Contents

Part I

Proposal context, positioning and objectives

1 Summary

This project builds upon and aims to advance *a remarkable connection* between computer science and mathematics that was independently found by the lead researchers in 2013/2014. On one side of this interdisciplinary correspondence is the **lambda calculus**, a formal model of computation originally introduced in the 1930s for motivations in logic, which remains highly influential on the design and analysis of modern programming languages. On the other side is the study of **graphs on surfaces** (or "maps"), a deep branch of mathematics with a long history. The unexpected discovery of a host of enumerative and bijective links between these two domains (see Table 1) presents a real opportunity for transferring knowledge and techniques in both directions, and for making new advances. Although previously unnoticed, these striking results can be analyzed within the context of older ideas bridging logic, geometry, and combinatorics. This research project aims to deepen our scientific understanding by using these new bijective connections:

1. *develop rigorous logical perspectives on maps and related combinatorial objects;* and
2. *develop precise quantitative perspectives on lambda calculus and related systems.*

family of lambda terms	**family of rooted maps**	**OEIS**[75]
linear	3-valent (genus $g \geq 0$)	A062980
ordered	planar 3-valent	A002005
unitless linear	bridgeless 3-valent ($g \geq 0$)	A267827
unitless ordered	bridgeless planar 3-valent	A000309
normal linear/$\sim$	(all maps of genus $g \geq 0$)	A000698
normal ordered	planar	A000168
normal unitless linear/$\sim$	bridgeless ($g \geq 0$)	A000699
normal unitless ordered	bridgeless planar	A000260

Table 1: Known correspondences between λ-terms and maps.

2 Background

In this section we briefly review some of the foundational background material as well as recent work necessary for understanding the context of our proposal.

2.1 Lambda calculus, types, categories, and linear logic

The lambda calculus was invented by Church in the late 1920s as part of an ambitious project to build a foundation for mathematics around the concept of function [23]. Although his original system turned out to be logically inconsistent, Church was able to extract from it two separate systems [28, 29] that remain of paramount interest to this day [3, 4], with a typed calculus for logic and an untyped calculus for pure computation. In both its typed and untyped forms, the characteristic feature of lambda calculus is so-called *(lambda) abstraction*, written $\lambda x.t$, which intuitively denotes a function that given an input value x produces output t. This intuition is formalized by the *β-reduction* rule $(\lambda x.t)(u) \to t[u/x]$, which reduces the application of an abstraction term $\lambda x.t$ to another term u by substituting u for every occurrence of the variable x in t.

In the 1960s and 1970s, especially through the work of Lawvere [56] and Lambek [53], a close connection was established between typed lambda calculus and the theory of cartesian closed categories, which are categories with both products $A \times B$ and internal function spaces $B \to C$, thereby inducing natural isomorphisms of hom-sets $\lambda : \mathrm{Hom}(A \times B, C) \xrightarrow{\sim} \mathrm{Hom}(A, B \to C)$. Around the same time, Dana Scott discovered the first non-trivial mathematical model of untyped lambda calculus, which he later axiomatized through the notion of a *reflexive object* in a cartesian closed category [73]. Such models were initially very surprising in that they provided a solution to the "paradoxical" equation $U \cong U \to U$, which for cardinality reasons is impossible to satisfy in the category of sets except for the trivial case of a one-element set $U = \{*\}$.

Jean-Yves Girard's formulation of linear logic [40] in the 1980s brought forth many new perspectives on lambda calculus, and in particular drew renewed attention to its linear subsystem, defined by the property that in any abstraction term $\lambda x.t$, the variable x must occur *exactly once* in t. Computationally this has the effect that the linear system is no longer Turing-complete but rather complete for polynomial-time [57], and categorically, that cartesian closed categories are replaced by the weaker class of symmetric monoidal closed categories [55]. Despite these restrictions, a still richer picture emerges upon consideration of *exponential modalities,* which faithfully embed the classical system into an extended linear system. For example, the ordinary function space may be decomposed in terms of the linear function space and an exponential modality, the so-called Girard translation $A \to B \cong\, !A \multimap B$. In the context of the present proposal, it is worth mentioning the theory of *generalized species* [37], which applies this decomposition to go beyond Joyal's original theory of combinatorial species [49], and has been used for example to construct new models of untyped lambda calculus.

