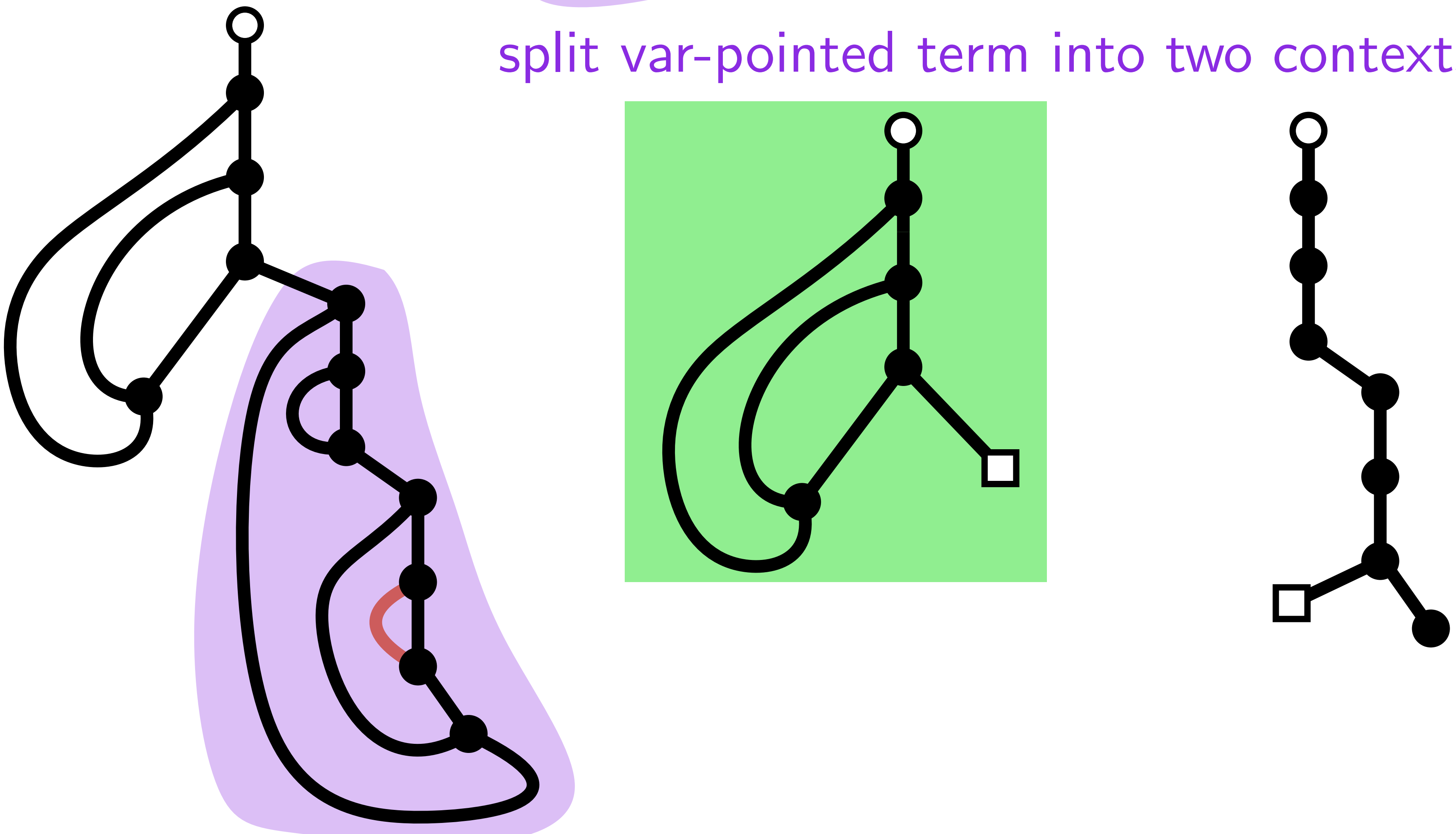

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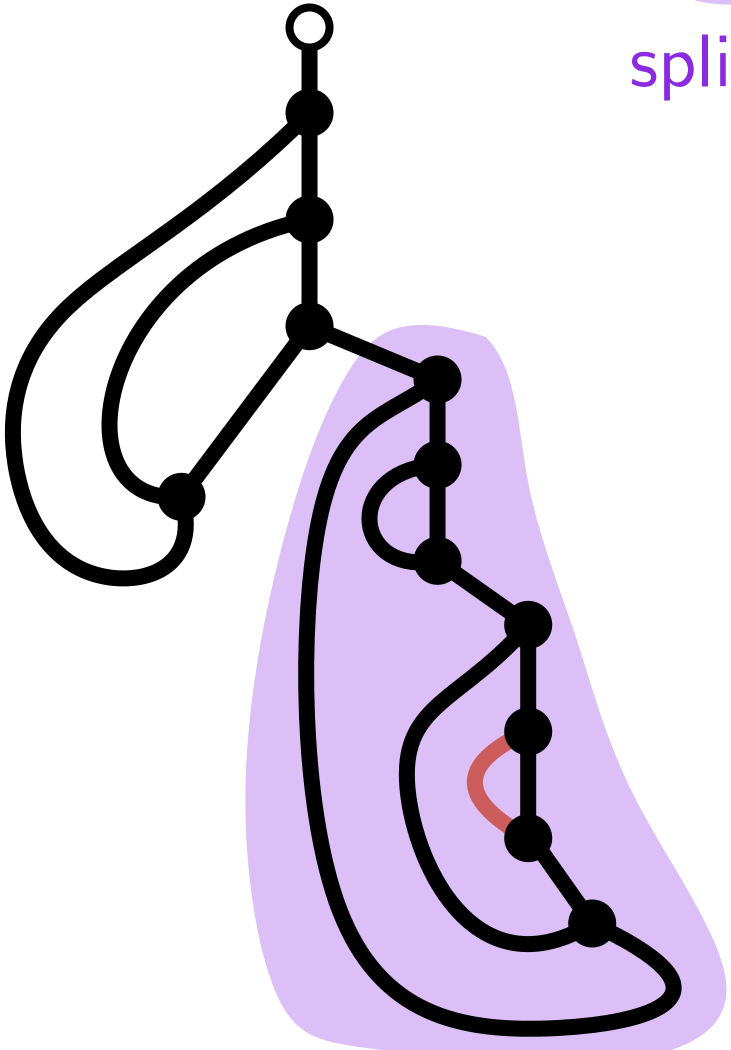

