5. Generalized Duality.

We consider the general form of the linear programming problem, allowing some constraints to be equalities, and some variables to be unrestricted ($-\infty < x_j < \infty$).

The General Maximum Problem. Find x_j, $j = 1, \ldots, n$, to maximize $\boldsymbol{x}^{\mathsf{T}}\boldsymbol{c}$ subject to

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i \quad \text{for } i = 1, \ldots, k$$
$$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad \text{for } i = k+1, \ldots, m$$

and

$$x_j \ge 0 \quad \text{for } j = 1, \ldots, \ell$$
$$x_j \text{ unrestricted} \quad \text{for } j = \ell+1, \ldots, n.$$

The dual to this problem is

The General Minimum Problem. Find y_i, $i = 1, \ldots, m$, to minimize $\boldsymbol{y}^{\mathsf{T}}\boldsymbol{b}$ subject to

$$\sum_{i=1}^{m} y_i a_{ij} \ge c_j \quad \text{for } j = 1, \ldots, \ell$$
$$\sum_{i=1}^{m} y_i a_{ij} = c_j \quad \text{for } j = \ell+1, \ldots, n$$

and

$$y_i \ge 0 \quad \text{for } i = 1, \ldots, k$$
$$y_i \text{ unrestricted} \quad \text{for } i = k+1, \ldots, m.$$

In other words, a strict equality in the constraints of one program corresponds to an unrestricted variable in the dual.

If the general maximum problem is transformed into a standard maximum problem by

1. replacing each equality constraint, $\sum_j a_{ij}x_j = b_i$, by two inequality constraints, $\sum_j a_{ij}x_j =\le b_i$ and $\sum_j(-a_{ij})x_j \le -b_i$, and

2. replacing each unrestricted variable, x_j, by the difference of two nonnegative variables, $x_j = x'_j - x''_j$ with $x'_j \ge 0$ and $x''_j \ge 0$,

and if the dual general minimum problem is transformed into a standard minimum problem by the same techniques, the transformed standard problems are easily seen to be duals by the definition of duality for standard problems. (Exercise 1.)

Therefore the theorems on duality for the standard programs in Section 2 are valid for the general programs as well. The Equilibrium Theorem seems as if it might require