A quick introduction to species, operads, and closed multicategories

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1. What is a species?

Origins

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Une théorie combinatoire des séries formelles

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This paper presents a combinatorial theory of formal power series. The combinatorial interpretation of formal power series is based on the concept of species of structures. A categorical approach is used to formulate it. A new proof of Cayley's formula for the number of labelled trees is given as well as a new combinatorial proof (due to G. Labelle) of Lagrange's inversion formula. Polya's enumeration theory of isomorphism classes of structures is entirely renewed. Recursive methods for computing cycle index polynomials are described. A combinatorial version of the implicit function theorem is stated and proved. The paper ends with general considerations on the use of coalgebras in combinatorics.

Origins

INTRODUCTION

Le but de ce travail est à la fois d'exposer, de clarifier et d'unifier le sujet. L'utilité des séries formelles dans les calculs combinatoires est bien établie. L'interprétation combinatoire de l'opération de *substitution* a fait l'objet de travaux assez récents (Bender et Goldman [1], Doubilet *et al.* [8], Foata et Schützenberger [12], Garsia et Joni [13], Gessel [14]). La première interprétation (probabiliste) de la substitution des séries de puissances remonte à Watson (Kendall [18]) (dans la théorie de processus en cascade).

La caractéristique principale de la théorie présentée ici est son degré de généralité et sa simplicité. Dans cette théorie, les objets combinatoires correspondant aux séries formelles sont les *espèces de structures*. L'accent est mis sur le *transport* des structures plutôt que sur leurs propriétés. Ce point de vue n'est pas sans évoquer celui d'Ehresmann [9] et contraste avec celui de Bourbaki [2]. Aux opérations combinatoires sur les séries formelles

For those interested in an English translation, see Brent Yorgey's project https://github.com/byorgey/series-formelles, which is still a work-in-progress but includes a good chunk of the paper, together with some commentary and beautiful diagrams.

Origins

correspondent des opérations sur les espèces de structures. Aux identités algébriques entre expressions formelles correspondent souvent des identités combinatoires. L'intuition et le calcul peuvent alors jouer sur deux plans dans un dialogue qui ressemble à celui qu'entretiennent l'algèbre et la géométrie. Il en résulte une sorte d'algèbre combinatoire analogue à l'algèbre géométrique de Grassman (et de Leibniz). La simplicité de la théorie est en grande partie due à l'usage qu'elle fait des *concepts* de la théorie des catégories [23] (les théories antérieures utilisant surtout la théorie des ensembles ordonnés et des partitions). De plus, elle met en évidence le fait fondamental qu'un très grand nombre de bijections construites sont *naturelles*, c'est-à-dire qu'elles ne dépendent pas d'un système de coordonnées introduit au moyen d'une énumération arbitraire.

Le travail contient quelques innnovations combinatoires comme le concept de *vertébré* et une nouvelle démonstration du résultat de Cayley [...]

Definition (version Joyal 1986)

A **(symmetric) species** is a functor $S : \mathcal{B} \to \mathrm{Set}$

where $\mathcal{B}$ is the category of finite sets and bijections

where Set is the category of sets and functions

Unpacking the definition:

not necessarily finite ↓

1. for any finite set A, a set S[A] of "structures on A"
2. for any bijection $f : A \cong B$, a function $S[f] : S[A] \to S[B]$
3. such that $S[g \circ f] = S[g] \circ S[f]$ and $S[id] = id$

an immediate consequence is that $S[f] : S[A] \cong S[B]$

Intuition: family of combinatorial objects carrying a finite set of labels, *invariant under relabelling.* (cf. Flajolet & Sedgewick's "labelled classes")

The groupoid of elements of a species

To any functor $T : \mathscr{C} \to \mathrm{Set}$ may be associated a **category of elements** el(T) equipped with a canonical projection functor $\mathrm{el}(T) \to \mathscr{C}$. The definition of el(S) reduces to the following in the case of a species $S : \mathcal{B} \to \mathrm{Set}$:

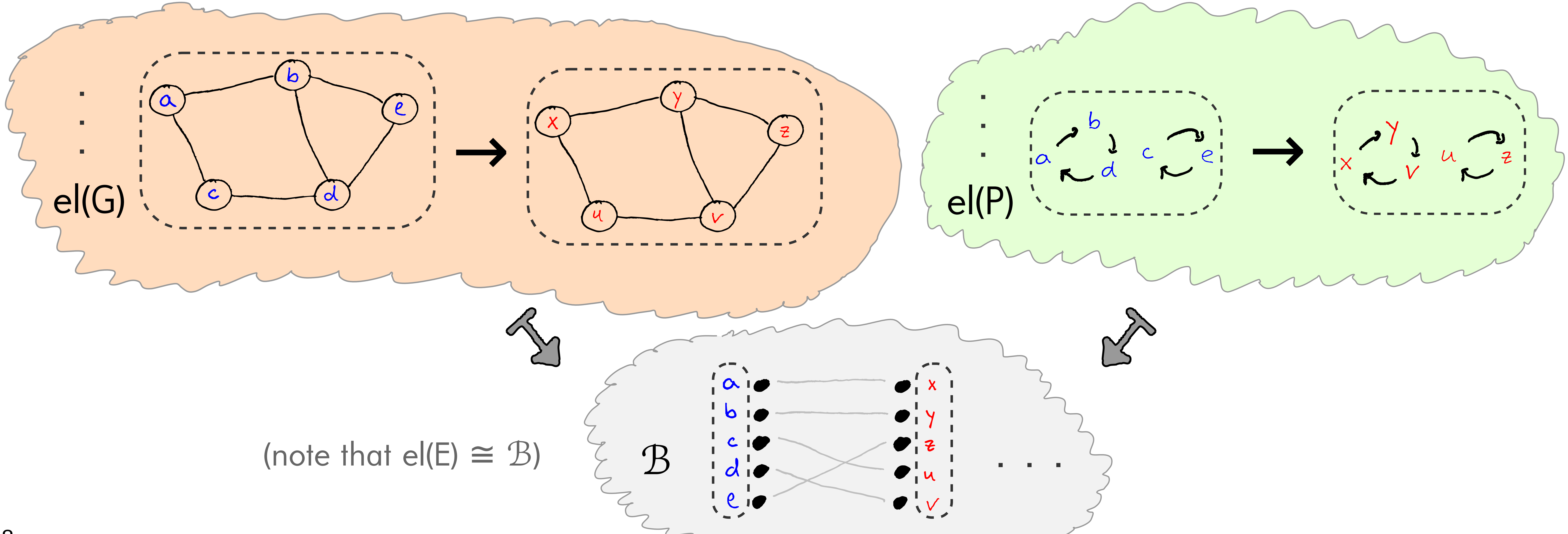
- objects are pairs (A,s) where A is a finite set and $s \in S[A]$ a structure on A
- morphisms $(A,s) \to (B,t)$ are bijections $f : A \cong B$ such that $S[f](s) = t$

For a species, el(S) is always a **groupoid** = category where every morphism is invertible, and the connected components of this groupoid may be seen as equivalence classes of structures modulo relabelling.

Conversely, one can also recover the functor $S : \mathcal{B} \to \mathrm{Set}$ from the functor $\mathrm{el}(S) \to \mathcal{B}$, and this *fibrational perspective* on species is at times helpful.

Some examples of species

- $G[A]$ = simple graphs = irr. sym. relations $R \subseteq A \times A$; $G[f] = R \mapsto f(R)$
- $P[A]$ = bijections $\varphi : A \cong A$; $P[f] = \varphi \mapsto f \circ \varphi \circ f^{-1}$
- $E[A] = \{*\}$; $E[f]$ = trivial



From (finitary) species to generating functions

A species S : $\mathcal{B}$ → Set is said to be **finitary** if Card S[A] < ∞ for all A.
In that case, we can associate to S an *exponential generating function:*

$$S(x) = \sum_{n \geq 0} \mathrm{Card}\, S[n] \frac{x^n}{n!}$$

The formula may be justified conceptually as computing the "groupoid cardinality" of each connected component of el(S).