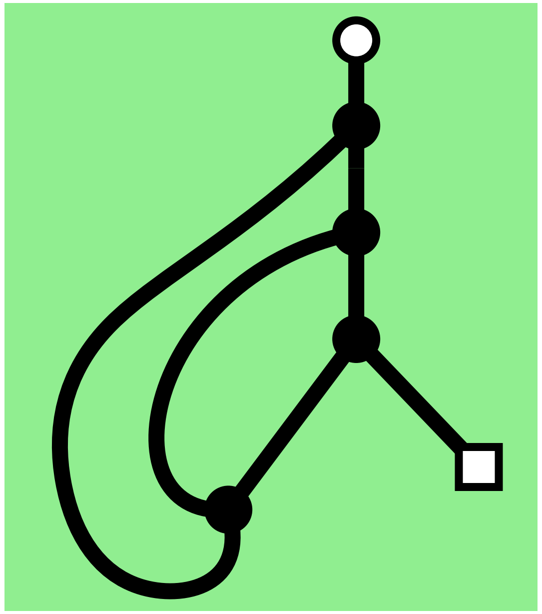

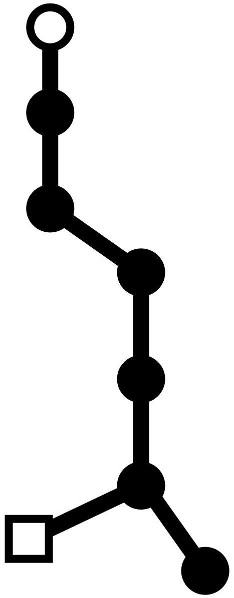

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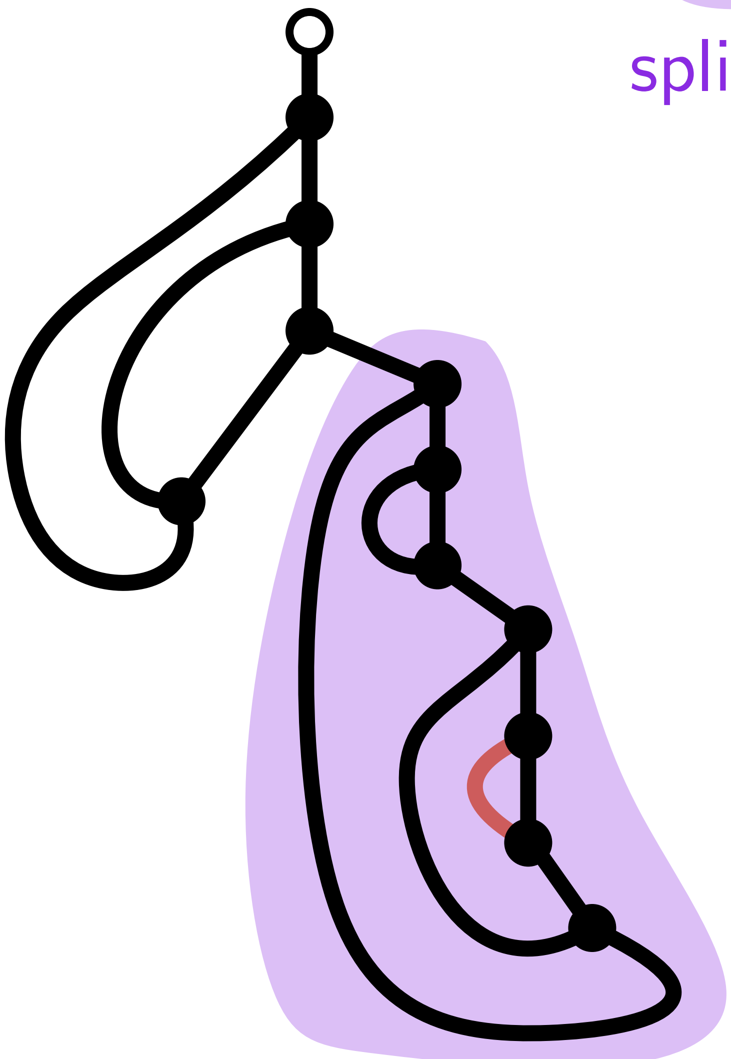

