MPRI 2010-11 Cours 2-7-1

Examen du 30-11-2010

1 Théorie des types

In Martin-Löf's Type Theory, one constructs a closed term :

$$f : \Pi x : N.\Pi y : N.\Sigma z : N.x = y + z \vee y = x + z$$

What can be the normal forms of :
- $\pi_1(f\ 5\ 3)$
- $(f\ 3\ 5)$
- $(f\ 5\ 5)$

2 Logique du 1er ordre

One reminds that normal λ-terms can always be written

$$\lambda x_1. \ldots \lambda x_n.(x\ t_1\ \ldots\ t_m)$$

One considers the first-order language built with a single proposition symbol A. Take the following proof of $[] \vdash A \Rightarrow A$.

$$\dfrac{\dfrac{\dfrac{}{[A \Rightarrow A] \vdash A \Rightarrow A}\ (\text{Ax})}{[] \vdash (A \Rightarrow A) \Rightarrow (A \Rightarrow A)}\ (\Rightarrow\text{-I}) \qquad \dfrac{\dfrac{}{[A] \vdash A}\ (\text{Ax})}{[] \vdash A \Rightarrow A}\ (\Rightarrow\text{-I})}{[] \vdash A \Rightarrow A}\ (\Rightarrow\text{-E})$$

a) Give the cut-free version of this proof.
b) What are these two proofs seen through the Curry-Howard isomorphism ?
c) Are there other cut-free proofs of $[] \vdash A \Rightarrow A$? Justify brievly.
d) We now add the rewrite rule $A \ \triangleright\ (A \Rightarrow A)$. We are thus now in deduction modulo. Give a proof of $[] \vdash A$. What can be said of the corresponding λ-term ?

3 Système T

Define in System T a function $\mathtt{eg} : N \to N \to N$
which returns 1 when its arguments are equal and 0 when they are not.

4 Is choosing constructive ?

We place ourselves in Heyting's arithmetic (that is intuitionistic arithmetic)
a) Explain how one can prove the proposition

$$\forall x.\forall y.x = y \vee \neg(x = y)$$

One may find inspiration in the previous question.
b) A proposition P is decidable when one can prove $P \vee \neg P$.
Show tha if A and B are decidable, then $A \wedge B$, $A \vee B$ et $A \Rightarrow B$ are decidable.
c) One now adds a choice operator, or "Hilbert operator" to arithmetic. That is for every proposition P (of the language of arithmetic) and every variable x one has :

- an object $\mathcal{E}(x.P)$ in the language,
- a family of axioms $P[x \setminus t] \Rightarrow P[x \setminus \mathcal{E}(x.P)]$ (for every object t).

Show that if $P[x \setminus t]$ is decidable, for any t, then $\exists x.P$ is decidable, in this theory.

d) Show that if $P[x \setminus t]$ is decidable, for any t, then $\forall x.P$ is decidable, in this theory.

e) What can one thus say about all propostions of this theory ? or this theory in general. Try to be precise and concise.

5 Déconstruire est-ce choisir ?

We go back to Martin-Löf's type theory. We give ourselves the excluded middle axiom :

$$\frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash EM_A : A + (A \to \bot)}$$

a) In a context where : A : Type, $a : A$ et $P : A \to$ Type,
build a term $\mathcal{E}(P) : A$ such that :

$$\Sigma x : A.(P\ x) \to (P\ \mathcal{E}(P)).$$

b) Does the construction of the previous question seem possible in type theory ? What is the general conclusion ?