Chapter 5

Preliminaries on polyhedra and linear and integer programming

This chapter surveys what we need on polyhedra and linear and integer programming. Most background can be found in Chapters 7–10, 14, 16, 19, 22, and 23 of Schrijver [1986b]. We give proofs of a few easy further results that we need in later parts of the present book.
The results of this chapter are mostly formulated for real space, but are maintained when restricted to rational space. So the symbol $\mathbb{R}$ can be replaced by the symbol $\mathbb{Q}$. In applying these results, we add the adjective *rational* when we restrict ourselves to rational numbers.

5.1. Convexity and halfspaces

A subset C of $\mathbb{R}^n$ is *convex* if $\lambda x + (1-\lambda)y$ belongs to C for all $x, y \in C$ and each λ with $0 \leq \lambda \leq 1$. A *convex body* is a compact convex set.

The *convex hull* of a set $X \subseteq \mathbb{R}^n$, denoted by conv.hull$X$, is the smallest convex set containing X. Then:

(5.1) $$\text{conv.hull}X = \{\lambda_1 x_1 + \cdots + \lambda_k x_k \mid k \geq 1, x_1, \ldots, x_k \in X, \lambda_1, \ldots, \lambda_k \in \mathbb{R}_+, \lambda_1 + \cdots + \lambda_k = 1\}.$$

A useful fundamental result was proved by Carathéodory [1911]:

Theorem 5.1 (Carathéodory's theorem)**.** *For any $X \subseteq \mathbb{R}^n$ and $x \in$ conv.hull$X$, there exist affinely independent vectors $x_1, \ldots, x_k$ in X with $x \in$ conv.hull$\{x_1, \ldots, x_k\}$.*

(Corollary 7.1f in Schrijver [1986b].)

A subset H of $\mathbb{R}^n$ is called an *affine halfspace* if $H = \{x \mid c^{\mathsf{T}} x \leq \delta\}$, for some $c \in \mathbb{R}^n$ with $c \neq \mathbf{0}$ and some $\delta \in \mathbb{R}$. If $\delta = 0$, then H is called a *linear halfspace*.

Let $X \subseteq \mathbb{R}^n$. The set conv.hull$X + \mathbb{R}^n_+$ is called the *up hull* of X, and the set conv.hull$X - \mathbb{R}^n_+$ the *down hull* of X.

5.2. Cones

A subset C of $\mathbb{R}^n$ is called a *(convex) cone* if $C \neq \emptyset$ and $\lambda x + \mu y \in C$ whenever $x, y \in C$ and $\lambda, \mu \in \mathbb{R}_+$. The cone *generated by* a set X of vectors is the smallest cone containing X:

(5.2) $\text{cone}X = \{\lambda_1 x_1 + \cdots + \lambda_k x_k \mid k \geq 0, \lambda_1, \ldots, \lambda_k \geq 0, x_1, \ldots, x_k \in X\}.$

There is a variant of Carathéodory's theorem:

Theorem 5.2. *For any $X \subseteq \mathbb{R}^n$ and $x \in \text{cone}X$, there exist linearly independent vectors $x_1, \ldots, x_k$ in X with $x \in \text{cone}\{x_1, \ldots, x_k\}$.*

A cone C is *polyhedral* if there is a matrix A such that

(5.3) $C = \{x \mid Ax \leq \mathbf{0}\}.$

Equivalently, C is polyhedral if it is the intersection of finitely many linear halfspaces.

Results of Farkas [1898,1902], Minkowski [1896], and Weyl [1935] imply that

(5.4) a convex cone is polyhedral if and only if it is finitely generated,

where a cone C is *finitely generated* if $C = \text{cone}X$ for some finite set X. (Corollary 7.1a in Schrijver [1986b].)

5.3. Polyhedra and polytopes

A subset P of $\mathbb{R}^n$ is called a *polyhedron* if there exists an $m \times n$ matrix A and a vector $b \in \mathbb{R}^m$ (for some $m \geq 0$) such that

(5.5) $P = \{x \mid Ax \leq b\}.$

So P is a polyhedron of and only if it is the intersection of finitely many affine halfspaces. If (5.5) holds, we say that $Ax \leq b$ *determines* P. Any inequality $c^{\mathsf{T}}x \leq \delta$ is called *valid* for P if $c^{\mathsf{T}}x \leq \delta$ holds for each $x \in P$.

A subset P of $\mathbb{R}^n$ is called a *polytope* if it is the convex hull of finitely many vectors in $\mathbb{R}^n$. Motzkin [1936] showed:

(5.6) a set P is a polyhedron if and only if $P = Q + C$ for some polytope Q and some cone C.

(Corollary 7.1b in Schrijver [1986b].) If $P \neq \emptyset$, then C is unique and is called the *characteristic cone* char.cone(P) of P. Then:

(5.7) $\text{char.cone}(P) = \{y \in \mathbb{R}^n \mid \forall x \in P \forall \lambda \geq 0 : x + \lambda y \in P\}.$

If $P = \emptyset$, then by definition its characteristic cone is char.cone$(P) := \{\mathbf{0}\}$.

(5.6) implies the following fundamental result (Minkowski [1896], Steinitz [1916], Weyl [1935]):

(5.8) a set P is a polytope if and only if P is a bounded polyhedron.

(Corollary 7.1c in Schrijver [1986b].)

A polyhedron P is called *rational* if it is determined by a rational system of linear inequalities. Then a rational polytope is the convex hull of a finite number of rational vectors.

5.4. Farkas' lemma

A system $Ax \leq b$ is called *feasible* (or *solvable*) if it has a solution x. Feasibility of a system $Ax \leq b$ of linear inequalities is characterized by *Farkas' lemma* (Farkas [1894,1898], Minkowski [1896]):

Theorem 5.3 (Farkas' lemma). *$Ax \leq b$ is feasible $\iff$ $y^{\mathsf{T}}b \geq 0$ for each $y \geq \mathbf{0}$ with $y^{\mathsf{T}}A = \mathbf{0}^{\mathsf{T}}$.*

(Corollary 7.1e in Schrijver [1986b].) Theorem 5.3 is equivalent to:

Corollary 5.3a (Farkas' lemma — variant). *$Ax = b$ has a solution $x \geq \mathbf{0}$ $\iff$ $y^{\mathsf{T}}b \geq 0$ for each y with $y^{\mathsf{T}}A \geq \mathbf{0}^{\mathsf{T}}$.*

(Corollary 7.1d in Schrijver [1986b].) A second equivalent variant is:

Corollary 5.3b (Farkas' lemma — variant). *$Ax \leq b$ has a solution $x \geq \mathbf{0}$ $\iff$ $y^{\mathsf{T}}b \geq 0$ for each $y \geq \mathbf{0}$ with $y^{\mathsf{T}}A \geq \mathbf{0}^{\mathsf{T}}$.*

(Corollary 7.1f in Schrijver [1986b].) A third equivalent, affine variant of Farkas' lemma is:

Corollary 5.3c (Farkas' lemma — affine variant). *Let $Ax \leq b$ be a feasible system of inequalities and let $c^{\mathsf{T}}x \leq \delta$ be an inequality satisfied by each x with $Ax \leq b$. Then for some $\delta' \leq \delta$, the inequality $c^{\mathsf{T}}x \leq \delta'$ is a nonnegative linear combination of the inequalities in $Ax \leq b$.*

(Corollary 7.1h in Schrijver [1986b].)

5.5. Linear programming

Linear programming, abbreviated to *LP*, concerns the problem of maximizing or minimizing a linear function over a polyhedron. Examples are

(5.9) $\max\{c^{\mathsf{T}}x \mid Ax \leq b\}$ and $\min\{c^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \geq b\}$.

If a supremum of a linear function over a polyhedron is finite, then it is attained as a maximum. So a maximum is finite if the value set is nonempty and has an upper bound. Similarly for infimum and minimum.

The *duality theorem of linear programming* says (von Neumann [1947], Gale, Kuhn, and Tucker [1951]):

Theorem 5.4 (duality theorem of linear programming)**.** *Let A be a matrix and b and c be vectors. Then*

(5.10) $$\max\{c^{\mathsf{T}}x \mid Ax \leq b\} = \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}\},$$

if at least one of these two optima is finite.

(Corollary 7.1g in Schrijver [1986b].) So, in particular, if at least one of the optima is finite, then both are finite.

