A cartesian bicategory of polynomial functors in homotopy type theory

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This is joint work with Eric Finster, Maxime Lucas and Thomas Seiller.

Part I

Polynomials and polynomial functors

In a nutshell

The situation:

- the category of polynomial functors is cartesian closed

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Our contributions:

- we have formalized polynomials in groupoids (or spaces) in HoTT/Agda
- we have shown that the resulting bicategory is cartesian closed
- we have provided a small axiomatization of the type $\mathbb{B}$ of natural numbers and bijections

Categorifying polynomials

A **polynomial** is a sum of monomials

$$P(X) = \sum_{0 \leq i < k} X^{n_i}$$

(no coefficients, but repetitions allowed)

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We can **categorify** this notion: replace natural numbers by elements of a set.

$$P(X) = \sum_{b \in B} X^{E_b}$$

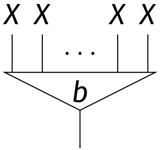
Polynomial functors

This data can be encoded as a **polynomial** P, which is a diagram in **Set**:

$$E \xrightarrow{p} B$$

where

- $b \in B$ is a monomial
- $E_b = P^{-1}(b)$ is the set of instances of X in the monomial b.



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It induces a **polynomial functor**

$$\begin{aligned} [\![P]\!] : \mathbf{Set} &\to \mathbf{Set} \\ X &\mapsto \sum_{b \in B} X^{E_b} \end{aligned}$$

Polynomial functors

For instance, consider the polynomial corresponding to the function

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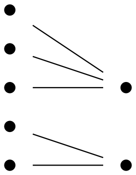

The associated polynomial functor is

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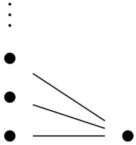
$$\mathbb{N} \xrightarrow{p} \mathbf{1}$$



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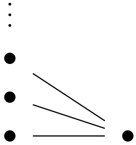

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A polynomial is **finitary** when each monomial is a finite product.

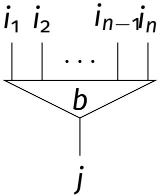
Polynomial functors: typed variant

We will more generally consider a “typed variant” of polynomials P

$$\begin{tikzcd} I & E \arrow[l, "s"'] \arrow[r, "p"] & B \arrow[r, "t"] & J \end{tikzcd}$$

this means that

- each monomial b has a “type $s(b) \in J$”,
- each occurrence of a variable $e \in E$ has a type $s(e) \in I$.



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It induces a **polynomial functor**

$$\begin{aligned} [\![P]\!](X) : \mathbf{Set}^I &\to \mathbf{Set}^J \\ (X_i)_{i \in I} &\mapsto \left(\sum_{b \in t^{-1}(j)} \prod_{e \in p^{-1}(b)} X_{s(e)} \right)_{j \in J} \end{aligned}$$

The category of polynomial functors

Given a set I, we have an “identity” polynomial functor:

$$\begin{tikzcd} I & I \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] & I \arrow[r, "\mathrm{id}"] & I \end{tikzcd}$$

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Proposition
The composite of two polynomial functors is again polynomial:

$$\begin{tikzcd} \mathbf{Set}^I \arrow[r, "{[\![P]\!]}"] \arrow[rr, bend right, "{[\![Q]\!] \circ [\![P]\!]}"'] & \mathbf{Set}^J \arrow[r, "{[\![Q]\!]}"] & \mathbf{Set}^K \end{tikzcd}$$

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Proof.
Basically the usual one:

$$\begin{aligned}[\![Q]\!] \circ [\![P]\!](X_i) &= \sum \prod \sum \prod X_i \\ &\cong \sum \sum \prod \prod X_i \\ &\cong \sum \prod X_i\end{aligned}$$

The category of polynomial functors

We can thus build a category **PolyFun** of sets and polynomial functors:

- an object is a set I,
- a morphism

  $$F : I \to J$$

  is a polynomial functor

  $$[\![P]\!] : \mathbf{Set}^I \to \mathbf{Set}^J$$

Polynomial vs polynomial functors

A *polynomial* P

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induces a *polynomial functor*

$$[\![P]\!] : \mathbf{Set}^I \to \mathbf{Set}^J$$

We have mentioned that composition is defined for polynomials. However, on polynomials, it is not strictly associative: we can build a *bicategory* **Poly** of sets an polynomial functors.