2.2 Graphs on surfaces and map enumeration

Graphs on surfaces or *maps* [54] are a very old object of study in mathematics, going back at least to Francis Guthrie's formulation of the Four Color Problem in 1852, and arguably much earlier to the study of platonic and archimedean solids as well as of tessellations of the plane. As with other natural concepts, maps admit several equivalent definitions. Topologically, a map may be defined as an embedding of a graph into a connected oriented surface such that the complement of the graph inside the surface is a union of simply connected regions (see Figure 1), considered up to deformation of the underlying surface; algebraically, as a set equipped with a transitive action of the group $G = \langle v, e, f \mid vef = e^2 = 1\rangle$ (the generators v, e, and f may be interpreted as *vertices, edges,* and *faces,* respectively), considered up to G-equivariant isomorphism; and combinatorially, as a connected graph equipped with a cyclic ordering of the edges around each vertex, considered up to order-respecting graph isomorphism. (Remarkably, all three definitions are equivalent, in the sense that they induce an equivalence of categories, cf. [48].) Certain families of maps play a special role in graph theory and other contexts. For example, *3-valent maps* are particularly important both in topology and algebra, corresponding to the duals of triangulations and to subgroups of the modular group $\mathrm{PSL}(2,\mathbb{Z}) \cong \langle x, y \mid x^3 = y^2 = 1\rangle$. Every map has a well-defined *genus* g corresponding to the number of holes of its underlying surface, which may also be calculated by the Euler characteristic formula $\#v - \#e + \#f = 2 - 2g$. In particular a *planar map* is a map of genus $g = 0$. A map is *bridgeless* if it remains connected upon the removal of any edge. Bridgeless maps play a role in the theory of map coloring: for example, the Four Color Theorem [78] is formally the statement that *every bridgeless planar map has a proper face-4-coloring,* and by a well-known reduction it is equivalent to the statement that *every bridgeless planar 3-valent map has a proper edge-3-coloring* (see Figure 2).

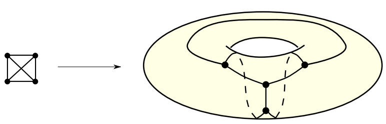

Figure 1: A graph living in a torus.

A *rooted map* is a map equipped with a distinguished root. The study of rooted maps was initiated by Tutte in a series of papers on the combinatorics of planar maps, starting in the early 1960s [81], and has by now developed into a highly active subfield of combinatorics [72, 21]. While the classical theory of maps is often formulated in terms of surfaces without boundary, it is possible to consider boundaries as distinguished faces representing "gaps" in the surface [35]. After removing these faces what is left is an *open graph* in the sense that some edges have "external" ends: see Figure 3 for such depictions of *rooted 3-valent maps* as trivalent maps on a surface with boundary, where we use a small circle to mark one external end as the root.

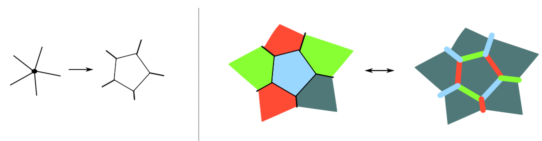

Figure 2: Tait's reduction of 4CT to edge-coloring of 3-valent maps.

2.3 Relating maps and λ-terms

A first connection between lambda calculus and map enumeration was established by Bodini, Gardy, and Jacquot [15] in the form of a bijection between linear lambda terms and rooted 3-valent maps of arbitrary genus. Independently, Zeilberger and Giorgetti [91] found a different bijection between β-normal ordered linear lambda terms and rooted planar maps of arbitrary vertex degree. One may wonder whether these connections form part of a larger pattern, and as witnessed by Table 1, indeed they do!

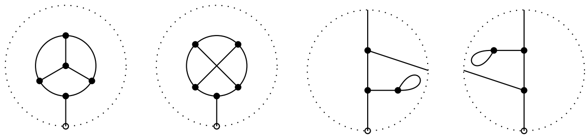

Figure 3: Some small rooted 3-valent maps.

We make a few remarks about these results established in followup work. The correspondences in the upper half of the table can all be obtained as the restrictions of a single natural bijection, which is understandable in complementary ways: algorithmically, the linear term corresponding to a 3-valent map can be computed efficiently via depth-first traversal [15]; conceptually, a linear term may be viewed as an endomorphism of a reflexive object, and the corresponding 3-valent map as a "string diagram" [87]; and algebraically, linear terms and 3-valent maps can be organized into isomorphic symmetric operads [88].

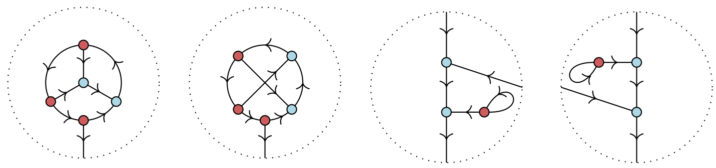

Figure 4: String diagrams of linear terms corresponding to the rooted 3-valent maps in Fig. 3, with red vertices marking lambda abstractions and blue vertices applications. For example, the diagram at the left represents the term B = $\lambda x.\lambda y.\lambda z.x(yz)$, and the next diagram C = $\lambda x.\lambda y.\lambda z.(xz)y$ (cf. [4, p.194]).

