Investigation on the typing of equality in type systems

PhD Defense

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Nov 25th, 2010

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With types, we can state general properties about these families like "vegetables are good for health".

Types in sciences

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- $\texttt{plus} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a function.

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Types can also be used to describe mathematical objects or data-structures, and these information can be helpful to avoid mistakes in programs:

- π is a real number, 1664 is a natural number ($\mathbb{N}$), $[1, 2, 3, 5, 7]$ is a list of natural numbers, ...
- `plus` : $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a function.
- `plus`$(\pi, 1664)$ is an *ill-typed* program since `plus` is expecting two natural numbers, while π is not one.

Proofs and Types (1)

- Types can be used to ensure the correctness of programs:

  "If `length` computes the length of a list, then we know that, if `l` is a valid list, then `length(l)` is valid (and has to be a natural number)."

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- If *when it's snowing, I'm cold* and *it's snowing*, then we can conclude that *I'm cold*.

Proof and Types (2)

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All of these proof assistants are based on particular *type theories* which ensure their correctness.

1. Pure Type Systems
2. Equivalence and Typed Reduction
3. Further extensions and conclusion

Simply Typed λ-Calculus

$$
\begin{array}{rcl}
M, N & ::= & x \mid \lambda x^A.M \mid M\ N \\
A, B & ::= & a \mid A \to B \\
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- STLC was presented by Church in the 1930's.
- Since then, several extensions have been studied: *"Type:Type", System-***F***, Calculus of Constructions . . .*
- Each extension has more expressive power than the previous one (polynomials, second order arithmetic . . .)
- All these type systems share a common core, but their meta-theory were studied one at a time.

What is a dependent type

To make a dependent version of `list`, we can add the *length* to the type:

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- concatenation of two non-dependent lists is of type
$$\mathtt{list} \rightarrow \mathtt{list} \rightarrow \mathtt{list}.$$
- the dependent version (`concat`) is of type
$\Pi n^{nat}.\Pi m^{nat}.\quad \mathtt{list}\ n \rightarrow \mathtt{list}\ m \rightarrow \mathtt{list}\ (n+m)$

$\Pi x^{A}.B$ is called a *dependent product*.

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†We write $A \to B$ when B does not depend on the input.

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- PTSs are an abstraction of Barendregt's λ-cube, presented independently by Berardi and Terlouw.
- To be able to deal with all the different type systems, PTSs have parameters that describe which type is valid: $\mathit{Sorts}, \mathcal{A}x$ and $\mathcal{R}el$.

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Some typing rules for PTSs

STLC		PTS
$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A.M : A \to B}$$	$\Rightarrow$	$$\frac{\Gamma \vdash \Pi x^A.B : s \qquad \Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A.M : \Pi x^A.B}$$
$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B}$$	$\Rightarrow$	$$\frac{\Gamma \vdash M : \Pi x^A.B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B[N/x]}$$

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$\dfrac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B}$	$\Rightarrow$	$\dfrac{\Gamma \vdash M : \Pi x^A.B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B[N/x]}$

$$\dfrac{\Gamma \vdash M : A \qquad A =_\beta B \qquad \Gamma \vdash B : s}{\Gamma \vdash M : B}\ \text{CONV}$$

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  $\Pi n^{nat}.\Pi m^{nat}.\quad \texttt{list}\ n \rightarrow \texttt{list}\ m \rightarrow \texttt{list}\ (n+m)$
- $\texttt{concat 4 4 l l} =_\beta [1,3,5,7,1,3,5,7]\ :\ \texttt{list}\ (4+4)$

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- `concat 4 4 l l` $=_\beta [1, 3, 5, 7, 1, 3, 5, 7]$ `:` `list` (4+4)

The conversion rule is here to *compute* at the level of types and change `list` (4+4) into `list` 8.

Facts about PTSs

For example:

Type Correctness

If $\Gamma \vdash M : T$ then there is $s \in \mathit{Sorts}$ such that $T \equiv s$ or $\Gamma \vdash T : s$.

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A more complex one:

Subject Reduction

If $\Gamma \vdash M : T$ and $M \to_\beta M'$ then $\Gamma \vdash M' : T$.

Needs Injectivity of Π-types: If $\Pi x^A.B =_\beta \Pi x^C.D$ then $A =_\beta C$ and $B =_\beta D$. (Easy by confluence of β-reduction)

Untyped conversion considered harmful ?