This suggests that 2-cells are an important part of the story!

Morphisms between polynomials

A morphism between two polynomials is

$$\begin{tikzcd}
I \arrow[d, equal] & E \arrow[l, "s"'] \arrow[r, "p"] \arrow[d, "\varepsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "t"] \arrow[d, "\beta"] & J \arrow[d, equal] \\
I & E' \arrow[l, "s'"] \arrow[r, "p'"'] & B' \arrow[r, "t'"'] & J
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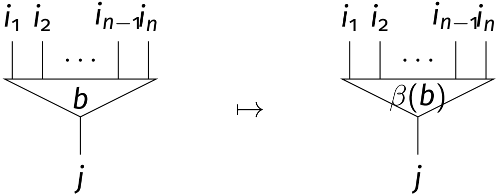
We send operations to operators, preserving typing and arities:



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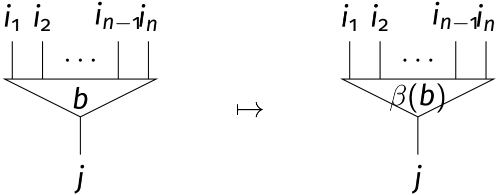

We can build a bicategory **Poly** of sets, polynomials and morphisms of polynomials

Morphisms between polynomial functors

A morphism between polynomial functors

$$[\![P]\!], [\![Q]\!] : \mathbf{Set}^I \to \mathbf{Set}^J$$

is a “suitable” natural transformation, and we can build a 2-category **PolyFun**.

Cartesian structure

The category **PolyFun** is cartesian. Namely, given two polynomial functors in **Poly**

$$P : I \to J \qquad\qquad Q : I \to K$$

i.e., in **Cat**,

$$[\![P]\!] : \mathbf{Set}^I \to \mathbf{Set}^J \qquad\qquad [\![Q]\!] : \mathbf{Set}^I \to \mathbf{Set}^K$$

we have, in **Cat**,

$$\langle P, Q \rangle : \mathbf{Set}^I \to \mathbf{Set}^J \times \mathbf{Set}^K \cong \mathbf{Set}^{J \sqcup K}$$

and the constructions preserve polynomiality: in **PolyFun**,

$$\langle P, Q \rangle : I \to (J \sqcup K)$$

Closed structure

For the closed structure, we can hope for the same: given, in **PolyFun**,

$$P : I \sqcup J \to K$$

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for LL-people: this looks like $!J \parr K$.

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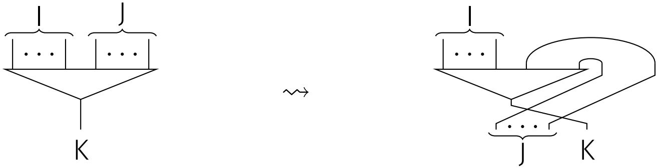
In terms of operations, the intuition behind the bijection

$$\mathbf{PolyFun}(I \sqcup J, K) \cong \mathbf{PolyFun}(I, \mathbf{Set}^{J} \times K)$$

is that we can formally transform operations as follows



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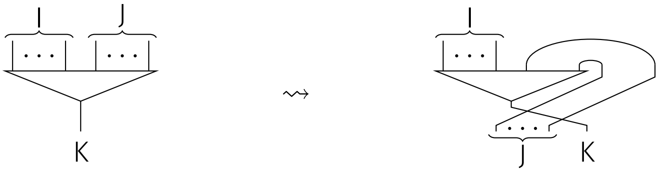

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Finitary polynomial functors are also known as **normal functors** (introduced by Girard).

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Theorem
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Remark (Girard)
The <u>2-</u>category **PolyFun** is <u>not</u> cartesian closed.

Failure of the cartesian closed structure

We would like to have an equivalence of categories

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which is induced by the polynomial

$$1 \longleftarrow 2 \longrightarrow 1 \longrightarrow 1$$

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The equivalence fails:

$$\mathbf{PolyFun}(0 \sqcup 1, 1) \not\simeq \mathbf{PolyFun}(0, \mathbb{N}/1 \times 1)$$

(two elements on the left, one on the right because 0 is initial)

Fixing the cartesian closed structure

The failure of the equivalence

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can be interpreted as being due to the fact that $2 \in \mathbb{N}/1$ has no non-trivial isomorphism.