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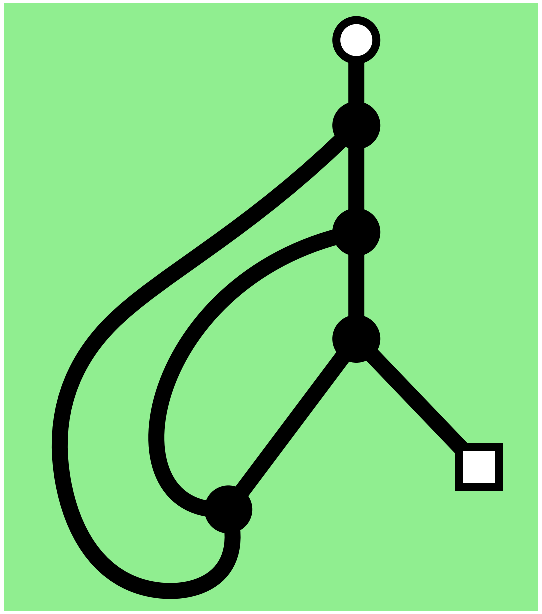

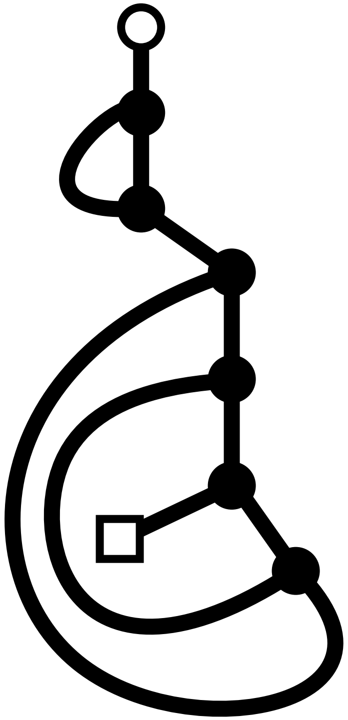

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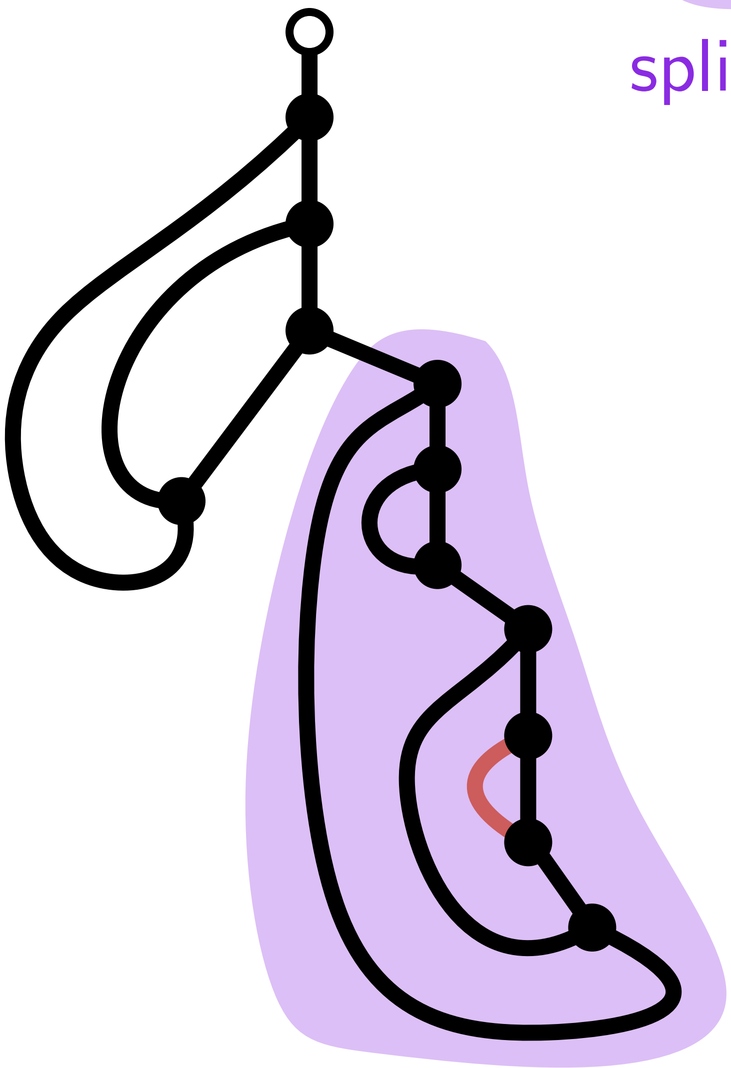