Examples:

$$G(x) = \sum_{n \geq 0} 2^{n(n-1)/2} \frac{x^n}{n!} \qquad P(x) = \sum_{n \geq 0} n! \frac{x^n}{n!} = \frac{1}{1-x} \qquad E(x) = \sum_{n \geq 0} \frac{x^n}{n!} = e^x$$

The category of species

Species can themselves be organized into a category Esp with morphisms $\varphi : S \to T$ given by **natural transformations**, that is, families of functions $\varphi[A] : S[A] \to T[A]$ such that $T[f] \circ \varphi[A] = \varphi[B] \circ S[f]$ for all $f : A \cong B$.

note this is equivalent to:

$$\begin{array}{ccc} el(S) & \to & el(T) \\ & \searrow \quad \swarrow & \\ & \mathcal{B} & \end{array}$$

Often, though not always, identities on generating functions can be lifted to *natural isomorphisms* of the corresponding species.

counterexample (Joyal): $L(x) = P(x)$ but $L \not\cong P$, where $L[A]$ = linear orders on A.

But...*the power of the approach relies on the fact that species live inside a category that includes many non-invertible morphisms!*

Categorifying operations on generating functions

Coproducts in Esp yield **sums** of egfs:

$$(S + T)[A] = S[A] \uplus T[A] \qquad (S+T)(x) = S(x) + T(x)$$

$$0[A] = \varnothing \qquad 0(x) = 0$$

A convolution tensor product on Esp yields **multiplication** of egfs:

$$(S \otimes T)[A] = \{(B,C,s,t) \mid A = B \uplus C,\ s \in S[B],\ t \in T[C]\} \qquad (S \otimes T)(x) = S(x)T(x)$$

$$1[A] = \{* \mid A = \varnothing\} \qquad 1(x) = 1$$

Esp also has categorical products, which yield the "Hadamard product" of egfs:

$$(S \,\&\, T)[A] = S[A] \times T[A] \qquad (S\&T)(x) = \sum_{n \geq 0} s_n t_n x^n \quad \text{where } s_n = [x^n]S(x), t_n = [x^n]T(x)$$

(the unit is the terminal species E)

Composition and differentiation

Esp has yet another monoidal structure, capturing **composition** of egfs:

$$(S \circ T)[A] = \{((A_b)_{b \in B}, s, (t_b)_{b \in B}) \mid A = \biguplus_{b \in B} A_b,\ s \in S[B],\ \forall b \in B.\ t_b \in T[A_b]\ \}$$

$$X[A] = \{f \mid f : A \cong \{*\}\}$$

$$(S \circ T)(x) = S(T(x)) \qquad X(x) = x$$

Observe that this monoidal product is *not* symmetric.

The **derivative** of a species is defined as follows:

$$S'[A] = S[A \uplus \{*\}] \qquad S'(x) = \partial_x S(x)$$

Many laws of differentiation lift to *natural isomorphisms*...

$$1' \cong 0 \quad X' \cong 1 \quad E' \cong E \quad (S + T)' \cong S' + T'$$

$$(S \otimes T)' \cong (S' \otimes T) + (S \otimes T') \qquad (S \circ T)' \cong (S' \circ T) \otimes T'$$

Monoidal closed structure

Interestingly, the cartesian product, the symmetric tensor product, and the non-symmetric composition product all have right closures:

$$-\&S \dashv S\Rightarrow- \qquad -\otimes S \dashv S\multimap- \qquad -\circ S \dashv S\triangleright-$$

In general these do not correspond to simple operations on egfs, but sometimes they can be simplified. In particular we have the following nice identity:

$$X \multimap S \cong S'$$

Relative species and multivariate gfs

The base category $\mathcal{B}$ somehow possesses a lot of "innate knowledge" of egfs, but may be replaced by other categories to categorify other kinds of generating functions. For example, ordinary generating functions are categorified by **ordered species**, defined as functors $S : \mathcal{O} \to \mathrm{Set}$ on the groupoid of linearly ordered finite sets and order-preserving bijections.