Note that the inequality $\leq$ in (5.10) is easy, since $c^{\mathsf{T}}x = y^{\mathsf{T}}Ax \leq y^{\mathsf{T}}b$. This is called *weak duality*.

There are several equivalent forms of the duality theorem of linear programming, like

(5.11)
$$\begin{aligned}
&\max\{c^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \leq b\} = \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \geq c^{\mathsf{T}}\},\\
&\max\{c^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax = b\} = \min\{y^{\mathsf{T}}b \mid y^{\mathsf{T}}A \geq c^{\mathsf{T}}\},\\
&\min\{c^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \geq b\} = \max\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \leq c^{\mathsf{T}}\},\\
&\min\{c^{\mathsf{T}}x \mid Ax \geq b\} = \max\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}\}.
\end{aligned}$$

Any of these equalities holds if at least one of the two optima is finite (implying that both are finite).

A most general formulation is: let $A, B, C, D, E, F, G, H, K$ be matrices and let a, b, c, d, e, f be vectors; then

(5.12)
$$\begin{aligned}
&\max\{d^{\mathsf{T}}x + e^{\mathsf{T}}y + f^{\mathsf{T}}z \mid x \geq \mathbf{0}, z \leq \mathbf{0},\\
&Ax + By + Cz \leq a,\\
&Dx + Ey + Fz = b,\\
&Gx + Hy + Kz \geq c\}\\
&= \min\{u^{\mathsf{T}}a + v^{\mathsf{T}}b + w^{\mathsf{T}}c \mid u \geq \mathbf{0}, w \leq \mathbf{0},\\
&u^{\mathsf{T}}A + v^{\mathsf{T}}D + w^{\mathsf{T}}G \geq d^{\mathsf{T}},\\
&u^{\mathsf{T}}B + v^{\mathsf{T}}E + w^{\mathsf{T}}H = e^{\mathsf{T}},\\
&u^{\mathsf{T}}C + v^{\mathsf{T}}F + w^{\mathsf{T}}K \leq f^{\mathsf{T}}\},
\end{aligned}$$

provided that at least one of the two optima is finite (cf. Section 7.4 in Schrijver [1986b]).

So there is a one-to-one relation between constraints in a problem and variables in its dual problem. The objective function in one problem becomes the right-hand side in the dual problem. We survey these relations in the following table:

maximize	minimize
$\leq$ constraint	variable ≥ 0
$\geq$ constraint	variable ≤ 0
$=$ constraint	unconstrained variable
variable ≥ 0	$\geq$ constraint
variable ≤ 0	$\leq$ constraint
unconstrained variable	$=$ constraint
right-hand side	objective function
objective function	right-hand side

Some LP terminology. Linear programming concerns maximizing or minimizing a linear function $c^{\mathsf{T}}x$ over a polyhedron P. The polyhedron P is called the *feasible region*, and any vector in P a *feasible solution*. If the feasible region is nonempty, the problem is called *feasible*, and *infeasible* otherwise. The function $x \to c^{\mathsf{T}}x$ is called the *objective function* or the *cost function*. Any feasible solution attaining the optimum value is called an *optimum solution*. An inequality $c^{\mathsf{T}}x \leq \delta$ is called *tight* or *active* for some x^* if $c^{\mathsf{T}}x^* = \delta$.

Equations like (5.10), (5.11), and (5.12) are called *linear programming duality equations*. The minimization problem is called the *dual problem* of the maximization problem (which problem then is called the *primal problem*), and conversely. A feasible solution of the dual problem is called a *dual solution*.

Complementary slackness. The following *complementary slackness conditions* characterize optimality of a pair of feasible solutions x, y of the linear programs (5.10):

(5.13) x and y are optimum solutions if and only if $(Ax)_i = b_i$ for each i with $y_i > 0$.

Similar conditions can be formulated for other pairs of dual linear programs (cf. Section 7.9 in Schrijver [1986b]).

Carathéodory's theorem. A consequence of Carathéodory's theorem (Theorem 5.1 above) is:

Theorem 5.5. *If the optimum value in the LP problems* (5.10) *is finite, then the minimum is attained by a vector $y \geq \mathbf{0}$ such that the rows of A corresponding to positive components of y are linearly independent.*

(Corollary 7.1l in Schrijver [1986b].)

5.6. Faces, facets, and vertices

Let $P = \{x \mid Ax \leq b\}$ be a polyhedron in $\mathbb{R}^n$. If c is a nonzero vector and $\delta = \max\{c^{\mathsf{T}}x \mid Ax \leq b\}$, the affine hyperplane $\{x \mid c^{\mathsf{T}}x = \delta\}$ is called a *supporting hyperplane* of P. A subset F of P is called a *face* if $F = P$ or if $F = P \cap H$ for some supporting hyperplane H of P. So

(5.14) F is a face of $P \iff F$ is the set of optimum solutions of $\max\{c^{\mathsf{T}}x \mid Ax \leq b\}$ for some $c \in \mathbb{R}^n$.

An inequality $c^{\mathsf{T}}x \leq \delta$ is said to *determine* or to *induce* face F of P if

(5.15) $F = \{x \in P \mid c^{\mathsf{T}}x = \delta\}$.

Alternatively, F is a face of P if and only if

(5.16) $F = \{x \in P \mid A'x = b'\}$

for some subsystem $A'x \leq b'$ of $Ax \leq b$ (cf. Section 8.3 in Schrijver [1986b]). So any face of a nonempty polyhedron is a nonempty polyhedron. We say that a constraint $a^{\mathsf{T}}x \leq \beta$ from $Ax \leq b$ is *tight* or *active* in a face F if $a^{\mathsf{T}}x = \beta$ holds for each $x \in F$.

An inequality $a^{\mathsf{T}}x \leq \beta$ from $Ax \leq b$ is called an *implicit equality* if $Ax \leq b$ implies $a^{\mathsf{T}}x = \beta$. Then:

Theorem 5.6. *Let $P = \{x \mid Ax \leq b\}$ be a polyhedron in $\mathbb{R}^n$. Let $A'x \leq b'$ be the subsystem of implicit inequalities in $Ax \leq b$. Then* $\dim P = n - \mathrm{rank} A'$.

(Cf. Section 8.2 in Schrijver [1986b].)

A *facet* of P is an inclusionwise maximal face F of P with $F \neq P$. An inequality determining a facet is called *facet-determining* or *facet-inducing*. Any facet has dimension one less than the dimension of P.

A system $Ax \leq b$ is called *minimal* or *irredundant* if each proper subsystem $A'x \leq b'$ has a solution x not satisfying $Ax \leq b$. If $Ax \leq b$ is irredundant and P is full-dimensional, then $Ax \leq b$ is the unique minimal system determining P, up to multiplying inequalities by positive scalars.

If $Ax \leq b$ is irredundant, then there is a one-to-one relation between the facets F of P and those inequalities $a^{\mathsf{T}}x \leq \beta$ in $Ax \leq b$ that are not implicit equalities, given by:

(5.17) $F = \{x \in P \mid a^{\mathsf{T}}x = \beta\}$

(cf. Theorem 8.1 in Schrijver [1986b]). This implies that each face $F \neq P$ is the intersection of facets.

A face of $P = \{x \mid Ax \leq b\}$ is called a *minimal face* if it is an inclusionwise minimal face. Any minimal face is an affine subspace of $\mathbb{R}^n$, and all minimal faces of P are translates of each other. They all have dimension $n - \mathrm{rank} A$.

If each minimal face has dimension 0, P is called *pointed*. A *vertex* of P is an element z such that $\{z\}$ is a minimal face. A polytope is the convex hull of its vertices.

For any element z of $P = \{x \mid Ax \leq b\}$, let $A_z x \leq b_z$ be the system consisting of those inequalities from $Ax \leq b$ that are satisfied by z with equality. Then:

Theorem 5.7. *Let $P = \{x \mid Ax \leq b\}$ be a polyhedron in $\mathbb{R}^n$ and let $z \in P$. Then z is a vertex of P if and only if* $\mathrm{rank}(A_z) = n$.

An *edge* of P is a bounded face of dimension 1. It necessarily connects two vertices of P. Two vertices connected by an edge are called *adjacent*. An *extremal ray* is a face of dimension 1 that forms a halfline.