What if the path between A and B is “ill-typed” ?

$$\frac{}{(\lambda x^{A'}.P)\ Q =_\beta P[Q/x]} \qquad \frac{\Gamma \vdash M : A \qquad A =_\beta B \qquad \Gamma \vdash B : s}{\Gamma \vdash M : B}$$

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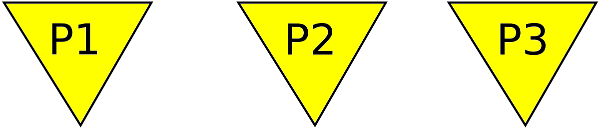
Let's consider *P* to be the following proof of $\Gamma \vdash M : B$.



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By Confluence:

$$\frac{\overset{\text{P1}}{\Gamma \vdash M : A} \quad \overset{\text{P'2}}{A \twoheadrightarrow_\beta C} \quad \overset{\text{P"2}}{B \twoheadrightarrow_\beta C} \quad \overset{\text{P3}}{\Gamma \vdash B : s}}{\Gamma \vdash M : B}$$

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By Type Correctness and Subject Reduction:

$$\dfrac{\dfrac{\overset{\text{P1}}{\Gamma \vdash M : A} \quad \overset{\text{P'2}}{A \twoheadrightarrow_\beta C} \quad \overset{\text{SR}}{\Gamma \vdash C : t}}{\Gamma \vdash M : C} \qquad \overset{\text{P"2}}{B \twoheadrightarrow_\beta C} \quad \overset{\text{P3}}{\Gamma \vdash B : s}}{\Gamma \vdash M : B}$$

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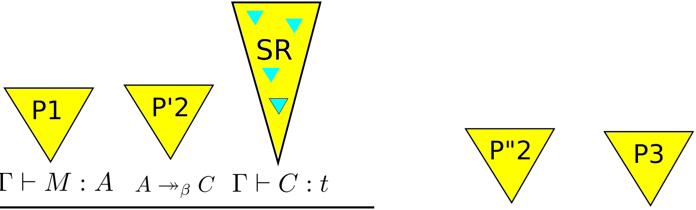
But Subject Reduction introduces new harmful conversions:



$$\dfrac{\dfrac{\Gamma \vdash M : A \quad A \twoheadrightarrow_\beta C \quad \Gamma \vdash C : t}{\Gamma \vdash M : C} \qquad B \twoheadrightarrow_\beta C \quad \Gamma \vdash B : s}{\Gamma \vdash M : B}$$

PTS with typed equality (PTSe)

To avoid this possibility, we can *type* every step of the conversion. They are called *semantical PTSs* or *PTSs with typed equality* [Geuvers93]. We have now two typing judgments:

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PTS with typed equality

Untyped β-equality is quite "small":

$$\frac{}{(\lambda x^A.M)\ N =_\beta M[N/x]}$$

$$\frac{A =_\beta A' \qquad M =_\beta M'}{\lambda x^A.M =_\beta \lambda x^{A'}.M'}$$

PTS with typed equality

Typed β-equality is notably "bigger":

$$\frac{\begin{array}{c}\Gamma, x : A \vdash_e M : B \qquad \Gamma \vdash_e N : A \\ \Gamma \vdash_e A : s \qquad \Gamma, x : A \vdash_e B : t \qquad (s, t, u) \in \mathcal{R}el\end{array}}{\Gamma \vdash_e (\lambda x^A.M)N =_\beta M[N/x] : B[N/x]}$$

$$\frac{\begin{array}{c}\Gamma \vdash_e A =_\beta A' : s \qquad \Gamma, x : A \vdash_e M =_\beta M' : B \\ \Gamma, x : A \vdash_e B : t \qquad (s, t, u) \in \mathcal{R}el\end{array}}{\Gamma \vdash_e \lambda x^A.M =_\beta \lambda x^{A'}.M' : \Pi x^A.B}$$

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To prove this as we did for PTSs, we need Π-Injectivity for typed equality judgments, which is a really difficult question for PTSe since it relies on (typed) property of *Confluence*, which relies on *Subject Reduction*,

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Strong Π-Injectivity : a counterexample

If $\Gamma \vdash_e \Pi x^A.B =_\beta \Pi x^C.D : u$ then there are s, t such that $\Gamma \vdash_e A =_\beta C : s$, $\Gamma(x : A) \vdash_e B =_\beta D : t$ and $(s, t, u) \in \mathcal{R}el$.