The lower half of the table remains less well-understood from a bijective standpoint, but has been nonetheless productive in building connections between lambda calculus and other topics (such as *chord diagrams* [31] and *Tamari intervals* [90]), via their mutual links with map enumeration. Moreover, rather than merely superficial, these connections appear to run surprisingly deep, with certain natural properties of maps corresponding to natural properties of lambda terms and vice versa. For example, planar maps correspond to terms that are *ordered,* that is, in which variables are used in the order they are bound, while bridgeless maps correspond to terms that have no closed subterms. (The latter we refer to as "unitless". Both of these properties also have natural categorical descriptions: ordered terms may be interpreted by morphisms in *non-symmetric* closed categories, and unitless terms by morphisms in *non-unital* closed categories.) Finally, these connections have been applied in quite unexpected ways, for instance to give a reformulation of the Four Color Theorem as a statement about typing of lambda terms [87, 88], see Figure 5 for an illustration.

It should be said that although these enumerative links have (surprisingly!) only been recently discovered, the use of graphical syntax for lambda terms itself is not new, going back at least to the 1970s [52, 86, 76], and indeed it fits within a rich tapestry of ideas connecting logic and algebra with geometry, from Penrose diagrams [67] to linear logic proof-nets [40]. The novelty of the combinatorial perspective (perhaps in a similar spirit to Joyal and Street's work on categorical string diagrams [50, 51]) is that it both places such connections in a broader context and also *makes them more precise,* for example by pointing to the existence of bijections that allow to directly transfer ideas and results between different domains.

2.4 Analytic combinatorics

The discipline of *analytic combinatorics,* according to Flajolet and Sedgewick [38], "aims to enable precise quantitative predictions of the properties of large combinatorial structures [...] through a careful

$$\dfrac{\dfrac{}{x : \beta \multimap \gamma \vdash x : \beta \multimap \gamma} \quad \dfrac{\dfrac{}{y : \alpha \multimap \beta \vdash y : \alpha \multimap \beta} \quad \dfrac{}{z : \alpha \vdash z : \alpha}}{y : \alpha \multimap \beta, z : \alpha \vdash y(z) : \beta}}{\dfrac{x : \beta \multimap \gamma, y : \alpha \multimap \beta, z : \alpha \vdash x(yz) : \gamma}{\dfrac{x : \beta \multimap \gamma, y : \alpha \multimap \beta \vdash \lambda z.x(yz) : \alpha \multimap \gamma}{\dfrac{x : \beta \multimap \gamma \vdash \lambda y.\lambda z.x(yz) : (\alpha \multimap \beta) \multimap (\alpha \multimap \gamma)}{\vdash \lambda x.\lambda y.\lambda z.x(yz) : (\beta \multimap \gamma) \multimap ((\alpha \multimap \beta) \multimap (\alpha \multimap \gamma))}}}}$$

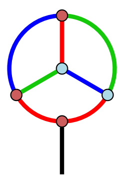

Figure 5: Principal typing derivation for the B term (cf. Figure 4), and the corresponding edge-coloring obtained by taking α, β, γ to be three distinct non-zero values of the Klein Four Group (here colored α = red, β = blue, γ = green) and interpreting $A \multimap B := -A + B$.

combination of symbolic enumeration methods and complex analysis". More precisely, the approach consists in first associating to a class of objects a formal enumerative generating series $f(z) = \sum a_n z^n$ where a_n is the number of objects of size n, and to then see this series (when possible) as an analytical function at the origin. Besides the so-called "symbolic method" for automatically constructing the generating function $f(z)$ from a formal specification of the class of combinatorial objects (which bears similarities with species theory), the great advantage of the complex analytic perspective lies in the use of Cauchy's theorem, which expresses the n-th coefficient of the series in the form of a contour integral $a_n = \frac{1}{2i\pi}\int_\gamma f(z)/z^{n+1}dz$. As a consequence, one obtains the important Flajolet–Odlysko theorems which correlate the growth of the coefficients a_n with the behavior of the associated function in the vicinity of its dominant singularity. The techniques of analytic combinatorics have found applications across diverse disciplines, particularly in computer science to the analysis of algorithms and data structures. More recently these techniques have been applied to the enumeration of lambda terms and maps of arbitrary genus [14, 16, 13].

3 Objectives and methodology

Our project is grouped in two interdisciplinary themes, which are divided below into several workpackages describing specific research directions and promising leads.

3.1 Logical perspectives on maps and related objects

The first group of workpackages seek to unify perspectives coming from lambda calculus and category theory with deep results and observations coming from combinatorics. The aim here is not only to build a better understanding of objects such as lambda terms and maps, but also to stimulate the development of new techniques and theories.

3.1.1 WP1a: A bilingual dictionary between graph theory and lambda calculus

As we saw in Section 2.3, under the correspondences of Table 1, certain natural properties of maps correspond to natural properties of lambda terms and vice versa: planarity corresponds to well-ordering of variables while bridgelessness corresponds to absence of closed subterms. It is natural to wonder how far this dictionary may be extended and applied. For instance, planarity of maps is a special case of bounded genus. Do the linear lambda terms of fixed genus or bounded genus $\leq g$ have a natural logical characterization, for $g > 0$?