The 1-*skeleton* of a pointed polyhedron P is the union of the vertices, edges, and extremal rays of P. If P is a polytope, the 1-skeleton is a topological graph. The *diameter* of P is the diameter of the associated (combinatorial) graph.

The *Hirsch conjecture* states that a d-dimensional polytope with m facets has diameter at most $m-d$. Naddef [1989] proved this for polytopes with $0,1$ vertices. We refer to Kalai [1997] for a survey of bounds on the diameter and on the number of pivot steps in linear programming.

5.7. Polarity

(For the results of this section, see Section 9.1 in Schrijver [1986b].) For any subset C of $\mathbb{R}^n$, the *polar* of C is

(5.18) $\qquad C^* := \{z \in \mathbb{R}^n \mid x^{\mathsf{T}} z \leq 1 \text{ for all } x \in C\}.$

If C is a cone, then C^* is again a cone, the *polar cone* of C, and satisfies

(5.19) $\qquad C^* := \{z \in \mathbb{R}^n \mid x^{\mathsf{T}} z \leq 0 \text{ for all } x \in C\}.$

Let C be a polyhedral cone; so $C = \{x \mid Ax \leq \mathbf{0}\}$ for some matrix A. Trivially, if C is generated by the vectors $x_1, \ldots, x_k$, then C^* is equal to the cone determined by the inequalities $x_i^{\mathsf{T}} z \leq 0$ for $i = 1, \ldots, k$. It is less trivial, and can be derived from Farkas' lemma, that:

(5.20) the polar cone C^* is equal to the cone generated by the transposes of the rows of A.

This implies

(5.21) $\qquad C^{**} = C$ for each polyhedral cone C.

So there is a symmetric duality relation between finite sets of vectors generating a cone and finite sets of vectors generating its polar cone.

5.8. Blocking polyhedra

(For the results of this section, see Section 9.2 in Schrijver [1986b].) A duality relation similar to polarity holds between convex sets 'of blocking type', and also between convex sets 'of antiblocking type'. This was shown by Fulkerson [1970b,1971a,1972a], who found several applications in combinatorial optimization.

We say that a subset P of $\mathbb{R}^n$ is *up-monotone* if $x \in P$ and $y \geq x$ imply $y \in P$. Similarly, P is *down-monotone* if $x \in P$ and $y \leq x$ imply $y \in P$.

Moreover, P is *down-monotone in* $\mathbb{R}^n_+$ if $x \in P$ and $\mathbf{0} \leq y \leq x$ imply $y \in P$. For any $P \subseteq \mathbb{R}^n$ we define

(5.22) $$P^{\uparrow} := \{y \in \mathbb{R}^n \mid \exists x \in P : y \geq x\} = P + \mathbb{R}^n_+ \text{ and}$$
$$P^{\downarrow} := \{y \in \mathbb{R}^n \mid \exists x \in P : y \leq x\} = P - \mathbb{R}^n_+.$$

$P^{\uparrow}$ is called the *dominant* of P. So P is up-monotone if and only if $P = P^{\uparrow}$, and P is down-monotone if and only if $P = P^{\downarrow}$.

We say that a convex set $P \subseteq \mathbb{R}^n$ is of *blocking type* if P is a closed convex up-monotone subset of $\mathbb{R}^n_+$. Each polyhedron P of blocking type is pointed. Moreover, P is a polyhedron of blocking type if and only if there exist vectors $x_1, \ldots, x_k \in \mathbb{R}^n_+$ such that

(5.23) $$P = \text{conv.hull}\{x_1, \ldots, x_k\}^{\uparrow};$$

and also, if and only if

(5.24) $$P = \{x \in \mathbb{R}^n_+ \mid Ax \geq \mathbf{1}\}$$

for some nonnegative matrix A.

For any polyhedron P in $\mathbb{R}^n$, the *blocking polyhedron* $B(P)$ of P is defined by

(5.25) $$B(P) := \{z \in \mathbb{R}^n_+ \mid x^{\mathsf{T}} z \geq 1 \text{ for each } x \in P\}.$$

Fulkerson [1970b,1971a] showed:

Theorem 5.8. *Let $P \subseteq \mathbb{R}^n_+$ be a polyhedron of blocking type. Then $B(P)$ is again a polyhedron of blocking type and $B(B(P)) = P$. Moreover, for any $x_1, \ldots, x_k \in \mathbb{R}^n_+$:*

(5.26) (5.23) *holds if and only if* $B(P) = \{z \in \mathbb{R}^n_+ \mid x_i^{\mathsf{T}} z \geq 1$ *for* $i = 1, \ldots, k\}$.

Here the only if part is trivial, while the if part requires Farkas' lemma.

Theorem 5.8 implies that for vectors $x_1, \ldots, x_k \in \mathbb{R}^n_+$ and $z_1, \ldots, z_d \in \mathbb{R}^n_+$ one has:

(5.27) $$\text{conv.hull}\{x_1, \ldots, x_k\} + \mathbb{R}^n_+ = \{x \in \mathbb{R}^n_+ \mid z_j^{\mathsf{T}} x \geq 1 \text{ for } j = 1, \ldots, d\}$$

if and only if

(5.28) $$\text{conv.hull}\{z_1, \ldots, z_d\} + \mathbb{R}^n_+ = \{z \in \mathbb{R}^n_+ \mid x_i^{\mathsf{T}} z \geq 1 \text{ for } i = 1, \ldots, k\}.$$

Two polyhedra P, R are called a *blocking pair (of polyhedra)* if they are of blocking type and satisfy $R = B(P)$. So if P, R is a blocking pair, then so is R, P.

5.9. Antiblocking polyhedra

(For the results of this section, see Section 9.3 in Schrijver [1986b].) The theory of antiblocking polyhedra is almost fully analogous to the blocking case and arises mostly by reversing inequality signs.

We say that a set $P \subseteq \mathbb{R}^n$ is of *antiblocking type* if P is a nonempty closed convex subset of $\mathbb{R}^n_+$ that is down-monotone in $\mathbb{R}^n_+$. Then P is a polyhedron of antiblocking type if and only if

(5.29) $\quad P = \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$

for some nonnegative matrix A and nonnegative vector b.

For any subset P of $\mathbb{R}^n$, the *antiblocking set* $A(P)$ of P is defined by

(5.30) $\quad A(P) := \{z \in \mathbb{R}^n_+ \mid x^{\mathsf{T}} z \leq 1 \text{ for each } x \in P\}.$

If $A(P)$ is a polyhedron we speak of the *antiblocking polyhedron*, and if $A(P)$ is a convex body, of the *antiblocking body.*

Fulkerson [1971a,1972a] showed:

Theorem 5.9. *Let $P \subseteq \mathbb{R}^n_+$ be of antiblocking type. Then $A(P)$ is again of antiblocking type and $A(A(P)) = P$.*

The antiblocking analogue of (5.26) is a little more complicated to formulate, but we need it only for full-dimensional polytopes. For any full-dimensional polytope $P \subseteq \mathbb{R}^n$ of antiblocking type and $x_1, \ldots, x_k \in \mathbb{R}^n_+$ we have:

(5.31) $\quad P = \text{conv.hull}\{x_1, \ldots, x_k\}^{\downarrow} \cap \mathbb{R}^n_+$ if and only if $A(P) = \{z \in \mathbb{R}^n_+ \mid x_i^{\mathsf{T}} z \leq 1 \text{ for } i = 1, \ldots, k\}.$

Two convex sets P, R are called an *antiblocking pair (of polyhedra)* if they are of antiblocking type and satisfy $R = A(P)$. So if P, R is an antiblocking pair, then so is R, P.

5.10. Methods for linear programming

The simplex method was designed by Dantzig [1951b] to solve linear programming problems. It is in practice and on average quite efficient, but no polynomial-time worst-case running time bound has been proved (most of the pivot selection rules that have been proposed have been proved to take exponential time in the worst case).

The simplex method consists of finding a path in the 1-skeleton of the feasible region, ending at an optimum vertex (in preprocessing, the problem first is transformed to one with a pointed feasible region). An important issue when implementing this is that the LP problem is not given by vertices and edges, but by linear inequalities, and that vertices are determined by a, not necessarily unique, 'basis' among the inequalities.