This suggests moving to **groupoids**!

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More precisely, we should replace $\mathbb{N}$ by the groupoid $\mathbb{B}$ of all symmetric groups.

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Kock has identified that if we perform all the usual constructions up to homotopy (slice, pullbacks, etc.), we recover a suitable setting to define polynomial functors.

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This requires properly defining and using all the usual constructions in a suitable 2-categorical sense.

Polynomial functors in groupoids

Given a polynomial P

$$E \xrightarrow{p} B$$

the induced polynomial functor

$$\begin{aligned} [\![P]\!] : \mathbf{Gpd} &\to \mathbf{Gpd} \\ X &\mapsto \int^{b \in B} E_b \end{aligned}$$

where E_b is the *homotopy fiber* of p at b and

$$\int^{b \in E} E_b = \sum_{b \in \pi_0(B)} X_b/\mathrm{Aut}(b)$$

where the quotient is to be taken homotopically…

Part II

Formalization in Agda

Homotopy type theory

There is a framework in which everything is constructed *up to homotopy* for free: **homotopy type theory**.

Let's formally develop the theory of polynomials in this setting.

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Some notations

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- `t ≡ u`: equality between terms `t` and `u`
- `A ≃ B`: equivalence between types `A` and `B`

Axiom:

- univalence: `(A ≡ B) ≃ (A ≃ B)`

Homotopy levels (type = space):

- propositions: `is-prop A = (x y : A) → x ≡ y`
- sets: `is-set A = (x y : A) → is-prop (x ≡ y)`
- groupoids: `is-groupoid A = (x y : A) → is-set (x ≡ y)`

Formalizing polynomials

A polynomial is

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

We are tempted to formalize it as

```
record Poly (I J : Type) : Type₁ where
  field
    B : Type
    E : Type
    t : B → J
    p : E → B
    s : E → I
```

but this is not very good because operations on those involve many handling of equalities

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The identity is

```
Id : Poly I I
Op Id i = ⊤
Pm Id i {j = j} tt = i ≡ j
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A polynomial is

$$\begin{tikzcd} I & E \arrow[l, "s"'] \arrow[r, "p"] & B \arrow[r, "t"] & J \end{tikzcd}$$

We formalize it as a **container**:

```
record Poly (I J : Type) : Type₁ where
  field
    Op : J → Type
    Pm : (i : I) → {j : J} → Op j → Type
```

We sometimes write

```
I ⇝ J = Poly I J
```

Composing polynomials

The polynomial functor induced by a polynomial `P` is

```
⟦_⟧ : I ⇝ J → (I → Type) → (J → Type)
⟦_⟧ P X j = Σ (Op P j) (λ c → (i : I) → (p : Pm P i c) → (X i))
```

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```

The composite of two functors is

```
_·_ : I ⇝ J → J ⇝ K → I ⇝ K
Op (P · Q) = ⟦ Q ⟧ (Op P)
Pm (_·_ P Q) i (c , a) = Σ J (λ j → Σ (Pm Q j c) (λ p → Pm P i (a j p)))
```

Morphisms of polynomials

The type of morphisms between two polynomials is

```
record Poly→ (P Q : Poly I J) : Type where
  field
    Op→ : {j : J} → Op P j → Op Q j
    Pm≃ : {i : I} {j : J} {c : Op P j} → Pm P i c ≃ Pm Q i (Op→ c)
```

A bicategory

Theorem
We can build a pre-bicategory of types, polynomials and their morphisms.

A bicategory

Theorem
We can build a pre-bicategory of types, polynomials and their morphisms.

Theorem
We can build a bicategory of groupoids, polynomials in groupoids and their morphisms.

Products

Theorem
This bicategory is cartesian.

Products

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This bicategory is cartesian.

The product is ⊔ on objects, left projection is

```
projl : (I ⊔ J) ⇝ I
Op projl i = ⊤
Pm projl (inl i) {i'} tt = i ≡ i'
Pm projl (inr j) {i'} tt = ⊥
```

and pairing is