To give another example, in graph theory, the property of being bridgeless is a special case of k-*edge-connectivity*, for $k = 2$. What does it mean for a lambda term to be, say, 3-edge-connected or 4-edge-connected? One of us presented a preliminary proposal in this direction [89], the rough idea being to allow "generalized" subterms of higher type (in the sense of *higher-order abstract syntax* [46]). Thus, if we let U be a reflexive object modelling linear lambda terms, then a 1-cut corresponds to an ordinary subterm of type U, a 2-cut to a subterm of type $U \to U$, a 3-cut to a subterm either of type $U \to (U \to U)$ or of type $(U \to U) \to U$, etc. See Figure 6 for an example. We believe that this provisional definition is promising, but much work remains to develop it from a mathematical perspective. For example, how should we formulate and prove the analogues of standard results in graph theory [82], such as the decomposition of a 2-edge-connected graph into 3-edge-connected components, of a 3-edge-connected graph into 4-edge-connected components, etc.

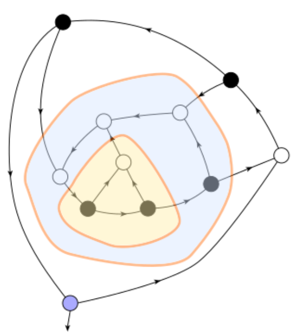

Figure 6: Example of a 3-edge-connected planar (= ordered linear) term $t = \lambda a.\lambda b.\lambda c.a(\lambda d.\lambda e.\lambda f.(b(cd))(ef))$. We have highlighted two different 3-cuts in t of type $(U \to U) \to U$ (in yellow) and of type $U \to (U \to U)$ (in blue).

Moreover, such logical characterizations of graph- or map-theoretic properties lead to questions of a general scientific nature:

First, is the lambda calculus perspective *useful,* for example by enabling us to derive old results about maps in a more systematic way, or even to derive new results? Here, preliminary evidence suggests the answer is yes. For instance, in very recent work [18, 19], we proved that the number of bridges in 3-valent maps of arbitrary genus asymptotically obeys a Poisson law with parameter 1, by proving the corresponding property for the number of closed subterms in linear lambda terms (i.e., as $n \to \infty$, a random closed linear lambda term of size $3n + 2$ has exactly k closed proper subterms with probability approaching $e^{-1}/k!$; in particular, it is bridgeless with probability $1/e$). Still, this is a question we aim to answer more definitively by proving deeper and more varied results.

Second, how do such graph- and map-theoretic properties interact with properties that are more traditionally studied in lambda calculus, like normalization and typing? The fact that edge-connectivity plays a central role in the theory of graph coloring [47] makes it not unreasonable to expect it to have similar relevance to typing of lambda terms. Likewise, one may wonder whether Mairson's result establishing PTIME-completeness of normalization in linear lambda calculus [57] may be adapted for the planar and/or bounded genus subsystems of lambda calculus, which a priori could have strictly lower complexity. Here, we should note that both β-reduction $(\lambda x.t)(u) \to t[u/x]$ as well as the dual rule of η-expansion $t \to \lambda x.t(x)$ have natural topological interpretations (see Figure 7), and that they do not increase the genus of the corresponding map. Although not directly related to the complexity of normalization, it is also worth mentioning recent work by Nguyên and Pradic [65], who used a variant of ordered linear lambda calculus to obtain a characterization of star-free languages within the framework of implicit complexity, relying on the planarity constraint in an essential way.

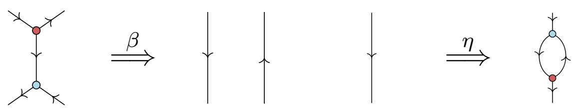

Figure 7: β-reduction and η-expansion of linear lambda terms as certain natural surgeries on trivalent graphs (corresponding to the "unzipping" and "bubbling" moves of [79]).

3.1.2 WP1b: Bijections with blossoming trees, walks in the quarter-plane, and more

The simple formulas for counting various families of planar maps originally obtained by Tutte through analytic means led to the desire for simpler explanations, and eventually to bijective accounts by placing maps in correspondence with various families of trees. For instance, Cori and Vauqelin's *well-labeled trees* [30] and Schaeffer's *balanced blossoming trees* [71] have been used to give alternative bijective proofs of Tutte's formula $\frac{2(2n)!3^n}{n!(n+2)!}$ for the number of rooted planar maps with n edges (A000168, cf. sixth row of Table 1). These structures have found further applications to random generation, as well as important applications in the study of large random maps, see [72] for a survey.

