

Equality is typable in Semi-Full Pure Type Systems

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July 11th, 2010

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- Reduction :

$$(\lambda x^A. M) \ N \xrightarrow{\beta} M[N/x] + \text{congruences}$$

PTS typing rules

$$\frac{}{\emptyset_{wf}} \quad \frac{\Gamma \vdash A : s \quad x \notin \text{Dom}(\Gamma)}{(\Gamma, x : A)_{wf}} \quad \frac{\Gamma_{wf} \quad (s, t) \in \mathcal{A}x}{\Gamma \vdash s : t} \quad \frac{\Gamma_{wf} \quad \Gamma(x) = A}{\Gamma \vdash x : A}$$

$$\frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : t \quad (s, t, u) \in \mathcal{R}el \quad \Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A. M : \Pi x^A. B}$$

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$$\frac{\Gamma \vdash M : \Pi x^A. B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B[x/N]} \quad \frac{\Gamma \vdash M : A \quad A \stackrel{\beta}{\equiv} B \quad \Gamma \vdash B : s}{\Gamma \vdash M : B}$$

Some known Type Systems

- Simply-Typed λ -Calculus:

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- Calculus of Constructions:

$$S = \{Prop, Type\} \quad Ax = \{(Prop, Type)\} \\ Rel = \{(s, Prop, Prop), (s, Type, Type)\}$$

Some special classes of PTS

- Functional: If $(s, t) \in \mathcal{A}x$ and $(s, t') \in \mathcal{A}x$ then $t = t'$.
If $(s, t, u) \in \mathcal{R}el$ and $(s, t, u') \in \mathcal{R}el$ then $u = u'$.

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 \hookrightarrow In those PTS, “any” products is typable.
- Semi-full PTS: If $(s, t, u) \in \mathcal{R}el$ then for all t' , there is u' such that $(s, t', u') \in \mathcal{R}el$.
 \hookrightarrow If the product $\Pi x^A. B$ is typable, then for any B' well-typed, $\Pi x^A. B'$ is also well-typed (or Π -functionality).

Facts about PTS

- Inversion lemmas :

e.g. if $\Gamma \vdash \lambda x^A.M : T$ then there are s, t, u and B such that

- $(s, t, u) \in \mathcal{R}el, T \stackrel{\beta}{\equiv} \Pi x^A.B$
- $\Gamma \vdash A : s$ and $\Gamma, x : A \vdash B : t$ and $\Gamma, x : A \vdash M : B$.

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If $\Gamma \vdash M : T$ then there is $s \in S$ such that $T = s$ or $\Gamma \vdash T : s$.

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If $\Pi x^A.B \equiv_{\beta} \Pi x^C.D$ then $A \equiv_{\beta} C$ and $B \equiv_{\beta} D$.

- Subject Reduction:

If $\Gamma \vdash M : T$ and $M \xrightarrow{\beta} M'$ then $\Gamma \vdash M' : T$.

Shape of types in PTS

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- $\forall s \in S, s \in T_s$
- $\forall A, B, \Pi x^A.B \in T_s$
- if $M \in T_s, MN \in T_s$ and $\lambda x^A.M \in T_s$

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 - $\forall A, B, \Pi x^A.B \in T_s$
 - if $M \in T_s, MN \in T_s$ and $\lambda x^A.M \in T_s$
- if $M \in T_v, \Gamma \vdash M : A$ and $\Gamma \vdash M : B$, then $A \stackrel{\beta}{\equiv} B$.
- if $M \in T_s, \Gamma \vdash M : A$ and $\Gamma \vdash M : B$, then $A \stackrel{\beta}{\twoheadrightarrow} \Pi x_1^{U_1} \dots x_n^{U_n}.s$ and $B \stackrel{\beta}{\twoheadrightarrow} \Pi x_1^{U_1} \dots x_n^{U_n}.t$.

Why do we want a typed equality ?

- In the conversion rules the intermediate steps are not checked.

$$\frac{\Gamma \vdash M : A \quad A \overset{\beta}{\equiv} B \quad \Gamma \vdash B : s}{\Gamma \vdash M : B}$$

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↪ all this lead to the definition of PTS *with Judgmental Equality*.