The first polynomial-time method for linear programming was given by Khachiyan [1979,1980], by adapting the 'ellipsoid method' for nonlinear programming of Shor [1970a,1970b,1977] and Yudin and Nemirovskiĭ [1976]. The method consists of finding a sequence of shrinking ellipsoids each containing at least one optimum solution, until we have an ellipsoid that is small enough so as to derive an optimum solution. The method however is practically quite infeasible.

Karmarkar [1984a,1984b] showed that 'interior point' methods can solve linear programming in polynomial time, and moreover that they have efficient implementations, competing with the simplex method. Interior point methods make a tour not along vertices and edges, but across the feasible region.

5.11. The ellipsoid method

While the ellipsoid method is practically infeasible, it turned out to have features that are useful for deriving complexity results in combinatorial optimization. Specifically, the ellipsoid method does not require listing all constraints of an LP problem a priori, but allows that they are generated when needed. In this way, one can derive the polynomial-time solvability of a number of combinatorial optimization problems. This should be considered as existence proofs of polynomial-time algorithms — the algorithms are not practical.

This application of the ellipsoid method was described by Karp and Papadimitriou [1980,1982], Padberg and Rao [1980], and Grötschel, Lovász, and Schrijver [1981]. The book by Grötschel, Lovász, and Schrijver [1988] is devoted to it. We refer to Chapter 6 of this book or to Chapter 14 of Schrijver [1986b] for proofs of the results that we survey below.

The ellipsoid method applies to classes of polyhedra (and more generally, classes of convex sets) which are described as follows.

Let Σ be a finite alphabet and let Π be a subset of the set Σ^* of words over Σ. In applications, we take for Π very simple sets like the set of strings representing a graph or the set of strings representing a digraph.

For each $\sigma \in \Pi$, let E_σ be a finite set and let P_σ be a rational polyhedron in $\mathbb{Q}^{E_\sigma}$. (When we apply this, E_σ is often the vertex set or the edge or arc set of the (di)graph represented by σ.) We make the following assumptions:

(5.32) (i) there is a polynomial-time algorithm that, given $\sigma \in \Sigma^*$, tests if σ belongs to Π and, if so, returns the set E_σ;
(ii) there is a polynomial p such that, for each $\sigma \in \Pi$, P_σ is determined by linear inequalities each of size at most $p(\text{size}(\sigma))$.

Here the *size* of a rational linear inequality is proportional to the sum of the sizes of its components, where the *size* of a rational number p/q (for integers

p, q) is proportional to $\log(|p|+1) + \log q$. Condition (5.32)(ii) is equivalent to (cf. Theorem 10.2 in Schrijver [1986b]):

(5.33) there is a polynomial q such that, for each $\sigma \in \Pi$, we can write $P_\sigma = Q + C$, where Q is a polytope with vertices each of input size at most $q(\text{size}(\sigma))$ and where C is a cone generated by vectors each of input size at most $q(\text{size}(\sigma))$.

(The *input size*[7] of a vector is the sum of the sizes of its components.) In most applications, the existence of the polynomial p in (5.32)(ii) or of the polynomial q in (5.33) is obvious.

We did not specify how the polyhedra P_σ are given algorithmically. In applications, they might have an exponential number of vertices or facets, so listing them would not be an algorithmic option. To handle this, we formulate two, in a sense dual, problems. An algorithm for either of them would determine the polyhedra P_σ.

First, the *optimization problem for* $(P_\sigma \mid \sigma \in \Pi)$ is the problem:

(5.34) given: $\sigma \in \Pi$ and $c \in \mathbb{Q}^{E_\sigma}$,
find: $x \in P_\sigma$ maximizing $c^{\mathsf{T}}x$ over P_σ or $y \in \text{char.cone}(P_\sigma)$ with $c^{\mathsf{T}}y > 0$, if either of them exists.

Second, the *separation problem for* $(P_\sigma \mid \sigma \in \Pi)$ is the problem:

(5.35) given: $\sigma \in \Pi$ and $z \in \mathbb{Q}^{E_\sigma}$,
find: $c \in \mathbb{Q}^{E_\sigma}$ such that $c^{\mathsf{T}}x < c^{\mathsf{T}}z$ for all $x \in P_\sigma$ (if such a c exists).

So c gives a separating hyperplane if $z \notin P_\sigma$.

Then the ellipsoid method implies that these two problems are 'polynomial-time equivalent':

Theorem 5.10. *Let $\Pi \subseteq \Sigma^*$ and let $(P_\sigma \mid \sigma \in \Pi)$ satisfy* (5.32). *Then the optimization problem for $(P_\sigma \mid \sigma \in \Pi)$ is polynomial-time solvable if and only if the separation problem for $(P_\sigma \mid \sigma \in \Pi)$ is polynomial-time solvable.*

(Cf. Theorem (6.4.9) in Grötschel, Lovász, and Schrijver [1988] or Corollary 14.1c in Schrijver [1986b].)

The equivalence in Theorem 5.10 makes that we call $(P_\sigma \mid \sigma \in \Pi)$ *polynomial-time solvable* if it satisfies (5.32) and the optimization problem (equivalently, the separation problem) for it is polynomial-time solvable.

Using simultaneous diophantine approximation based on the basis reduction method given by Lenstra, Lenstra, and Lovász [1982], Frank and Tardos [1985,1987] extended these results to strong polynomial-time solvability:

[7] We will use the term *size* of a vector for the sum of its components.

Theorem 5.11. *The optimization problem and the separation problem for any polynomial-time solvable system of polyhedra are solvable in strongly polynomial time.*

(Theorem (6.6.5) in Grötschel, Lovász, and Schrijver [1988].)

For polynomial-time solvable classes of polyhedra, the separation problem can be strengthened so as to obtain a facet as separating hyperplane:

Theorem 5.12. *Let* $(P_\sigma \mid \sigma \in \Pi)$ *be a polynomial-time solvable system of polyhedra. Then the following problem is strongly polynomial-time solvable:*

(5.36) given: $\sigma \in \Pi$ and $z \in \mathbb{Q}^{E_\sigma}$,
find: $c \in \mathbb{Q}^{E_\sigma}$ and $\delta \in \mathbb{Q}$ such that $c^{\mathsf{T}} z > \delta$ and such that $c^{\mathsf{T}} x \leq \delta$ is facet-inducing for P_σ (if it exists).

(Cf. Theorem (6.5.16) in Grötschel, Lovász, and Schrijver [1988].) Also a weakening of the separation problem turns out to be equivalent, under certain conditions. The *membership problem for* $(P_\sigma \mid \sigma \in \Pi)$ is the problem:

(5.37) given $\sigma \in \Pi$ and $z \in \mathbb{Q}^{E_\sigma}$, does z belong to P_σ?

Theorem 5.13. *Let* $(P_\sigma \mid \sigma \in \Pi)$ *be a system of full-dimensional polytopes satisfying* (5.32)*, such that there is a polynomial-time algorithm that gives for each* $\sigma \in \Pi$ *a vector in the interior of* P_σ*. Then* $(P_\sigma \mid \sigma \in \Pi)$ *is polynomial-time solvable if and only if the membership problem for* $(P_\sigma \mid \sigma \in \Pi)$ *is polynomial-time solvable.*

(This follows from Corollary (4.3.12) and Theorem (6.3.2) in Grötschel, Lovász, and Schrijver [1988].)

The theorems above imply:

Theorem 5.14. *Let* $(P_\sigma \mid \sigma \in \Pi)$ *and* $(Q_\sigma \mid \sigma \in \Pi)$ *be polynomial-time solvable classes of polyhedra, such that for each* $\sigma \in \Pi$*, the polyhedra* P_σ *and* Q_σ *are in the same space* $\mathbb{R}^{E_\sigma}$*. Then also* $(P_\sigma \cap Q_\sigma \mid \sigma \in \Pi)$ *and* $(\mathrm{conv.hull}(P_\sigma \cup Q_\sigma) \mid \sigma \in \Pi)$ *are polynomial-time solvable.*

(Corollary 14.1d in Schrijver [1986b].)

Corollary 5.14a. *Let* $(P_\sigma \mid \sigma \in \Pi)$ *be a polynomial-time solvable system of polyhedra, all of blocking type. Then also the system of blocking polyhedra* $(B(P_\sigma) \mid \sigma \in \Pi)$ *is polynomial-time solvable.*

(Corollary 14.1e in Schrijver [1986b].) Similarly:

Corollary 5.14b. *Let* $(P_\sigma \mid \sigma \in \Pi)$ *be a polynomial-time solvable system of polyhedra, all of antiblocking type. Then also the system of antiblocking polyhedra* $(A(P_\sigma) \mid \sigma \in \Pi)$ *is polynomial-time solvable.*

(Corollary 14.1e in Schrijver [1986b].)