Ordered species are actually an instance of Joyal's more general notion of *relative species,* defined as functors $S : \mathrm{el}(T) \to \mathrm{Set}$ on the category of elements of another species T. Note that $\mathcal{O} \cong \mathrm{el}(L) \equiv \mathbb{N}$.

Similarly one can categorify **multivariate gfs** as functors over a cartesian product of categories. For example, a functor over $\mathcal{O} \times \mathcal{B}$ yields a bivariate gf that is ordinary in the first parameter and exponential in the second.

Functional equations and inductive definitions

In combinatorics, a gf is often specified by a "functional equation", e.g., the ogf for strings of balanced parentheses and the egf for rooted non-planar trees:

$$B(x) = 1 + xB(x)^2 \qquad\qquad T(x) = xe^{T(x)}$$

In functional programming languages, types are often specified by recursive equations, e.g., the type of binary trees with labelled nodes, or of "rose trees":

```
data Bin a = Leaf | Branch a (Bin a) (Bin a)
data Rose a = Node a [Rose a]
```

Joyal's Implicit Species Theorem ties these things together!

The Implicit Species Theorem 2*

Theorem. Let $F(X,Y)$ be a finitary bivariate species satisfying $F(0,Y) \cong n$ for some $n \in \mathbb{N}$. Then there exists a finitary species S, unique up to isomorphism, satisfying $S \cong F(X,S)$ with $S(0) \cong n$.

Examples:

$$L \cong 1 + X \otimes L \qquad B \cong 1 + X \otimes B \otimes B \qquad T \cong X \otimes (E \circ T)$$

$$\Lambda \cong X^2 + X \otimes \Lambda^2 + 2 \otimes X^4 \otimes \partial_X \Lambda$$

*this nice simple formulation is due to Brent Yorgey, see Theorem 5.4.3 of *Combinatorial Species and Labelled Structures,* PhD thesis, University of Pennsylvania, 2014.

2. What is an operad?

Idea, origins, and applications

From https://ncatlab.org/nlab/show/operad:

> An ***operad*** is a gadget used to describe algebraic structures in symmetric monoidal categories. It is
>
> - a bunch of abstract *operations* of arbitrarily many arguments;
> - equipped with a notion of how to compose these;
> - subject to evident associativity and unitality conditions.

Originally introduced by Peter May (1972) for motivations in algebraic topology.

Have found applications in category theory, combinatorics, CS, etc, see textbooks:

Tom Leinster, *Higher Operads, Higher Categories,* 2004
Miguel A. Méndez, *Set Operads in Combinatorics and Computer Science,* 2015
Donald Yau, *Operads of Wiring Diagrams,* 2018
Samuele Giraudo, *Nonsymmetric Operads in Combinatorics,* 2018