PTSe typing rules (1)

$$\frac{}{\emptyset_{wfe}} \quad \frac{\Gamma \vdash_e A : s \quad x \notin \text{Dom}(\Gamma)}{(\Gamma, x : A)_{wfe}} \quad \frac{\Gamma_{wfe} \quad (s, t) \in \mathcal{A}x}{\Gamma \vdash_e s : t} \quad \frac{\Gamma_{wfe} \quad \Gamma(x) = A}{\Gamma \vdash_e x : A}$$

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PTSe typing rules (2)

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$$\frac{\Gamma \vdash_e M = M' : \Pi x^A. B \quad \Gamma \vdash_e N = N' : A}{\Gamma \vdash_e MN = M'N' : B[x/N]}$$

$$\frac{\Gamma \vdash_e A = A' : s \quad \Gamma, x : A \vdash_e B = B' : t \quad (s, t, u) \in \mathcal{R}el}{\Gamma \vdash_e \Pi x^A. B = \Pi x^{A'}. B' : u}$$

$$\frac{\Gamma \vdash_e A = A' : s \quad \Gamma, x : A \vdash_e M = M' : B \quad \Gamma, x : A \vdash_e B : t \quad (s, t, u) \in \mathcal{R}el}{\Gamma \vdash_e \lambda x^A. M = \lambda x^{A'}. M' : \Pi x^A. B}$$

PTSe typing rules (3)

$$\frac{\Gamma \vdash_e M = M' : A \quad \Gamma \vdash_e A = B : s}{\Gamma \vdash_e M = M' : B}$$

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Are both systems the same ?

Easy part of the equivalence

We prove by mutual induction that

- If $\Gamma \vdash_e M : T$ then $\Gamma \vdash M : T$.
- If $\Gamma \vdash_e M = N : T$ then $\Gamma \vdash M : T$, $\Gamma \vdash N : T$ and $M \overset{\beta}{\equiv} N$.
- If Γ_{wf_e} then Γ_{wf} .

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Here we just “lose” some information, nothing complicated.

The other way around needs a way to “type” a β -equivalence into a judgmental equality:

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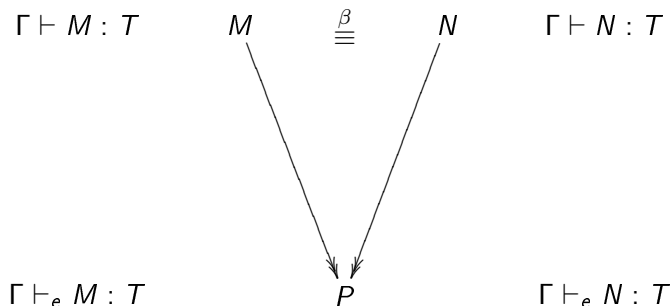
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But can we ?

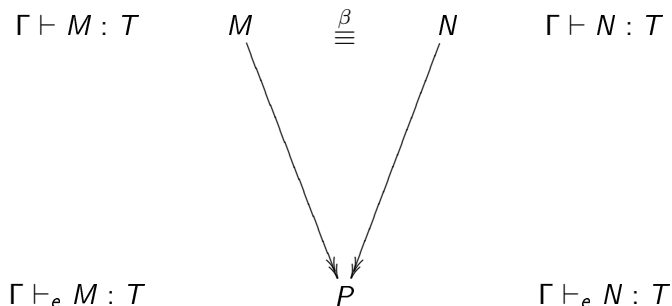
How do we do this ?