In the context of our project, it is natural to try to relate such structures to lambda terms. Of course, we know it must be possible to derive bijections between these objects via their mutual correspondence with maps, but we have reason to hope that a careful analysis of their relationship will be moreover *illuminating.* Indeed blossoming trees themselves already bear a strong resemblance to lambda terms. Consider the process of transforming a balanced blossoming tree to a rooted 4-valent planar map: moving counterclockwise along the boundary of the tree, each black leaf is paired with a uniquely determined white leaf to form a rooted 4-valent planar map equipped with a canonical 2-orientation of its edges (i.e., such that every vertex has two incoming edges and two outgoing edges, see Figure 8 for an example), and finally the orientation is forgetten. The fact that the orientation is canonical means that the process may be reversed to go from rooted 4-valent planar maps to balanced blossoming trees. This appears quite analogous to the process of transforming the syntactic tree of a linear lambda term to a rooted 3-valent (not necessarily planar) map: each lambda abstraction is paired with the unique variable it binds to form a rooted 3-valent map equipped with a canonical (2,1)-orientation of its edges (i.e., such that every vertex either has two incoming edges and one outgoing edge or one incoming edge and two outgoing edges), and finally this orientation is forgetten. The fact that this orientation is canonical means that the process may be reversed to go from a rooted 3-valent map to a linear lambda term (cf. Figures 3 and 4, as well as [88, §3.3]).

In a different angle of attack, Bernardi [9] gave the first bijective proof of a result by Kreweras, that the number of *plane lattice walks* that start and end at the origin, remain in the non-negative quadrant,

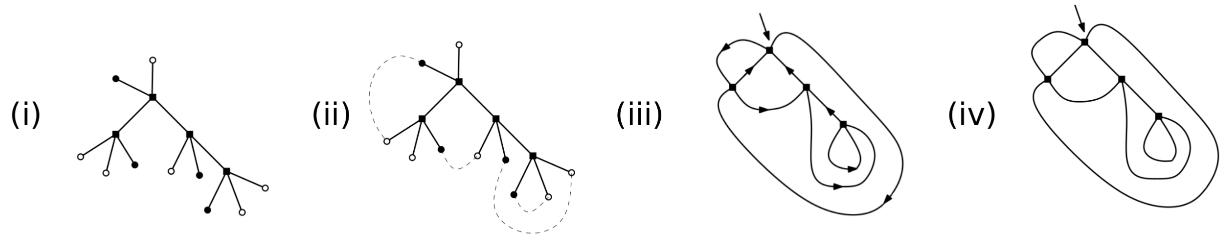

Figure 8: (i) An unrooted balanced blossoming tree T. (ii) The canonical matching of black and white leaves of T. (iii) The associated rooted 4-valent planar map equipped with a canonical 2-orientation. (iv) The result of forgetting the orientation. (Diagrams taken from Figure 1.17 of [72].)

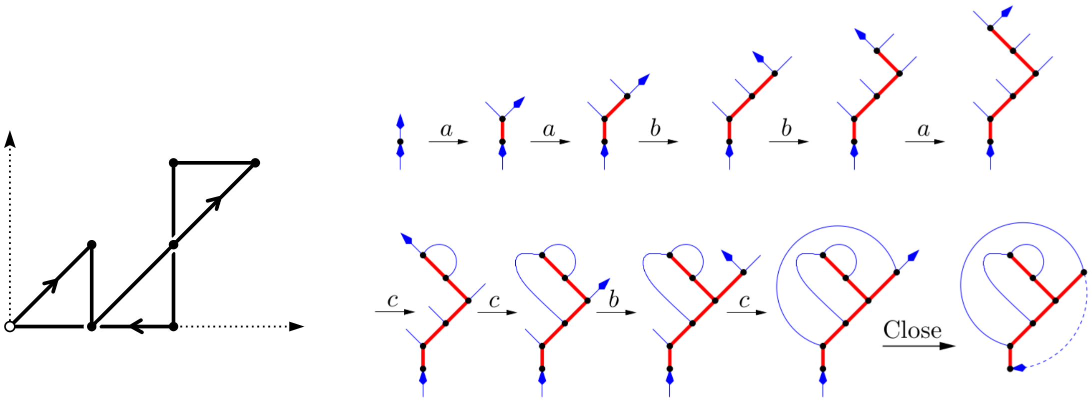

Figure 9: A Kreweras walk in the quarter-plane, and the construction of the corresponding bridgeless planar 3-valent map equipped with a distinguished depth tree. (Diagrams on the right taken from Figure 19 of [9]. Note that the map is constructed by reading the walk in reverse, with the labels indicating the types of steps $a = \leftarrow$, $b = \downarrow$, and $c = \nearrow$.)