Also the following holds:

Theorem 5.15. *Let* $(P_\sigma \mid \sigma \in \Pi)$ *be a polynomial-time solvable system of polyhedra, where each* P_σ *is a polytope. Then the following problems are strongly polynomial-time solvable:*

(5.38)
(i) *given* $\sigma \in \Pi$*, find an internal vector, a vertex, and a facet-inducing inequality of* P_σ*;*
(ii) *given* $\sigma \in \Pi$ *and* $x \in P_\sigma$*, find affinely independent vertices* $x_1, \ldots, x_k$ *of* P_σ *and write* x *as a convex combination of* $x_1, \ldots, x_k$*;*
(iii) *given* $\sigma \in \Pi$ *and* $c \in \mathbb{R}^{E_\sigma}$*, find facet-inducing inequalities* $c_1^\mathsf{T} x \leq \delta_1, \ldots, c_k^\mathsf{T} x \leq \delta_k$ *of* P_σ *with* $c_1, \ldots, c_k$ *linearly independent, and find* $\lambda_1, \ldots, \lambda_k \geq 0$ *such that* $\lambda_1 c_1 + \cdots + \lambda_k c_k = c$ *and* $\lambda_1 \delta_1 + \cdots + \lambda_k \delta_k = \max\{c^\mathsf{T} x \mid x \in P_\sigma\}$ *(i.e., find an optimum dual solution).*

(Corollary 14.1f in Schrijver [1986b].)

The ellipsoid method can be applied also to nonpolyhedral convex sets, in which case only approximative versions of the optimization and separation problems can be shown to be equivalent. We only need this in Chapter 67 on the convex body $\mathrm{TH}(G)$, where we refer to the appropriate theorem in Grötschel, Lovász, and Schrijver [1988].

5.12. Polyhedra and NP and co-NP

An appropriate polyhedral description of a combinatorial optimization problem relates to the question NP≠co-NP. More precisely, unless NP=co-NP, the polyhedra associated with an NP-complete problem cannot be described by 'certifiable' inequalities. (These insights go back to observations in the work of Edmonds of the 1960s.)

Again, let $(P_\sigma \mid \sigma \in \Pi)$ be a system of polyhedra satisfying (5.32). Consider the decision version of the optimization problem:

(5.39) given $\sigma \in \Pi$, $c \in \mathbb{Q}^{E_\sigma}$, and $k \in \mathbb{Q}$, is there an $x \in P_\sigma$ with $c^\mathsf{T} x > k$?

Then:

Theorem 5.16. *Problem* (5.39) *belongs to* co-NP *if and only if for each* $\sigma \in \Pi$*, there exists a collection* $\mathcal{I}_\sigma$ *of inequalities determining* P_σ *such that the problem:*

(5.40) *given* $\sigma \in \Pi$*,* $c \in \mathbb{Q}^{E_\sigma}$*, and* $\delta \in \mathbb{Q}$*, does* $c^\mathsf{T} x \leq \delta$ *belong to* $\mathcal{I}_\sigma$*,* *belongs to* NP.

Proof. To see necessity, we can take for $\mathcal{I}_\sigma$ the collection of *all* valid inequalities for P_σ. Then co-NP-membership of (5.39) is equivalent of NP-membership of (5.40).

To see sufficiency, a negative answer to question (5.39) can be certified by giving inequalities $c_i^\mathsf{T} x \leq \delta_i$ from $\mathcal{I}_\sigma$ and $\lambda_i \in \mathbb{Q}_+$ $(i = 1, \ldots, k)$ such that $c = \lambda_1 c_1 + \cdots + \lambda_k c_k$ and $\delta \geq \lambda_1 \delta_1 + \cdots + \lambda_k \delta_k$. As we can take $k \leq |E_\sigma|$, and as each inequality in $\mathcal{I}_\sigma$ has a polynomial-time checkable certificate (as (5.40) belongs to NP), this gives a polynomial-time checkable certificate for the negative answer. Hence (5.39) belongs to co-NP. ∎

This implies for NP-complete problems:

Corollary 5.16a. *Let* (5.39) *be* NP*-complete and suppose* NP≠co-NP*. For each $\sigma \in \Pi$, let $\mathcal{I}_\sigma$ be a collection of inequalities determining P_σ. Then problem* (5.40) *does not belong to* NP*.*

Proof. If problem (5.40) would belong to NP, then by Theorem 5.16, problem (5.39) belongs to co-NP. If (5.39) is NP-complete, this implies NP=co-NP. ∎

Roughly speaking, this implies that if (5.39) is NP-complete and NP≠co-NP, then P_σ has 'difficult' facets, that is, facets which have no polynomial-time checkable certificate of validity for P_σ.

(Related work on the complexity of facets was reported in Karp and Papadimitriou [1980,1982] and Papadimitriou and Yannakakis [1982,1984].)

5.13. Primal-dual methods

As a generalization of similar methods for network flow and transportation problems, Dantzig, Ford, and Fulkerson [1956] designed the 'primal-dual method' for linear programming. The general idea is as follows. Starting with a dual feasible solution y, the method searches for a primal feasible solution x satisfying the complementary slackness condition with respect to y. If such a primal feasible solution x is found, x and y form a pair of optimum solutions (by (5.13)). If no such primal solution is found, the method prescribes a modification of y, after which the method iterates.

The problem now is how to find a primal feasible solution x satisfying the complementary slackness condition, and how to modify the dual solution y if no such primal solution is found. For general linear programs this problem can be seen to amount to another linear program, generally simpler than the original linear program. To solve the simpler linear program we could use any LP method. In many combinatorial applications, however, this simpler linear program is a simpler combinatorial optimization problem, for which direct methods are available. Thus, if we can describe a combinatorial optimization problem as a linear program, the primal-dual method gives us a scheme for reducing one combinatorial problem to an easier combinatorial problem. The efficiency of the method depends on the complexity of the easier problem and on the number of primal-dual iterations.

We describe the primal-dual method more precisely. Suppose that we wish to solve the LP problem

(5.41) $$\min\{c^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax = b\},$$

where A is an $m \times n$ matrix, with columns $a_1, \ldots, a_n$, and where $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. The dual problem is

(5.42) $$\max\{y^{\mathsf{T}}b \mid y^{\mathsf{T}}A \leq c^{\mathsf{T}}\}.$$

The primal-dual method consists of repeating the following *primal-dual iteration*. Suppose that we have a feasible solution y_0 for problem (5.42). Let A' be the submatrix of A consisting of those columns a_j of A for which $y_0^{\mathsf{T}}a_j = c_j$ holds. To find a feasible primal solution satisfying the complementary slackness, solve the *restricted linear program*

(5.43) $$x' \geq \mathbf{0}, A'x' = b.$$

If such an x' exists, by adding components 0, we obtain a vector $x \geq \mathbf{0}$ such that $Ax = b$ and such that $x_j = 0$ if $y_0^{\mathsf{T}}a_j < c_j$. By complementary slackness ((5.13)), it follows that x and y_0 are optimum solutions for problems (5.41) and (5.42).

On the other hand, if no x' satisfying (5.43) exists, by Farkas' lemma (Corollary 5.3a), there exists a y' such that ${y'}^{\mathsf{T}}A' \leq 0$ and ${y'}^{\mathsf{T}}b > 0$. Let α be the largest real number satisfying

(5.44) $$(y_0 + \alpha y')^{\mathsf{T}}A \leq c^{\mathsf{T}}.$$

(Note that $\alpha > 0$.) Reset $y_0 := y_0 + \alpha y'$, and start the iteration anew. (If $\alpha = \infty$, (5.42) is unbounded, hence (5.41) is infeasible.)

This describes the primal-dual method. It reduces problem (5.41) to (5.43), which often is an easier problem.