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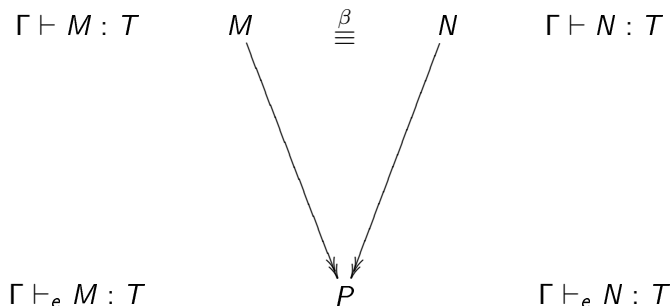


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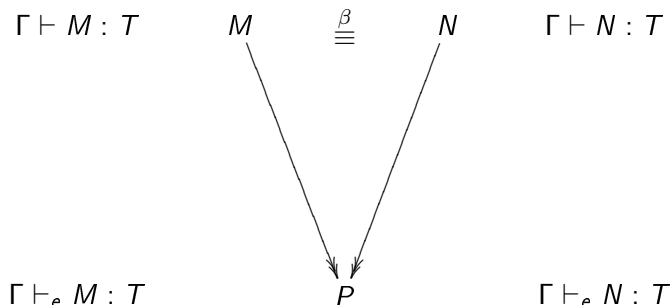
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- Is P welltyped in PTSe ?

How do we do this ?



- P is welltyped in PTS by *Subject Reduction*.
- Is P welltyped in PTSe ?
- How do we type $M = P$ and $N = P$ in PTSe ?

The need of Subject Reduction

To do so, we need to prove that PTSe have the *Subject Reduction* property:

Subject Reduction:

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We only have some partials results:

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- for semi-full and full PTS : V. Siles and H. Herbelin [10] “Equality is typable in Semi-Full Pure Type Systems”.
- But the question is still open for general PTS !

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 - Prove that TPOSR is *Church-Rosser*.
 - Prove that TPOSR has *Subject-Reduction*.
 - Prove that TPOSR is equivalent to PTS **and** PTSe.

TPOSR typing rules (1)

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TPOSR typing rules (1)

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TPOSR typing rules (2)

$$\frac{\Gamma \vdash A \triangleright A' : s \quad \Gamma, x : A \vdash B \triangleright B' : t \quad \Gamma, x : A \vdash M \triangleright M' : B \quad \Gamma \vdash N \triangleright N' : A \quad (s, t, u) \in \mathcal{R}el}{\Gamma \vdash (\lambda x^A.M)_{(x)B} N \triangleright M'[x/N'] : B[x/N]}$$

$$\frac{\Gamma \vdash M \triangleright N : A \quad \Gamma \vdash A \triangleright B : s}{\Gamma \vdash M \triangleright N : B}$$

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We do not keep track of the sort (it requires *Type Uniqueness*).

From TPOSR to PTS and PTSe

Let's consider the $|\cdot|$ function that removes all annotations on applications, we can easily prove the following lemmas:

From TPOSR to PTS

If $\Gamma \vdash M \triangleright N : T$ then $|\Gamma| \vdash |M| : |T|$, $|\Gamma| \vdash |N| : |T|$ and $|M| \xrightarrow{\beta} |N|$.

From TPOSR to PTSe

If $\Gamma \vdash M \triangleright N : T$ then $|\Gamma| \vdash |M| = |N| : |T|$.

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As easy as before by induction, we just remove some information in the derivations.

First step: Church-Rosser

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The main issues are the critical pairs involving the application rules: we are unable to apply some induction hypothesis

- the induction hypothesis over B requires a context “ $\Gamma, x : A$ ”
- we only have an hypothesis “ $\Gamma, x : C \vdash B \triangleright B' : s$ ”
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\hookrightarrow So we need a way to equal A and C .

Functional vs Semi-Full

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By applying the previous lemma to N''' :

- first case : $\Gamma \vdash A \equiv C$
- second case : A and C only differ by their last sort s and t

Back to the Untyped World, second case

If we erase all the equalities we have so far, by *untyped Confluence* we can conclude that:

- $\Pi_{x|A|.|B|} \stackrel{\beta}{\equiv} \Pi_{x|C|.|B|}.$

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With this, we can now finish to prove everything up to *Subject Reduction*

Validity of Annotations

To close the equivalence, we need to prove that the additional annotations on applications did not change the typing system, that is:

Validity of Annotations

If $\Gamma \vdash M : T$, then $\Gamma^* \vdash M^* \triangleright M^* : T^*$

(for all Γ^*, M^*, T^* such than $|\Gamma^*| = \Gamma$, $|M^*| = M$ and $|T^*| = T$).