Definition

Super slick version: *an operad is a monoid in (Esp, $\circ$, X)!*

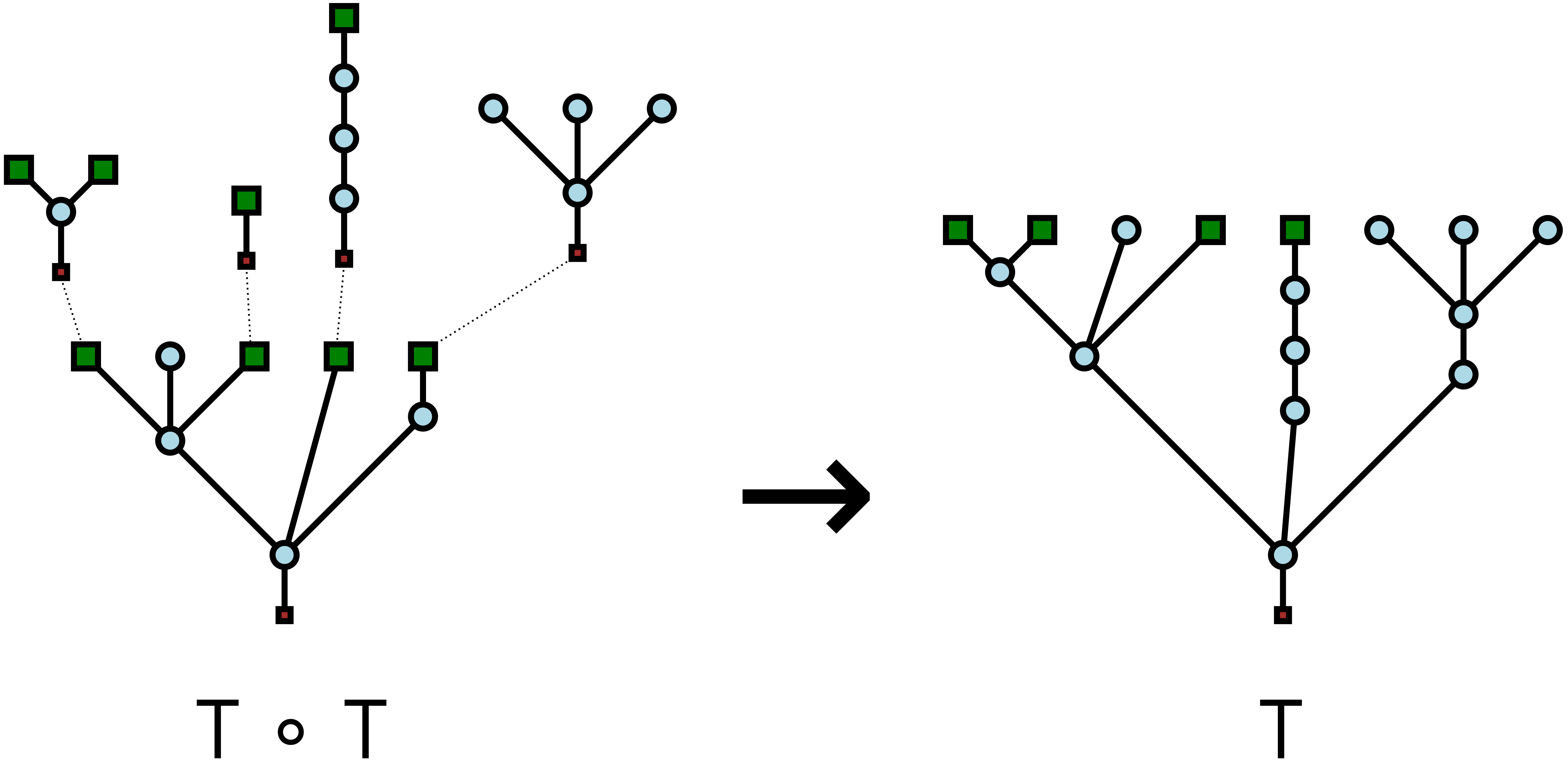
That is, an operad is a species T equipped with a pair of morphisms

$$T \circ T \to T \qquad X \to T$$

satisfying associativity and unitality laws.

Working in Esp (the category of $\mathcal{B}$-species) gives us **symmetric operads**, but the same definition works in different categories of species to recover different flavors of operads. In particular, **planar (= non-symmetric) operads** can be defined as monoids in the category of $\mathcal{O}$-species.

Paradigmatic example: trees w/labelled leaves



Free operads

The **free operad** over a species S is the initial algebra for the functor

$$R \mapsto X + S \circ R$$

and hence satisfies $T_S \cong X + S \circ T_S$. The operations of T_S can be seen as trees with labelled leaves and with nodes labelled by structures in S.

Even if S is finitary, T_S need not be (e.g., take $S = 1 + X^2$). However, it can be made finitary by introducing an extra variable tracking nodes…

$$T_S \cong X + Z \otimes (S \circ T_S)$$

resulting in a corresponding functional equation for the bivariate gf:

$$T_S(z,x) = x + z \cdot S(T_S(z,x))$$

Colored operads = multicategories

A classical operad only keeps track of the arities of operations.