and take $3n$ steps of the form $\nearrow$, $\leftarrow$, or $\downarrow$ (see left side of Figure 9), is counted by the simple formula $\frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}$. Bernardi's proof makes use of planar 3-valent maps equipped with a certain kind of spanning tree called a *depth tree* (see right side of Figure 9), and his paper likewise provided a new proof of Tutte's formula $\frac{2^{n+1}(3n)!}{n!(2n+2)!}$ for the number of rooted bridgeless planar 3-valent maps with $2n$ vertices (A000309, cf. fourth row of Table 1). Around the same time, he also gave the first bijective proof that the number of planar maps of size n rooted by a general spanning tree is counted by the simple formula $\mathcal{C}_n\mathcal{C}_{n+1}$ (where $\mathcal{C}_n$ is the n-th Catalan number), by defining a bijection between such tree-rooted maps and shuffles of parenthesis systems, the latter being equivalently defined as walks in the quarter-plane using only steps of the form $\uparrow$, $\rightarrow$, $\leftarrow$, or $\downarrow$ [10]. Again, there are clear motivations for trying to understand Bernardi's constructions (which incidentally have found unexpected recent applications in probability theory [74, 45]) from a lambda calculus point-of-view, and our preliminary investigations are promising. For example, the notion of depth tree precisely captures *well-scoping* in lambda calculus, that is, the condition that a bound variable is not referenced outside the body of a lambda abstraction. These observations open up the exciting possibility of studying more general lattice walks from a lambda calculus perspective, and vice versa, with the aim of transporting ideas and results across these two rich disciplines.

Lastly, we mention yet another surprising connection with the combinatorics of Tamari lattices, posets whose Hasse diagrams are the 1-skeleta of the fundamental polytopes known as *associahedra* [64, 24]. In 2006, Chapoton [25] wrote an influential paper demonstrating a link between the number of intervals in the Tamari lattice of order n and Tutte's formula $\frac{2(4n+1)!}{(n+1)!(3n+2)!}$ for the number of 3-connected planar triangulations with $2n$ triangles (A000260, cf. last row of Table 1). Although Chapoton's proof was purely symbolic, a first bijective proof was found by Bernardi and Bonichon [11], and another one more recently by Fang [36], who also related 3-connected planar triangulations and Tamari intervals with bridgeless planar maps and *closed flows on forests* [27]. Again, a relationship between Tamari intervals and certain lambda terms is suggested by these mutual correspondences and we have even found a direct bijection, but the deeper implications of this connection are still unclear (cf. [90, p.5]).

3.1.3 WP1c: Category-theoretic and operadic views of combinatorics

As recalled in Section 2, there are many longstanding and fruitful connections between lambda calculus and category theory, going back at least to the late 1960s, which has moreover helped to build bridges between logic and other fields such as physics and topology [2]. There is also a modern trend of using category-theoretic concepts to better understand techniques from combinatorics, particularly building on the fundamental concept of *species* introduced by Joyal [49, 7], as well as the related concept of *operad* [62, 42].

In this spirit, different members of the team have explored links...

- between the combinatorics of lattice walks and the algebra of the monoidal category of Sup-Lattices [69, 70];
- between enumeration of pattern-avoiding syntax trees and non-symmetric operads [44];
- between the combinatorics of opetopes (which model higher categories) and proof theory [77];
- between enumeration of Tamari intervals and skew monoidal categories [90, 85];

...and more. In broad terms, we see this project as an opportunity to make new advances, taking advantage of the precise connections already mentioned as well as our team's unique interdisciplinary composition. More than a specific set of problems, this workpackage therefore represents a unifying outlook that we will apply over the course of the project.

Still, we would like to highlight one particularly natural group of problems that arise in this context. The bijective results already obtained, as well as the research we shall develop within WP1a and WP1b, suggest the existence of a strong connection between various types of geometric and combinatorial objects (maps, trees, walks, ...) and various theories of closed and monoidal categories (skew, braided, symmetric, ...). We shall investigate in particular whether these geometric and combinatorial objects may be organized into closed or monoidal categories that are free in some sense. Indeed, the bijections that were discovered by the PIs suggest that the functor from some free structured category to the category of these objects is full and faithful. The kind of research we aim to is somewhat parallel to and inspired from the foundational work by Joyal and Street relating string diagrams to braided monoidal categories [50, 51]. Therefore, as a second step, we shall investigate further this analogy and establish (or refute) the existence of a continuous path leading from the work just mentioned to our research. If possible, we shall establish a general framework by which to tackle this kind of problem.

3.2 Quantitative perspectives on lambda calculus and related systems

The second group of workpackages turns from the conceptual to the quantitative, building on some of our team members' expertise in analytic combinatorics as well as in sophisticated type systems and denotational semantics to build a deeper understanding of fine-grained, statistical properties of lambda calculus and related systems.

3.2.1 WP2a: Asymptotic analysis of parameters in lambda calculus and maps

As mentioned in Section 2.4, the powerful techniques of analytic combinatorics have been recently applied towards enumeration of lambda terms and maps, for example to estimate the asymptotic number of pure (not necessarily linear) lambda terms weighted by length of their De Bruijn representation [16], as well as to estimate the distributions of various parameters in large random maps of arbitrary genus [13]. In WP1a we also referenced more recent work [18, 19] on estimating the asymptotic distribution of parameters such as number of closed subterms of linear lambda terms = bridges in 3-valent maps, as well as number of free variables = external vertices.