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By using the identity function as a *coercion*, one can restrict the different types of a term. Let's consider a particular PTS with (u, v) and (u, v') in the definition of its axioms:

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4. $\emptyset \vdash_e \Pi x^{M_1}.u =_\beta \Pi x^{M_2}.u : w$

By injectivity, we would be able to get $\emptyset \vdash_e M_1 =_\beta M_2 : s$ for some s, which is impossible.

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If we assume that this equality enjoys Π-injectivity, then it is enough to prove *Subject Reduction* for PTSe. Sadly, there is no known proof of that at the moment.

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Another way to prove Subject Reduction for PTSe would be to use the Subject Reduction we have for PTSs. We need to prove some kind of equivalence between both systems.

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A more practical reason why we are looking for this equivalence is about *proof assistants*. Usually, the implementation is done with an untyped equality, whereas the consistency proof is done with a typed equality. Such an equivalence would bring closer the implementation from its theory.

The Big Question

Are PTSs and PTSe the same systems ?

[Geuvers93]

Easy part of the equivalence

We prove by mutual induction that

- If $\Gamma \vdash_e M : T$ then $\Gamma \vdash M : T$.
- If $\Gamma \vdash_e M =_\beta N : T$ then $\Gamma \vdash M : T$, $\Gamma \vdash N : T$ and $M =_\beta N$.
- If Γ_{wf_e} then Γ_{wf}.

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- If Γ_{wf_e} then Γ_{wf}.

Here we just “lose” some information, nothing complicated.

Difficult part of the equivalence

The other way around needs a way to “type” a β-equivalence into a judgmental equality:

- If $\Gamma \vdash M : T$ then $\Gamma \vdash_e M : T$.
- If $\Gamma \vdash M : T$, $\Gamma \vdash N : T$ and $M =_\beta N$ then $\Gamma \vdash_e M =_\beta N : T$.
- If Γ_{wf} then Γ_{wf_e}.

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- Is P well-typed in PTSe ?
- How do we type $M =_\beta P$ and $N =_\beta P$ in PTSe ?

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- A first *syntactical* criterion was shown for a subclass of PTSs [Adams06] called *functional* PTSs, by adding annotations inside the syntax of terms.
- By using the same intermediate system, Herbelin and I extended this result to other subclasses of PTSs called *semi-full* and *full*.

Extension of the TPOSR solution

Adams introduced an additional annotation inside the applications:

$$M, N, A, B \quad ::= \quad x \mid \lambda x^A.M \mid M_{(x)B}\ N \mid \Pi x^A.B \mid s$$

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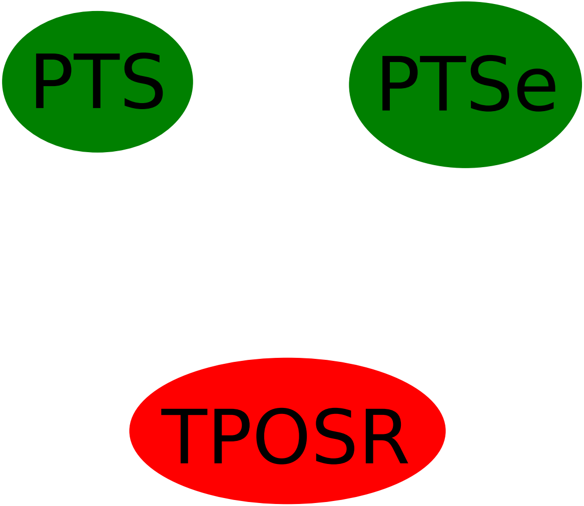
Also, its system called *Typed Parallel One Step Reduction* is no longer based on equality but on *reduction*:

$$\frac{\Gamma \vdash M \rhd N : A \qquad \Gamma \vdash A \cong B : s}{\Gamma \vdash M \rhd N : B}$$

$$\frac{\begin{array}{c}\Gamma \vdash A \rhd A' : s \qquad \Gamma, x : A \vdash B \rhd B' : t \\ \Gamma, x : A \vdash M \rhd M' : B \qquad \Gamma \vdash N \rhd N' : A \qquad (s, t, u) \in \mathcal{R}el\end{array}}{\Gamma \vdash (\lambda x^A.M)_{(x)B} N \rhd M'[N'/x] : B[N/x]}$$