But there is also a "colored" version of operads, where each operation is assigned a list of input colors and a unique output color, and composition requires that the colors match.

Another name for a colored operad is a **multicategory**.

A multicategory is just like a category, except that the morphisms can have multiple inputs. Indeed, there is the following analogy:

monoid : category :: operad :: multicategory

Multicategories & sequent calculus

There is a close relationship between multicategories and sequent calculi for intuitionistic logics. Indeed, this was one of Joachim Lambek's inspirations for defining multicategories!

3. MULTICATEGORIES

I had pointed out in 1961 that Gentzen's sequent calculus was essentially the same as Bourbaki's (1958) method of bilinear maps. For argument's sake, let us assume we are dealing with $R-R$–bimodules for a given ring R. Bourbaki had described a canonical bilinear map $AB \to A \otimes B$ which induced a natural isomorphism between *bilinear* maps $AB \to C$ and *linear* maps $A \otimes B \to C$. This was exploited to obtain all properties of the tensor product.

[...]

Once the analogy between the methods of Gentzen and Bourbaki has been realized, it is quite natural to ask that the sequents in the Gentzen style presentation of the syntactic calculus be interpreted as some kind of multilinear maps. This was done in my 1968 paper, to be followed by a formal definition of " multicategories" in 1969.

3. What is a closed multicategory?

Definition

A multicategory is said to be **closed** if for any pair of objects ("colors") A and B, there is an object $A \multimap B$ equipped with a binary morphism

$$\mathrm{app} : A \multimap B, A \to B$$

such that for any (n+1)-ary morphism

$$f : A_1, \ldots, A_n , A \to B$$

there exists a unique n-ary morphism

$$\mathrm{lam}(f) : A_1, \ldots, A_n \to A \multimap B$$

such that $\mathrm{app} \circ_1 \mathrm{lam}(f) = f$.

Examples of closed multicategories

Any monoidal closed category. In particular:

Set = the multicategory of sets and n-ary functions.
(A closed cartesian multicategory.)

Vec = the multicategory of vector spaces and multilinear maps.
(A closed symmetric multicategory.)

Also, the multicategory of species, defined in three different ways!

The paradigmatic example: typed lambda calculus.

Free closed multicategories

(c = cartesian, s = symmetric, p = planar)

The **free closed** (c/s/p-)**multicategory** over a set S may be constructed as follows:

- objects are the elements of S, viewed as types;
- morphisms $A_1, \ldots, A_n \to B$ are $\equiv_{\beta\eta}$ classes of typed (g/l/o-)λ-terms $x_1{:}A_1, \ldots, x_n{:}A_n \vdash t{:}B$;   (g = general, l = linear, o = ordered)
- composition is defined by substitution.

Alternatively, it may be constructed directly from terms in normal form, with composition defined by normalizing substitution. This is nicer since these normal forms admit a simple inductive description!

In particular, we can enumerate morphisms in free closed multicategories using cut-free sequent calculi, à la Lambek.

A puzzling fact

Let M be the free closed planar multicategory over {*}.

Write M[n] for the set of morphisms $A_1, \ldots, A_k \to *$ in M where the $A_1 \ldots A_k$ contain a total of 2n+1 occurrences of *.

Let m_n = Card M[n].

Fact: *m_n is the number of rooted planar maps on 2n+1 darts.*

This is a consequence of ZG 2015, but we did not give a good explanation.

Can the new bijection by Wenjie Fang (2022) [talk this afternoon!] provide a path towards a proper understanding of this fact?

Food for thought

Let T be *the restriction of M to first-order types.*

where $\text{order}(*) = 0$
$\text{order}(A \multimap B) = \max(1 + \text{order}(A), \text{order}(B))$

Let t_n = Card T[n].

Fact: *t_n is the number of rooted planar trees on 2n+1 darts.*

The proof is easy. Interestingly, the type of t:A can be seen as encoding a tree by a Łukasiewicz path.

Can this easy fact and the puzzling fact be given a unified explanation?