All you ever wanted to know about Pure Type Systems

Vincent Siles

PPS - INRIA - Ecole Polytechnique

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1. What is a Pure Type System ?
2. An Alternative Presentation: PTS with Judgmental Equality
3. Equivalence

Definitions

- PTSs are a way to have general results over families of type systems (System **F**, Calculus of Constructions, Simply-Typed λ-Calculus,...).

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- The validity of typing judgments relies on two sets:
  - *Ax* is used to type sorts .
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- Reduction :

$$(\lambda x^A.M)\ N \ \xrightarrow{\beta}\ M[N/x] + \text{congruences}$$

Typing Rules

$$\frac{}{\emptyset_{wf}} \qquad \frac{\Gamma \vdash A : s \quad x \notin Dom(\Gamma)}{(\Gamma, x : A)_{wf}} \qquad \frac{\Gamma_{wf} \quad (s,t) \in \mathcal{A}x}{\Gamma \vdash s : t} \qquad \frac{\Gamma_{wf} \quad \Gamma(x) = A}{\Gamma \vdash x : A}$$

$$\frac{\begin{array}{cc} \Gamma \vdash A : s & \Gamma, x : A \vdash B : t \\ (s,t,u) \in \mathcal{R}el & \Gamma, x : A \vdash M : B \end{array}}{\Gamma \vdash \lambda x^A.M : \Pi x^A.B}$$

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$$\frac{\Gamma \vdash M : \Pi x^A.B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B[x/N]} \qquad \frac{\Gamma \vdash M : A \qquad A \overset{\beta}{\equiv} B \qquad \Gamma \vdash B : s}{\Gamma \vdash M : B}$$

Example: How to get the STλC

PTS instantiation for Simply-Typed λ-Calculus:

$S = \{$  ,  $\}$ $Ax = \{($  ,  $)\}$ $Rel = \{($  ,  ,  $)\}$

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- $$\frac{[\,] \vdash\ :\ \qquad x :\ \vdash\ :\ }{[\,] \vdash \Pi x.\ : !!!}$$ $\implies (\ \rightarrow\ )$ is not a valid.

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- $[\,] \vdash$ 🍎 $:$ 👶 Ok
- $$\frac{[\,] \vdash A : 🍎 \qquad x : A \vdash B : 🍎}{[\,] \vdash \Pi x^A . B : 🍎}$$ $\longrightarrow$ we need a term typed by 🍎

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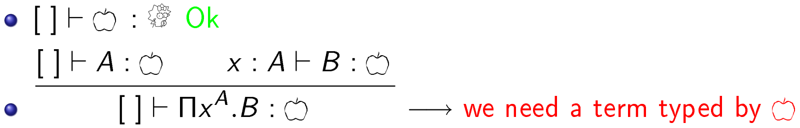
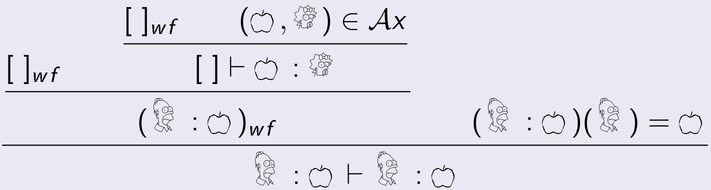


As usual, we introduce *base types* as a fresh variables:



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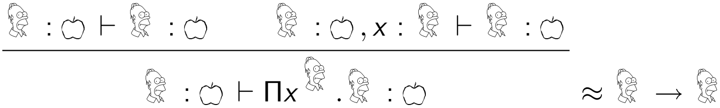
- 

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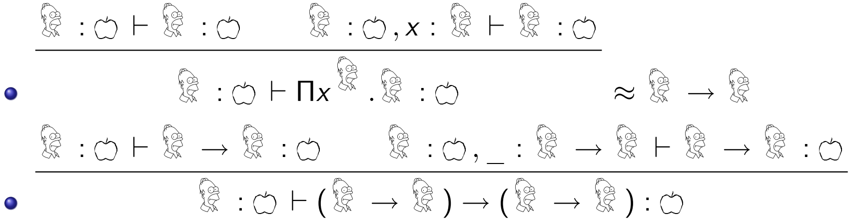

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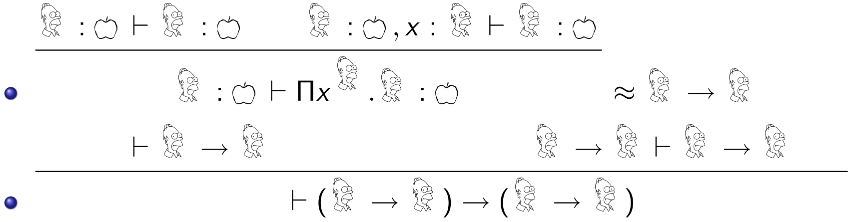

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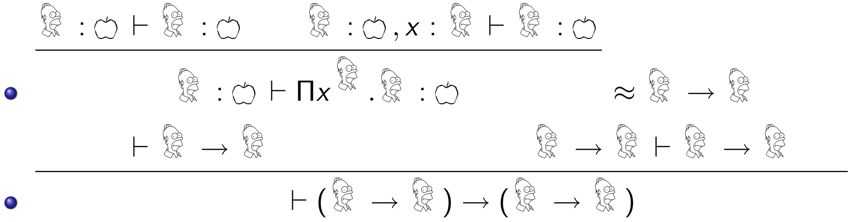

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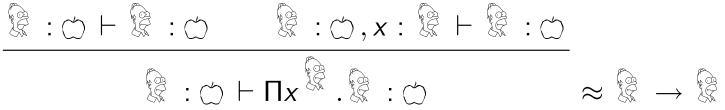
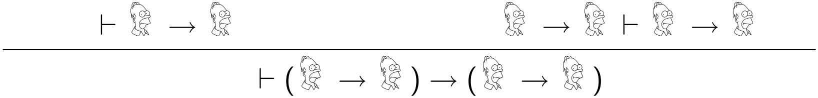
- 
- 
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With those rules, we can only build *non-dependent* types.