The primal-dual method can equally well be considered as a *gradient method*. Suppose that we wish to solve problem (5.42), and we have a feasible solution y_0. This y_0 is not optimum if and only if there exists a vector y' such that ${y'}^{\mathsf{T}}b > 0$ and y' is a *feasible direction* at y_0 (that is, $(y_0 + \alpha y')^{\mathsf{T}}A \leq c^{\mathsf{T}}$ for some $\alpha > 0$). If we let A' consist of those columns of A in which $y_0^{\mathsf{T}}A \leq c^{\mathsf{T}}$ has equality, then y' is a feasible direction if and only if ${y'}^{\mathsf{T}}A' \leq 0$. So y' can be found by solving (5.43).

5.14. Integer linear programming

A vector $x \in \mathbb{R}^n$ is called *integer* if each component is an integer, i.e., if x belongs to $\mathbb{Z}^n$. Many combinatorial optimization problems can be described as maximizing a linear function $c^{\mathsf{T}}x$ over the *integer* vectors in some polyhedron $P = \{x \mid Ax \leq b\}$.

So this type of problems can be described as:

$$\max\{c^{\mathsf{T}}x \mid Ax \leq b; x \in \mathbb{Z}^n\}. \tag{5.45}$$

Such problems are called *integer linear programming*, or *ILP*, problems. They consist of maximizing a linear function over the intersection $P \cap \mathbb{Z}^n$ of a polyhedron P with the set $\mathbb{Z}^n$ of integer vectors.

Clearly, always the following inequality holds:

$$\max\{c^{\mathsf{T}}x \mid Ax \leq b; x \text{ integer}\} \leq \max\{c^{\mathsf{T}}x \mid Ax \leq b\}. \tag{5.46}$$

It is easy to make an example where strict inequality holds. This implies, that generally one will have strict inequality in the following duality relation:

$$\begin{aligned} &\max\{c^{\mathsf{T}}x \mid Ax \leq b; x \text{ integer}\} \\ &\leq \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}; y^{\mathsf{T}}A = c^{\mathsf{T}}; y \text{ integer}\}. \end{aligned} \tag{5.47}$$

No polynomial-time algorithm is known to exist for solving an integer linear programming problem in general. In fact, the general integer linear programming problem is NP-complete (since the satisfiability problem is easily transformed to an integer linear programming problem). However, for special classes of integer linear programming problems, polynomial-time algorithms have been found. These classes often come from combinatorial problems.

5.15. Integer polyhedra

A polyhedron P is called an *integer polyhedron* if it is the convex hull of the integer vectors contained in P. This is equivalent to: P is rational and each face of P contains an integer vector. So a polytope P is integer if and only if each vertex of P is integer. If a polyhedron $P = \{x \mid Ax \leq b\}$ is integer, then the linear programming problem

$$\max\{c^{\mathsf{T}}x \mid Ax \leq b\} \tag{5.48}$$

has an integer optimum solution if it is finite. Hence, in that case,

$$\max\{c^{\mathsf{T}}x \mid Ax \leq b;\ x \text{ integer}\} = \max\{c^{\mathsf{T}}x \mid Ax \leq b\}. \tag{5.49}$$

This in fact characterizes integer polyhedra, since:

Theorem 5.17. *Let P be a rational polyhedron in $\mathbb{Q}^n$. Then P is integer if and only if for each $c \in \mathbb{Q}^n$, the linear programming problem* $\max\{c^{\mathsf{T}}x \mid Ax \leq b\}$ *has an integer optimum solution if it is finite.*

A stronger characterization is (Edmonds and Giles [1977]):

Theorem 5.18. *A rational polyhedron P in $\mathbb{Q}^n$ is integer if and only if for each $c \in \mathbb{Z}^n$ the value of* $\max\{c^{\mathsf{T}}x \mid x \in P\}$ *is an integer if it is finite.*

(Corollary 22.1a in Schrijver [1986b].) We also will use the following observation:

Theorem 5.19. *Let P be an integer polyhedron in $\mathbb{R}^n_+$ with $P+\mathbb{R}^n_+ = P$ and let $c \in \mathbb{Z}^n_+$ be such that $x \leq c$ for each vertex x of P. Then $P \cap \{x \mid x \leq c\}$ is an integer polyhedron again.*

Proof. Let $Q := P \cap \{x \mid x \leq c\}$ and let R be the convex hull of the integer vectors in Q. We must show that $Q \subseteq R$.

Let $x \in Q$. As $P = R + \mathbb{R}^n_+$ there exists a $y \in R$ with $y \leq x$. Choose such a y with $y_1 + \cdots + y_n$ maximal. Suppose that $y_i < x_i$ for some component i. Since $y \in R$, y is a convex combination of integer vectors in Q. Since $y_i < x_i \leq c_i$, at least one of these integer vectors, z say, has $z_i < c_i$. But then the vector $z' := z+\chi^i$ belongs to R. Hence we could increase y_i, contradicting the maximality of y. ∎

We call a polyhedron P *box-integer* if $P \cap \{x \mid d \leq x \leq c\}$ is an integer polyhedron for each choice of integer vectors d, c. The set $\{x \mid d \leq x \leq c\}$ is called a *box*.

A $0,1$ *polytope* is a polytope with all vertices being 0,1 vectors.

5.16. Totally unimodular matrices

Total unimodularity of matrices is an important tool in integer programming. A matrix A is called *totally unimodular* if each square submatrix of A has determinant equal to 0, $+1$, or -1. In particular, each entry of a totally unimodular matrix is 0, $+1$, or -1.

An alternative way of characterizing total unimodularity is by requiring that the matrix is integer and that each nonsingular submatrix has an integer inverse matrix. This implies the following easy, but fundamental result:

Theorem 5.20. *Let A be a totally unimodular $m \times n$ matrix and let $b \in \mathbb{Z}^m$. Then the polyhedron*

(5.50) $$P := \{x \mid Ax \leq b\}$$

is integer.

(Cf. Theorem 19.1 in Schrijver [1986b].) It follows that each linear programming problem with integer data and totally unimodular constraint matrix has integer optimum primal and dual solutions:

Corollary 5.20a. *Let A be a totally unimodular $m \times n$ matrix, let $b \in \mathbb{Z}^m$, and let $c \in \mathbb{Z}^n$. Then both optima in the LP duality equation*

(5.51) $$\max\{c^{\mathsf{T}}x \mid Ax \leq b\} = \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}\}$$

have integer optimum solutions (if the optima are finite).

(Corollary 19.1a in Schrijver [1986b].) Hoffman and Kruskal [1956] showed that this property is close to a characterization of total unimodularity.

Corollary 5.20a implies:

Corollary 5.20b. *Let A be an $m \times n$ matrix, let $b \in \mathbb{Z}^m$, and let $c \in \mathbb{R}^n$. Suppose that*

(5.52) $$\max\{c^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \leq b\}$$

has an optimum solution x^* such that the columns of A corresponding to positive components of x^* form a totally unimodular matrix. Then (5.52) *has an integer optimum solution.*

Proof. Since x^* is an optimum solution, we have

(5.53) $$\max\{c^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \leq b\} = \max\{c'^{\mathsf{T}}x' \mid x' \geq \mathbf{0}, A'x' \leq b\},$$

where A' and c' are the parts of A and c corresponding to the support of x^*. As A' is totally unimodular, the right-hand side maximum in (5.53) has an integer optimum solution x'^*. Extending x'^* by components 0, we obtain an integer optimum solution of the left-hand side maximum in (5.53). ∎

We will use the following characterization of Ghouila-Houri [1962b] (cf. Theorem 19.3 in Schrijver [1986b]):

Theorem 5.21. *A matrix M is totally unimodular if and only if each collection R of rows of M can be partitioned into classes R_1 and R_2 such that the sum of the rows in R_1, minus the sum of the rows in R_2, is a vector with entries $0, \pm 1$ only.*

5.17. Total dual integrality

Edmonds and Giles [1977] introduced the powerful notion of total dual integrality. It is not only useful as a tool to derive combinatorial min-max relation, but also it gives an efficient way of expressing a whole bunch of min-max relations simultaneously.

A system $Ax \leq b$ in n dimensions is called *totally dual integral*, or just *TDI*, if A and b are rational and for each $c \in \mathbb{Z}^n$, the dual of maximizing $c^{\mathsf{T}}x$ over $Ax \leq b$:

(5.54) $$\min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}\}$$

has an integer optimum solution y, if it is finite.