Summary of the proof



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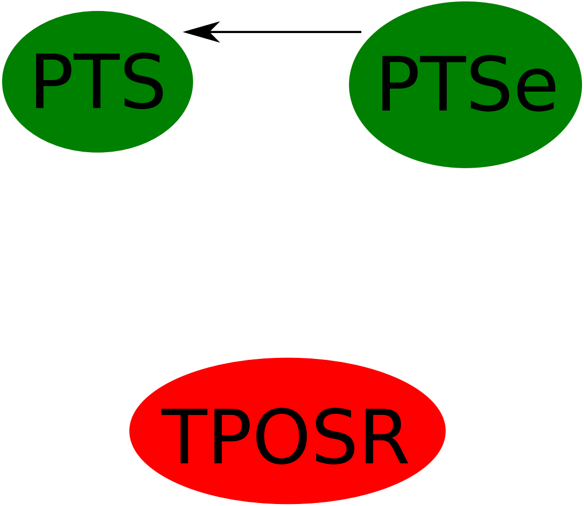

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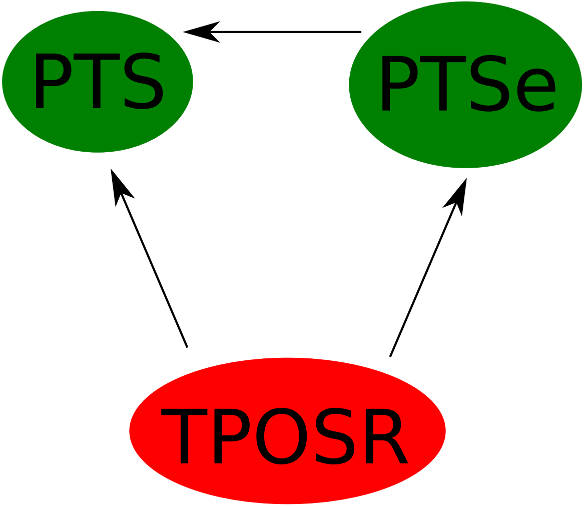

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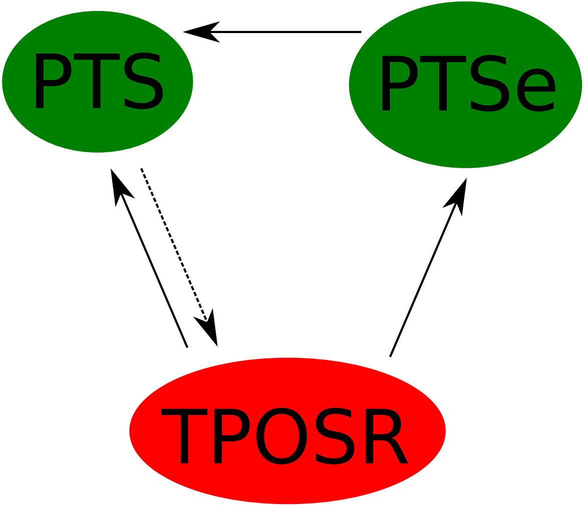

Road map to a proof of equivalence

The idea is to prove that:

- TPOSR's equality is *Confluent*.
- TPOSR's equality has *Injectivity of* Π*-types*.
- TPOSR has *Subject-Reduction*.
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$\Uparrow$

tricky part

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Typed Confluence and Injectivity

Diamond property for PTS_{atr}

If $\Gamma \vdash M \rhd N : A$ and $\Gamma \vdash M \rhd P : B$ then there is Q such that $\Gamma \vdash N \rhd Q : A, B$ and $\Gamma \vdash P \rhd Q : A, B$.

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The proof is easier than the proof for TPOSR because of the additional annotations. However, these annotations will give us extra work in the following properties.

As a direct consequence:

Π-Injectivity for PTS_{atr}

If $\Gamma \vdash \Pi x^A.B \cong \Pi x^C.D$ then $\Gamma \vdash A \cong C$ and $\Gamma, x : A \vdash B \cong D$.

Typed Subject Reduction and annotations

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Subject Reduction for PTS_{atr}

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The proof is almost the same as the usual one for PTSs. Some additional work is required for the β case: we need to provide the A_0 that links both annotations.

$$\frac{\dots \qquad \Gamma \vdash A_0 \rhd^+ A : s \qquad \Gamma \vdash A_0 \rhd^+ A' : s}{\Gamma \vdash (\lambda x^A.M)_{\Pi x^{A'}.B} N \rhd M'[N'/x] : B[N/x]}$$

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If $\Gamma \vdash M : T$, then there is Γ^*, M^* and T^* such that $\Gamma^* \vdash M^* \rhd M^* : T^*$, where $|\Gamma^*| \equiv \Gamma$, $|M^*| \equiv M$ and $|T^*| \equiv T$.