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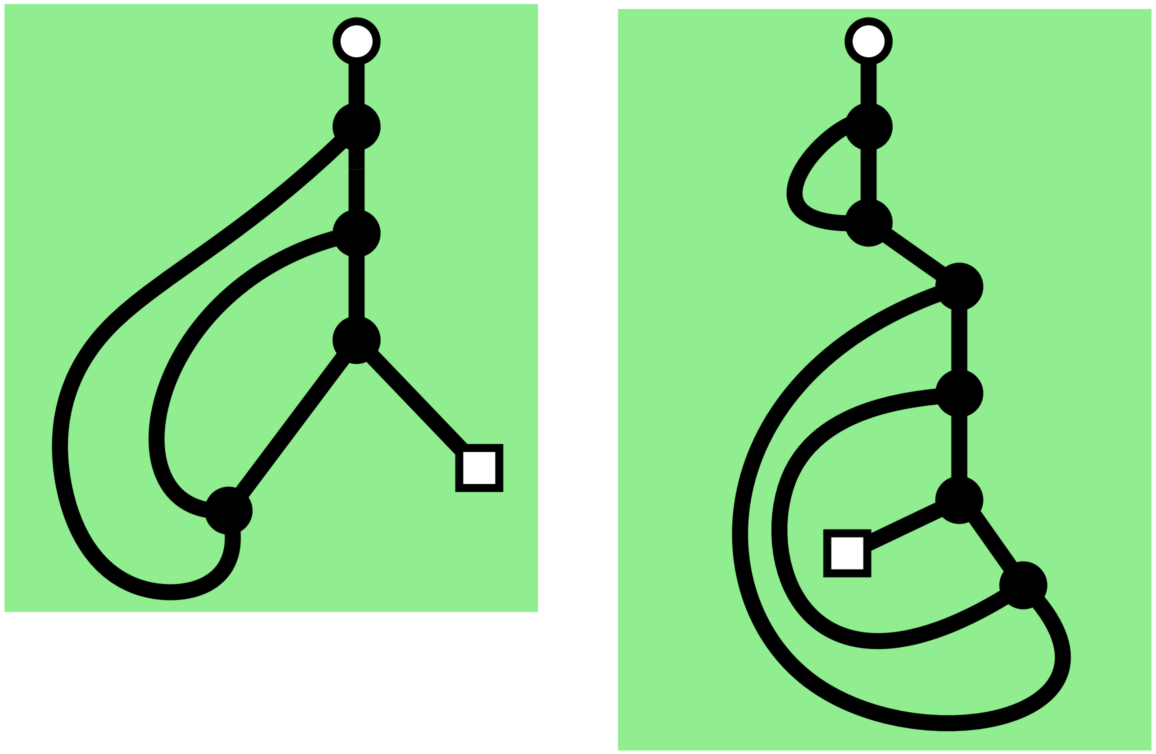

Some open problems

- Bijective interpretation of G&J rec. for general genus

$$o(0,g) = 1$$

$$o(k+1,g) = 2(3k+2)t(k,g)$$

$$(k+1)t(k,g) = \begin{array}{c} 2k(3k-2)o(k-1,g-1) \\ + \\ \displaystyle\sum_{\substack{i+j=k \\ h+\ell=g}}^{n} o(i,h)o(j,\ell) \end{array}$$

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- Genus for λ-terms?

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- Genus for λ-terms?

Thank you!

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