```
pair : (I ⇝ J) → (I ⇝ K) → I ⇝ (J ⊔ K)
Op (pair P Q) (inl j) = Op P j
Op (pair P Q) (inr k) = Op Q k
Pm (pair P Q) i {inl j} c = Pm P i c
```

Defining the exponential

In order to define the 1-categorical closure, the plan was:

$$\mathbf{Set} \qquad \rightsquigarrow \qquad \mathbf{Set}_{\mathrm{fin}} \qquad \rightsquigarrow \qquad \mathbb{N}$$

Defining the exponential

In order to define the 1-categorical closure, the plan was:

$$\mathbf{Set} \quad \rightsquigarrow \quad \mathbf{Set}_{\mathrm{fin}} \quad \rightsquigarrow \quad \mathbb{N}$$

For the 2-categorical closure the plan is

$$\mathbf{Gpd} \quad \rightsquigarrow \quad \mathbf{Gpd}_{\mathrm{fin}} \quad \rightsquigarrow \quad \mathbb{B}$$

Here, $\mathbb{B}$ is the groupoid with $n \in \mathbb{N}$ as objects and Σ_n as automorphisms on n.

Finite types

We write `Fin n` for the canonical finite type with `n` elements:
its constructors are `0` to `n-1`.

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its constructors are `0` to `n-1`.

```
data Fin : ℕ → Set where
  zero : {n : ℕ}              → Fin (suc n)
  suc  : {n : ℕ} (i : Fin n) → Fin (suc n)
```

Finite types

The predicate of being **finite** is

```
is-finite : Type → Type
is-finite A = Σ ℕ (λ n → ∥ A ≃ Fin n ∥)
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The type of finite types is

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FinType : Type₁
FinType = Σ Type is-finite
```

(note that this is a *large* type)

Finitary polynomials

A polynomial is **finitary** when, for each operation, the total space of its parameters is finite:

```
is-finitary : (P : I ⇝ J) → Type
is-finitary P = {j : J} (c : Op P j) → is-finite (Σ I (λ i → Pm P i c))
```

A small model for finite types

The type of **integers** is

```
data ℕ : Type where
  zero : ℕ
  suc  : ℕ → ℕ
```

A small model for finite types

The type 𝔹 is

```
data 𝔹 : Type where
  obj      : ℕ → 𝔹
  hom      : {m n : ℕ} (α : Fin m ≃ Fin n) → obj m ≡ obj n
  id-coh   : (n : ℕ) → hom {n = n} ≃-refl ≡ refl
  comp-coh : {m n o : ℕ} (α : Fin m ≃ Fin n) (β : Fin n ≃ Fin o) →
             hom (≃-trans α β) ≡ hom α · hom β
```

(this is a small higher inductive type!)

A small model for finite types

The type $\mathbb{B}$ is

```
data 𝔹 : Type where
  obj      : ℕ → 𝔹
  hom      : {m n : ℕ} (α : Fin m ≃ Fin n) → obj m ≡ obj n
  id-coh   : (n : ℕ) → hom {n = n} ≃-refl ≡ refl
  comp-coh : {m n o : ℕ} (α : Fin m ≃ Fin n) (β : Fin n ≃ Fin o) →
             hom (≃-trans α β) ≡ hom α · hom β
```

(this is a small higher inductive type!)

Theorem
`FinType ≃ 𝔹`.

The closure

We define

```
Exp : Type → Type₁
Exp I = I → Type
```

Theorem
Ignoring size issues, for polynomials we have

```
(I ⊔ J) ⇝ K  ≃  I ⇝ (Exp J × K)
```

The closure

We define

```
Exp : Type → Type₁
Exp I = Σ (I → Type) (λ F → is-finite (Σ I F))
```

Theorem
Ignoring size issues, for finitary polynomials we have

```
(I ⊔ J) ⇝ K  ≃  I ⇝ (Exp J × K)
```

The closure

We define

```
Exp : Type → Type₁
Exp I = Σ FinType (λ N → fst N → I)
```

Theorem
Ignoring size issues, for finitary polynomials we have

```
(I ⊔ J) ⇝ K  ≃  I ⇝ (Exp J × K)
```

The closure

We define

```
Exp : Type → Type
Exp I = Σ 𝔹 (λ b → 𝔹-to-Fin b → A)
```

Theorem
For finitary polynomials we have

```
(I ⊔ J) ⇝ K  ≃  I ⇝ (Exp J × K)
```

The exponential

Note that

```
Exp : Type → Type
Exp I = Σ 𝔹 (λ b → 𝔹-to-Fin b → A)
```

is the free pseudo-commutative monoid!