However, it should be noted that in these contexts the associated generating series often turn out to be non-analytic (i.e., convergent only at 0), which makes the direct use of most of the standard theorems of analytic combinatorics impossible. Up to now this has been resolved by the introduction of various patches, such as Borel resummation, Bender linearization, etc. Moreover, the combinatorial classes associated to lambda terms and maps are often described by non-linear differential equations, or functional equations that are difficult to manipulate. For such classes, there is no well-established theorem to study the asymptotics of size, and even fewer tools to study the distribution of parameters. A general theory or set of tools would therefore be of great value.

Along different lines, the combinatorics of reduction in lambda calculus is still a wide-open territory, despite some impressive results such as that asymptotically almost all pure lambda terms are strongly normalizing [32] and that almost all simply typed terms have a long β-reduction sequence [1]. For example, the linear setting appears to raise the prospects for making more precise quantitative statements about the asymptotic distribution of statistics such as *length of longest β-reduction sequence* $\ell_\beta(t)$ starting at a term $t = t_0 \to^\beta t_1 \to^\beta t_2 \to^\beta \cdots \to^\beta t_n$, or *distance to β-normal form* $d_\beta(t)$, that is, the length of the shortest β-reduction sequence $t = t_0 \to^\beta t_1 \to^\beta t_2 \to^\beta \cdots \to^\beta t_n = v \not\to^\beta$ from t to its unique β-normal form v. Indeed, by linearity and confluence these statistics are equivalent (each β-reduction decreases the size of a linear term by 3 on the way to its unique normal form, so $\ell_\beta(t) = d_\beta(t) = \frac{|t|-|v|}{3}$), and experimentally they appear to be normally distributed: see Figure 10!

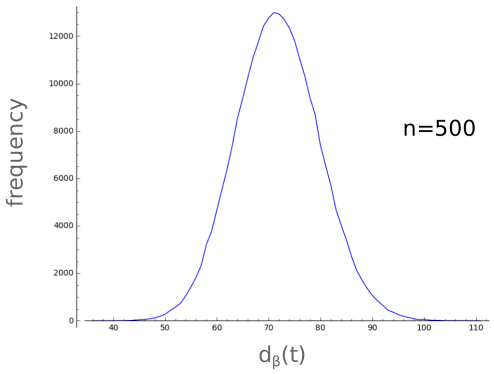

Figure 10: Histogram of distance to β-normal form (or equivalently length of longest β-reduction sequence), for randomly sampled closed linear terms of size $3 \cdot 500 + 2$.

Currently, it is not obvious what strategy could be used to prove this conjecture. However, such questions about the combinatorics of normalization potentially have nice connections with a line of research on *quantitative semantics* going back to Girard's normal functor semantics [41], which can in turn be seen as a precursor to Fiore et al.'s theory of generalized species [37] mentioned in Section 2.1. More recently, such execution time-sensitive semantics have also been tied to (non-idempotent) *intersection type systems* [33, 8], linking with WP2c, as well as to categorical models of *differential lambda calculus* [58, 66], linking with WP1c.

3.2.2 WP2b: Random sampling and experimental lambda calculus

The consortium of this project includes specialists in random generation of combinatorial structures (including under the versatile Boltzmann model [34, 17, 5]), and the ability to efficiently sample lambda terms satisfying varying constraints would open up a space for what could be called "experimental lambda calculus". As a concrete example of what we mean by this, consider Figure 10 again. The underlying data of the histogram was derived by first generating 262700 random linear terms of size 1502 using the uniform sampling algorithm of [15] (which has been implemented in the LinLam library for Haskell), and then simply computing how many steps each term takes to normalize. Note that asymptotically there are $t_n \sim \frac{3 \cdot 6^n n!}{\pi}$ closed linear terms of size $3n + 2$, so of course exhaustive enumeration would have been impossible here with $t_{500} \approx 1.4 \cdot 10^{1523}$. We plan to develop efficient, parameterized samplers for various classes of terms, including for ordered linear terms using the bijections of WP1b. Naturally, such samplers will also help us to investigate the statistical properties of different classes and formulate conjectures, tying with WP2a.