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- Thanks to Subject Reduction of PTS$_{atr}$, the conversion rule is no longer a problem, but we still have to compute some valid Γ^*, M^* and T^*.

Annotations and Typing

The proof is done by induction:

$$\frac{\Gamma \vdash A : s \qquad \Gamma, x : A \vdash B : t \qquad (s, t, u) \in \mathcal{R}el}{\Gamma \vdash \Pi x^A.B : u}$$

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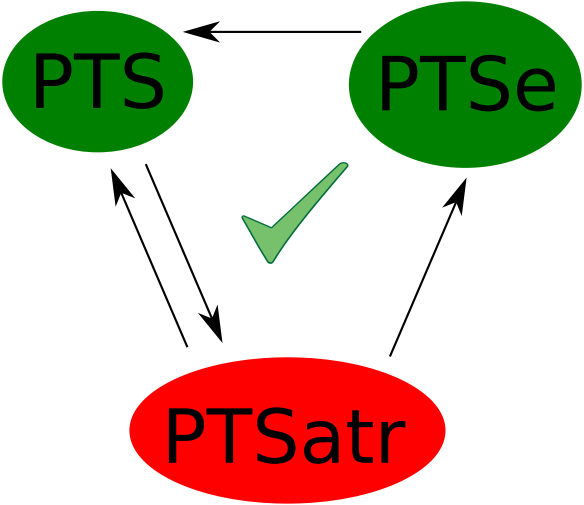
- We need a way to glue things together:

Erased Conversion

If $|A| \equiv |B|$, and if A and B are well-formed types in PTS_{atr}, then $\Gamma \vdash A \cong B$.

The proof of this lemma is very technical, and the most difficult proof of this thesis.

Complete Equivalence



Consequences of the equivalence

Complete Equivalence:

$$\begin{cases} \Gamma \vdash_e M : T & \textit{iff} \quad \Gamma \vdash M : T \\ \Gamma \vdash_e M =_\beta N : T & \textit{iff} \quad \Gamma \vdash M : T, \Gamma \vdash N : T \text{ and } M =_\beta N \\ \Gamma_{wf} & \textit{iff} \quad \Gamma_{wf_e} \end{cases}$$

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Proving Subject Reduction for PTSe is now trivial:

- If $\Gamma \vdash_e M : T$ and $M \to_\beta N$, then $\Gamma \vdash M : T$.
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Corollary: Weak Π-Injectivity

If $\Gamma \vdash_e \Pi x^A.B =_\beta \Pi x^C.D$ then $\Gamma \vdash_e A =_\beta C$ and $\Gamma, x : A \vdash_e B =_\beta D$.

By the way

Proved in Coq

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$$\Updownarrow$$

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Following ideas from [Lengrand06], we tried to switch to a view of PTSs based on *Sequent Calculus*, but failed at finding a final answer to this problem.

To infinity . . . and beyond !

In order to apply such a result to full scale type systems like the ones behind proof assistants, we need to extend the theory.

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- Adding inductive types to have more usable datatypes.
- Trying to add η-expansion to the conversion.
- Adding universes *à la* Martin-Löf: towards CC_ω and *CIC*.

The Calculus of Constructions with Universes: CC_ω

We tried several different extensions in order to deal with subtyping in the CC_ω type system [Miquel01] but we didn't yet find a proper solution.

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The two main issues we faced were quite different:

- By keeping the same annotation process, the *Erased Confluence* lemma is no longer true, so we can't annotate a PTS into PTS_{atr}.
- All the attempts to fix the way we deal with the annotations have broken the *Church-Rosser* lemma.

We still need to find the right way to deal with universes and subtyping.

Conclusion

Implementation contributions:

- Extension of the formalization started by Barras of the meta theory of PTSs and PTSe by adding the full meta theory of PTS_{atr} and the complete proof of equivalence.
- Formalization of a part of [Lengrand06] about PTSs in *Sequent Calculus* with some extensions.

Theoretical contributions:

- A new system with typed reduction that enjoys all the good properties of usual PTSs, without relying on normalization.
- The right notion of equality at the level of types for PTSe, which enjoys the *Injectivity of* Π*-types*.
- A final answer to the link between PTS and PTSe: they are completely equivalent.

We finally have a **unified** theory of Pure Type Systems.