By extension, a system $A'x \leq b', A''x = b''$ is defined to be TDI if the system $A'x \leq b', A''x \leq b'', -A''x \leq -b''$ is TDI. This is equivalent to requiring that A', A'', b', b'' are rational and for each $c \in \mathbb{Z}^n$ the dual of maximizing $c^\mathsf{T}x$ over $A'x \leq b'$, $A''x = b''$ has an integer optimum solution, if finite.

Problem (5.54) is the problem dual to $\max\{c^\mathsf{T}x \mid Ax \leq b\}$, and Edmonds and Giles showed that total dual integrality implies that also this primal problem has an integer optimum solution, if b is integer. In fact, they showed Theorem 5.18, which implies (since if (5.54) has an integer optimum solution, the optimum value is an integer):

Theorem 5.22. *If $Ax \leq b$ is TDI and b is integer, then $Ax \leq b$ determines an integer polyhedron.*

So total dual integrality implies 'primal integrality'. For combinatorial applications, the following observation is useful:

Theorem 5.23. *Let A be a nonnegative integer $m \times n$ matrix such that the system $x \geq \mathbf{0}, Ax \geq \mathbf{1}$ is TDI. Then also the system $\mathbf{0} \leq x \leq \mathbf{1}, Ax \geq \mathbf{1}$ is TDI.*

Proof. Choose $c \in \mathbb{Z}^n$. Let c_+ arise from c by setting negative components to 0. By the total dual integrality of $x \geq \mathbf{0}$, $Ax \geq \mathbf{1}$, there exist integer optimum solutions x, y of

(5.55) $$\min\{c_+^\mathsf{T}x \mid x \geq \mathbf{0}, Ax \geq \mathbf{1}\} = \max\{y^\mathsf{T}\mathbf{1} \mid y \geq \mathbf{0}, y^\mathsf{T}A \leq c_+^\mathsf{T}\}.$$

As A is nonnegative and integer and as $c_+ \geq \mathbf{0}$, we may assume that $x \leq \mathbf{1}$. Moreover, we can assume that $x_i = 1$ if $(c_+)_i = 0$, that is, if $c_i \leq 0$.

Let $z := c - c_+$. So $z \leq \mathbf{0}$. We show that x, y, z are optimum solutions of

(5.56) $$\begin{aligned}&\min\{c^\mathsf{T}x \mid \mathbf{0} \leq x \leq \mathbf{1}, Ax \geq \mathbf{1}\}\\ &= \max\{y^\mathsf{T}\mathbf{1} + z^\mathsf{T}\mathbf{1} \mid y \geq \mathbf{0}, z \leq \mathbf{0}, y^\mathsf{T}A + z^\mathsf{T} \leq c^\mathsf{T}\}.\end{aligned}$$

Indeed, x is feasible, as $\mathbf{0} \leq x \leq \mathbf{1}$ and $Ax \geq \mathbf{1}$. Moreover, y, z is feasible, as $y^\mathsf{T}A + z^\mathsf{T} \leq c_+^\mathsf{T} + z^\mathsf{T} = c^\mathsf{T}$. Optimality of x, y, z follows from

(5.57) $$c^\mathsf{T}x = c_+^\mathsf{T}x + z^\mathsf{T}x = y^\mathsf{T}\mathbf{1} + z^\mathsf{T}x = y^\mathsf{T}\mathbf{1} + z^\mathsf{T}\mathbf{1}.$$ ∎

In certain cases, to obtain total dual integrality one can restrict oneself to nonnegative objective functions:

Theorem 5.24. *Let A be a nonnegative $m \times n$ matrix and let $b \in \mathbb{R}^m_+$. Then $x \geq \mathbf{0}, Ax \leq b$ is TDI if and only if $\min\{y^\mathsf{T}b \mid y \geq \mathbf{0}, y^\mathsf{T}A \geq c^\mathsf{T}\}$ is attained by an integer optimum solution (if finite), for each $c \in \mathbb{Z}^n_+$.*

Proof. Necessity is trivial. To see sufficiency, let $c \in \mathbb{Z}^n$ with $\min\{y^\mathsf{T}b \mid y \geq \mathbf{0}, y^\mathsf{T}A \geq c^\mathsf{T}\}$ finite. Let it be attained by y. Let c_+ arise from c by setting negative components to 0. Then

(5.58) $$\min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \geq c_+^{\mathsf{T}}\} = \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \geq c^{\mathsf{T}}\},$$

since $y^{\mathsf{T}}A \geq \mathbf{0}$ if $y \geq \mathbf{0}$. As the first minimum has an integer optimum solution, also the second minimum has an integer optimum solution. ∎

Total dual integrality is maintained under setting an inequality to an equality (Theorem 22.2 in Schrijver [1986b]):

Theorem 5.25. *Let $Ax \leq b$ be TDI and let $A'x \leq b'$ arise from $Ax \leq b$ by adding $-a^{\mathsf{T}}x \leq -\beta$ for some inequality $a^{\mathsf{T}}x \leq \beta$ in $Ax \leq b$. Then also $A'x \leq b'$ is TDI.*

Total dual integrality is also maintained under translation of the solution set, as follows directly from the definition of total dual integrality:

Theorem 5.26. *If $Ax \leq b$ is TDI and $w \in \mathbb{R}^n$, then $Ax \leq b - Aw$ is TDI.*

For future reference, we prove:

Theorem 5.27. *Let $A_{11}, A_{12}, A_{21}, A_{22}$ be matrices and let b_1, b_2 be column vectors, such that the system*

(5.59) $$\begin{aligned} A_{1,1}x_1 + A_{1,2}x_2 &= b_1, \\ A_{2,1}x_1 + A_{2,2}x_2 &\leq b_2 \end{aligned}$$

is TDI and such that $A_{1,1}$ is nonsingular. Then also the system

(5.60) $$(A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2})x_2 \leq b_2 - A_{2,1}A_{1,1}^{-1}b_1$$

is TDI.

Proof. We may assume that $b_1 = \mathbf{0}$, since by Theorem 5.26 total dual integrality is invariant under replacing (5.59) by

(5.61) $$\begin{aligned} A_{1,1}x_1 + A_{1,2}x_2 &= b_1 - A_{1,1}A_{1,1}^{-1}b_1 = \mathbf{0}, \\ A_{2,1}x_1 + A_{2,2}x_2 &\leq b_2 - A_{2,1}A_{1,1}^{-1}b_1. \end{aligned}$$

Let x_2 minimize $c^{\mathsf{T}}x_2$ over (5.60), for some integer vector c of appropriate dimension. Define $x_1 := -A_{1,1}^{-1}A_{1,2}x_2$. Then x_1, x_2 minimizes $c^{\mathsf{T}}x_2$ over (5.59), since any solution x_1', x_2' of (5.59) satisfies $x_1' = -A_{1,1}^{-1}A_{1,2}x_2'$, and therefore x_2' satisfies (5.60); hence $c^{\mathsf{T}}x_2' \geq c^{\mathsf{T}}x_2$.

Let y_1, y_2 be an integer optimum solution of the problem dual to maximizing $c^{\mathsf{T}}x_2$ over (5.59). So y_1, y_2 satisfy

(5.62) $$y_1^{\mathsf{T}}A_{1,1} + y_2^{\mathsf{T}}A_{2,1} = \mathbf{0},\ y_1^{\mathsf{T}}A_{1,2} + y_2^{\mathsf{T}}A_{2,2} = c^{\mathsf{T}},\ y_2^{\mathsf{T}}b_2 = c^{\mathsf{T}}x_2.$$

Hence

(5.63) $$y_2^{\mathsf{T}}(A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2}) = y_2^{\mathsf{T}}A_{2,2} + y_1^{\mathsf{T}}A_{1,2} = c^{\mathsf{T}}$$

and

(5.64) $y_2^{\mathsf{T}} b_2 = c^{\mathsf{T}} x_2.$

So y_2 is an integer optimum solution of the problem dual to maximizing $c^{\mathsf{T}} x_2$ over (5.60). ∎

This has as consequence (where a_0 is a column vector):

Corollary 5.27a. *If $x_0 = \beta$, $a_0 x_0 + Ax \leq b$ is TDI, then $Ax \leq b - \beta a_0$ is TDI.*

Proof. This is a special case of Theorem 5.27. ∎

We also have:

Theorem 5.28. *Let $A = [a_1 \;\; a_2 \;\; A'']$ be an integer $m \times n$ matrix and let $b \in \mathbb{R}^m$. Let A' be the $m \times (n-1)$ matrix $[a_1 + a_2 \;\; A'']$. Then $A'x' \leq b$ is TDI if and only if $Ax \leq b, x_1 - x_2 = 0$ is TDI.*

Proof. To see necessity, choose $c \in \mathbb{Z}^n$. Let $c' := (c_1 + c_2, c_3, \ldots, c_n)^{\mathsf{T}}$. Then

(5.65) $\mu := \max\{c^{\mathsf{T}} x \mid Ax \leq b, x_1 - x_2 = 0\} = \max\{{c'}^{\mathsf{T}} x' \mid A'x' \leq b\}.$

Let $y \in \mathbb{Z}_+^m$ be an integer optimum dual solution of the second maximum. So $y^{\mathsf{T}} A' = c'$ and $y^{\mathsf{T}} b = \mu$. Then $y^{\mathsf{T}} a_1 + y^{\mathsf{T}} a_2 = c_1 + c_2$. Hence $y^{\mathsf{T}} A = c^{\mathsf{T}} + \lambda(1, -1, 0, \ldots, 0)$ for some $\lambda \in \mathbb{Z}$. So y, λ form an integer optimum dual solution of the first maximum.