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(for all Γ^*, M^*, T^* such than $|\Gamma^*| = \Gamma$, $|M^*| = M$ and $|T^*| = T$).

Since there are several ways to annotate a term, the induction can be quite tricky without the following lemma:

Erased Context Conversion

If $\Gamma_1 \vdash M \triangleright N : A$, $|\Gamma_1| = |\Gamma_2|$ and Γ_2 *wf*, then $\Gamma_2 \vdash M \triangleright N : A$.

Erased Conversion: the second pitfall

To prove this conversion lemma, we need a more general lemma which is easily done for functional PTS, but strangely hard for semi-full:

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If $\Gamma \vdash M \triangleright^? : S$, $\Gamma \vdash N \triangleright^? : T$ and $|M| = |N|$, then there is P such that:

- $\Gamma \vdash M \triangleright^+ P : S$
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$$\begin{array}{ll} |M| = |M'| & |N| = |N'| \\ \Gamma \vdash M \triangleright^+ M_0 : \Pi_{x^A}.B & \Gamma \vdash M' \triangleright^+ M_0 : \Pi_{x^{A'}}.B' \\ \Gamma \vdash N \triangleright^+ N_0 : A & \Gamma \vdash N' \triangleright^+ N_0 : A' \\ \Gamma \vdash M_{(x)B} N \triangleright? : B[x/N] & \Gamma \vdash M'_{(x)B'} N' \triangleright? : B'[x/N'] \end{array}$$

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(Very Simplified) Shape of Terms in TPOSR in Ts

If $\Gamma \vdash M \triangleright ? : \prod x_1^{U_1} \dots x_n^{U_n}.s$ then $\Gamma \vdash M \triangleright^+ \lambda x_1^{U_1} \dots x_n^{U_n}.P : \prod x_1^{U_1} \dots x_n^{U_n}.s$
and $\Gamma, x_1 : U_1, \dots, x_n : U_n \vdash P \triangleright P : s$

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By combining Shape of Types and Terms, we can prove that $n > 1$ in our problematic case, thus we can erase the troublesome annotation by performing a β -reduction step first.

Solution to the pitfall

$$\begin{array}{lll}
 \Gamma \vdash M_{(x)B} N & \triangleright^+ & M_0 \text{ }_{(x)B} N \quad : B[x/N] \\
 & \triangleright^+ & (\lambda x^C \lambda \Delta. K)_{(x)B} N \quad : B[x/N] \\
 & \triangleright^+ & \lambda \Delta [x/N]. K[x/N] \quad : B[x/N] \\
 & \triangleright^+ & \lambda \Delta [x/N_0]. K[x/N_0] \quad : B[x/N] \\
 \Gamma \vdash M'_{(x)B'} N' & \triangleright^+ & M_0 \text{ }_{(x)B'} N' \quad : B'[x/N'] \\
 & \triangleright^+ & (\lambda x^{C'} \lambda \Delta. K)_{(x)B'} N' \quad : B'[x/N'] \\
 & \triangleright^+ & \lambda \Delta [x/N']. K[x/N'] \quad : B'[x/N'] \\
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 \end{array}$$

With *Subject Reduction* and *Validity of Annotations*, we are now able to prove that $\text{PTS} \Rightarrow \text{TPOSR}$, and so:

$$\text{PTSe} \Rightarrow \text{PTS} \Rightarrow \text{TPOSR} \Rightarrow \text{PTSe}$$

Possible Extensions of the Proof

There are several ways to enhance the system:

- Change the conversion rule (with η for example).
- Extend the conversion rule with cumulativity : the road to subtyping.

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Possible solutions: adding *Strengthening* as a primitive rule as in ICC, restrict to normalizing systems, only add η -expansion...

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However, even if we manage to prove this, the approach used to proof *Validity of Annotations* do not scale to subtyping, so a new way to prove it still needs to be found.

Conclusion: Where are we ?

What do we have so far:

- + A more precise proof of *Church-Rosser* for TPOSR which works for all useful PTS.

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- Dealing with η -conversion is still the same nightmare
- Subtyping forces us to throw away the *Shape of Types* approach to *Validity of Annotations* and redo it from scratch.

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Thank you for your time. Any questions ?