At the same time, we plan to study the dynamics of lambda calculus and other rewriting systems from an analytic perspective, developing techniques to rigorously establish properties such as the aforementioned conjecture, as well as properties of more sophisticated evaluation strategies for general lambda terms. Such strategies include *randomized evaluation* [22]. Similar in nature to random walks, randomized evaluation of general lambda terms (see Figure 11) deals with new phenomenona related to erasing, copying and postponing of future possible execution paths. Such behaviors may appear exotic from the point-of-view of traditional combinatorics, although they have connections with well-studied

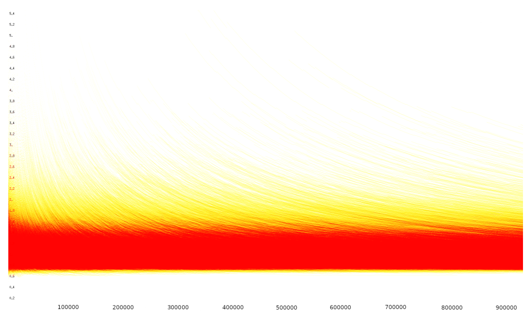

Figure 11: Experimental distribution of the size of the non-linear term $Y\ \lambda c.\lambda x.\lambda y.c(cxy)(cxy)$ (where $Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$ is a standard fixed-point combinator) over a million iterations of randomized evaluation; the graph is renormalized by a factor of $n\log(n)\log^2(n)$, the conjectured asymptotic size.

problems such as compression [39]. Still, we will certainly need to develop new analytic tools, which conversely may be able to illuminate other facets of lambda calculus and functional programming that are still not completely understood, such as lazy evaluation.

3.2.3 WP2c: Typed enumeration

With a few notable exceptions [63], most work on the combinatorics of lambda calculus has focused on enumeration of untyped terms. There are good conceptual reasons for taking this as a starting point, in line with the view "à la Curry" of typed terms as refinements of untyped terms [3, Ch. 1]. Still, there are both practical and theoretical reasons to be interested in typed enumeration. For example, in modern languages programs are often annotated with types, and one may be interested in enumerating or sampling all programs of a given type (in the spirit of program synthesis [68]). From a mathematical perspective, types add an interesting dimension that requires new tools on the side of the combinatorics.

On the one hand, as already mentioned, there is a tight connection between typing lambda terms and coloring maps (cf. Figure 5), meaning that the problem of typed enumeration is apparently closely related to the problem of enumerating colorings. Actually, that problem occupied Tutte over much of his career, from his work in the late 1940s and '50s on chromatic polynomials [80] to his work in the 1970s and '80s on "chromatic sums" [83], and formed part of his original motivation for studying enumeration of (uncolored) planar maps (cf. [84, Ch. 10]). More recently, Tutte's work has been revisited from a modern perspective by Bousquet-Mélou et al. [20, 12], and it would be fascinating to try to translate this work to the setting of typed lambda calculus.

On the other hand, types are usually considered to have a more rarefied algebraic structure than colors. Categorically, Curry-style type systems may be interpreted as functors $\mathcal{D} \to \mathcal{T}$ from a category whose morphisms are typing derivations to a category whose morphisms are terms [61], so that typing reduces to finding an appropriate "lifting" of a morphism in $\mathcal{T}$ to a morphism in $\mathcal{D}$ (see Figure 12). In the case of type systems for lambda calculus, $\mathcal{D}$ and $\mathcal{T}$ are often assumed to be cartesian or symmetric monoidal closed categories, perhaps with some additional structure, such as that required to interpret intersection types.

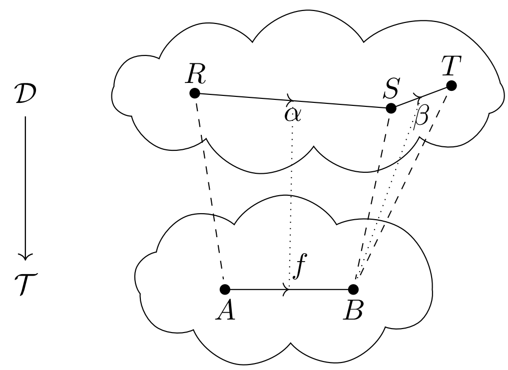

Figure 12: Type systems as functors. Here the morphism $\alpha : R \to S$ in $\mathcal{D}$ may be considered abstractly as a "typing derivation" for the morphism $f : A \to B$, and the morphism $\beta : S \to T$ as a subtyping derivation over the identity morphism on B, cf. [61].

As a somewhat more subtle approach that seems to relieve a bit of the tension between coloring and typing, one may consider replacing categories by *colored operads* (a.k.a. "multicategories") and considering operadic functors $\mathcal{D} \to \mathcal{T}$, typically where $\mathcal{T}$ has a single color (i.e., is an ordinary operad). Notably, Mazza et al. have used this view as a way of organizing intersection type systems and obtaining modular proofs of normalization for varieties of lambda calculi [59, 60]. In this vein, it is worth mentioning that although the use of operads in combinatorics is by now well-established [26, 62], colored operads have only recently begun to find applications – including in work by one of us [43] – and our project therefore represents a perfect opportunity to further develop this line of research.

Finally, in the context of analytic combinatorics, some of us [14, 6] have used infinite, algebraic systems of multivariate generating functions to determine statistical properties of different parameters of lambda terms (cf. WP2a). Such infinite specification systems appear closely related to typing, a connection that also deserves further exploration.

Part IV
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(Where open access versions are available we have indicated “oa” below, with a clickable link in the pdf version of this document.)

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