Facts about β-conversion

Some basic properties of β-reduction:

- Church-Rosser property:

if $M \xrightarrow{\beta}\!\!\!\!\rightarrow N$ and $M \xrightarrow{\beta}\!\!\!\!\rightarrow P$ then there is M' such that $N \xrightarrow{\beta}\!\!\!\!\rightarrow M'$ and $P \xrightarrow{\beta}\!\!\!\!\rightarrow M'$.

- Confluence:

if $M \stackrel{\beta}{\equiv} N$ then there is P such that $M \xrightarrow{\beta}\!\!\!\!\rightarrow P$ and $N \xrightarrow{\beta}\!\!\!\!\rightarrow P$.

- Injectivity of Products:

If $\Pi x^A.B \stackrel{\beta}{\equiv} \Pi x^C.D$ then $A \stackrel{\beta}{\equiv} C$ and $B \stackrel{\beta}{\equiv} D$.

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- Inversion lemmas :

e.g. if $\Gamma \vdash \lambda x^A.M : T$ then there are s, t, u and B such that

- $(s, t, u) \in \mathcal{R}el$, $T \stackrel{\beta}{\equiv} \Pi x^A.B$
- $\Gamma \vdash A : s$ and $\Gamma, x : A \vdash B : t$ and $\Gamma, x : A \vdash M : B$.

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If $\Gamma \vdash M : T$ then there is $s \in S$ such that $T = s$ or $\Gamma \vdash T : s$.

- Subject Reduction:

If $\Gamma \vdash M : T$ and $M \xrightarrow{\beta} M'$ then $\Gamma \vdash M' : T$.

Facts about PTS

- Shape of Types (Jutting [93]):

If $\Gamma \vdash M : A$ and $\Gamma \vdash M : B$, then

- either $A \overset{\beta}{\equiv} B$
- or $A \overset{\beta}{\twoheadrightarrow} \Pi x^{A_1} \ldots x^{A_n}.s$ and $B \overset{\beta}{\twoheadrightarrow} \Pi x^{A_1} \ldots x^{A_n}.t$

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- Not-Fact Normalization: there are some non-terminating PTS ($\circlearrowleft : \circlearrowleft$).
- Not-Fact Type Checking / Inference : type checking dependent types is undecidable.
- Not-Fact Expansion Postponement : replace conversion with reduction only:

If $\Gamma \vdash M : T$ then $\Gamma \vdash' M : T'$ and $T \overset{\beta}{\twoheadrightarrow} T'$.

Why do we want a typed equality ?

- In the conversion rules the intermediate steps are not checked.

$$\frac{\Gamma \vdash M : A \qquad A \overset{\beta}{\equiv} B \qquad \Gamma \vdash B : s}{\Gamma \vdash M : B}$$

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$\hookrightarrow$ all this lead to the definition of PTS *with Judgmental Equality*.

PTSe typing rules (1)

$$\frac{}{\emptyset_{wf_e}} \qquad \frac{\Gamma \vdash_e A : s \quad x \notin Dom(\Gamma)}{(\Gamma, x : A)_{wf_e}} \qquad \frac{\Gamma_{wf_e} \quad (s,t) \in \mathcal{A}x}{\Gamma \vdash_e s : t} \qquad \frac{\Gamma_{wf_e} \quad \Gamma(x) = A}{\Gamma \vdash_e x : A}$$

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PTSe typing rules (3)

$$\frac{\Gamma \vdash_e M = M' : A \qquad \Gamma \vdash_e A = B : s}{\Gamma \vdash_e M = M' : B}$$

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Are both systems the same ?

Easy part of the equivalence

We proove by mutual induction that

- If $\Gamma \vdash_e M : T$ then $\Gamma \vdash M : T$.
- If $\Gamma \vdash_e M = N : T$ then $\Gamma \vdash M : T$, $\Gamma \vdash N : T$ and $M \overset{\beta}{\equiv} N$.
- If Γ_{wf_e} then Γ_{wf}.

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Here we just "loose" some information, nothing complicated.

Hard Part

The other way around needs a way to "type" a β-equivalence into a judgmental equality:

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But can we ?

How do we do this ?

$$\Gamma \vdash M : T \qquad M \qquad \stackrel{\beta}{\equiv} \qquad N \qquad \Gamma \vdash N : T$$

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$$\begin{tikzcd} \Gamma \vdash M : T & M \arrow[dr, twoheadrightarrow] & \overset{\beta}{\equiv} & N \arrow[dl, twoheadrightarrow] & \Gamma \vdash N : T \\ \Gamma \vdash_e M : T & & P & & \Gamma \vdash_e N : T \end{tikzcd}$$

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- P is welltyped in PTS by *Subject Reduction.*
- Is P welltyped in PTSe ?
- How do we type $M = P$ and $N = P$ in PTSe ?

The need of Subject Reduction

To do so, we need to proove that PTSe have the *Subject Reduction* property:

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- for semi-full and full PTS : V. Siles and H. Herbelin [10] “Equality is typable in Semi-Full Pure Type Systems”.
- But the question is still open for general PTS !

That's all folks !

Thank you for your time.