To see sufficiency, choose $c' = (c_2, \ldots, c_n)^{\mathsf{T}} \in \mathbb{Z}^{n-1}$. Define $c := (0, c_2, \ldots, c_n)^{\mathsf{T}}$. Again we have (5.65). Let $y \in \mathbb{Z}_+^m$, $\lambda \in \mathbb{Z}$ constitute an integer optimum dual solution of the first maximum, where λ corresponds to the constraint $x_1 - x_2 = 0$. So $y^{\mathsf{T}} A + \lambda(1, -1, 0, \ldots, 0) = c$ and $y^{\mathsf{T}} b = \mu$. Hence $y^{\mathsf{T}} A' = c^{\mathsf{T}}$, and therefore, y is an integer optimum dual solution of the second maximum. ∎

Let A be a rational $m \times n$ matrix and let $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$. Consider the following series of inequalities (where a vector z is *half-integer* if $2z$ is integer):

(5.66)
$$\begin{aligned} &\max\{c^{\mathsf{T}} x \mid Ax \leq b, x \text{ integer}\} \leq \max\{c^{\mathsf{T}} x \mid Ax \leq b\} \\ &= \min\{y^{\mathsf{T}} b \mid y \geq \mathbf{0}, y^{\mathsf{T}} A = c^{\mathsf{T}}\} \\ &\leq \min\{y^{\mathsf{T}} b \mid y \geq \mathbf{0}, y^{\mathsf{T}} A = c^{\mathsf{T}}, y \text{ half-integer}\} \\ &\leq \min\{y^{\mathsf{T}} b \mid y \geq \mathbf{0}, y^{\mathsf{T}} A = c^{\mathsf{T}}, y \text{ integer}\}. \end{aligned}$$

Under certain circumstances, equality in the last inequality implies equality throughout:

Theorem 5.29. *Let $Ax \leq b$ be a system with A and b rational. Then $Ax \leq b$ is TDI if and only if*

(5.67) $\min\{y^{\mathsf{T}} b \mid y \geq \mathbf{0}, y^{\mathsf{T}} A = c^{\mathsf{T}}, y \textit{ half-integer}\}$

is finite and is attained by an integer optimum solution y, for each integer vector c with $\max\{c^{\mathsf{T}}x \mid Ax \leq b\}$ *finite.*

Proof. Necessity follows directly from (5.66). To see sufficiency, choose $c \in \mathbb{Z}^n$ with $\max\{c^{\mathsf{T}}x \mid Ax \leq b\}$ finite. We must show that $\min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}\}$ is attained by an integer optimum solution.

For each $k \geq 1$, define

(5.68) $$\alpha_k = \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = kc^{\mathsf{T}}, y \text{ integer}\}.$$

This is well-defined, as $\max\{kc^{\mathsf{T}}x \mid Ax \leq b\}$ is finite.

The condition in the theorem gives that, for each $t \geq 0$,

(5.69) $$\frac{\alpha_{2^t}}{2^t} = \alpha_1.$$

This can be shown by induction on t, the case $t = 0$ being trivial. If $t \geq 1$, then

(5.70) $$\begin{aligned} \alpha_{2^t} &= \min\{y^{\mathsf{T}}b \mid y^{\mathsf{T}}A = 2^t c^{\mathsf{T}}, y \in \mathbb{Z}_+^m\} \\ &= 2\min\{y^{\mathsf{T}}b \mid y^{\mathsf{T}}A = 2^{t-1}c^{\mathsf{T}}, y \in \tfrac{1}{2}\mathbb{Z}_+^m\} \\ &= 2\min\{y^{\mathsf{T}}b \mid y^{\mathsf{T}}A = 2^{t-1}c^{\mathsf{T}}, y \in \mathbb{Z}_+^m\} = 2\alpha_{2^{t-1}}, \end{aligned}$$

implying (5.69) by induction.

Now $\alpha_{k+l} \leq \alpha_k + \alpha_l$ for all k, l. Hence we can apply Fekete's lemma, and get:

(5.71) $$\begin{aligned} &\min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}\} = \min_k \frac{\alpha_k}{k} = \lim_{k\to\infty} \frac{\alpha_k}{k} = \lim_{t\to\infty} \frac{\alpha_{2^t}}{2^t} \\ &= \alpha_1. \end{aligned}$$ ∎

The following analogue of Carathéodory's theorem holds (Cook, Fonlupt, and Schrijver [1986]):

Theorem 5.30. *Let $Ax \leq b$ be a totally dual integral system in n dimensions and let $c \in \mathbb{Z}^n$. Then $\min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \geq c^{\mathsf{T}}\}$ has an integer optimum solution y with at most $2n - 1$ nonzero components.*

(Theorem 22.12 in Schrijver [1986b].)

We also will need the following substitution property:

Theorem 5.31. *Let $A_1x \leq b_1, A_2x \leq b_2$ be a TDI system with A_1 integer, and let $A_1' \leq b_1'$ be a TDI system with*

(5.72) $$\{x \mid A_1x \leq b_1\} = \{x \mid A_1'x \leq b_1'\}.$$

Then the system $A_1'x \leq b_1', A_2x \leq b_2$ is TDI.

Proof. Let $c \in \mathbb{Z}^n$ with

(5.73) $$\begin{aligned} &\max\{c^{\mathsf{T}}x \mid A_1'x \leq b_1', A_2x \leq b_2\} \\ &= \min\{y^{\mathsf{T}}b_1' + z^{\mathsf{T}}b_2 \mid y, z \geq \mathbf{0}, y^{\mathsf{T}}A_1' + z^{\mathsf{T}}A_2 = c^{\mathsf{T}}\} \end{aligned}$$

finite. By (5.72), also

(5.74) $$\max\{c^{\mathsf{T}}x \mid A_1x \leq b_1, A_2x \leq b_2\} = \min\{y^{\mathsf{T}}b_1 + z^{\mathsf{T}}b_2 \mid y,z \geq \mathbf{0}, y^{\mathsf{T}}A_1 + z^{\mathsf{T}}A_2 = c^{\mathsf{T}}\}$$

is finite. Hence, since $A_1x \leq b_1, A_2x \leq b_2$ is TDI, the minimum in (5.74) has an integer optimum solution y,z. Set $d := y^{\mathsf{T}}A_1$. Then, as d is an integer vector,

(5.75) $$y^{\mathsf{T}}b_1 = \min\{u^{\mathsf{T}}b_1 \mid u \geq \mathbf{0}, u^{\mathsf{T}}A_1 = d^{\mathsf{T}}\} = \max\{d^{\mathsf{T}}x \mid A_1x \leq b_1\} = \max\{d^{\mathsf{T}}x \mid A'_1x \leq b'_1\} = \min\{v^{\mathsf{T}}b'_1 \mid v \geq \mathbf{0}, v^{\mathsf{T}}A'_1 = d^{\mathsf{T}}\}$$

is finite. Hence, since $A'_1x \leq b'_1$ is TDI, the last minimum in (5.75) has an integer optimum solution v. Then v,z is an integer optimum solution of the minimum in (5.73). ∎

A system $Ax \leq b$ is called *totally dual half-integral* if A and b are rational and for each $c \in \mathbb{Z}^n$, the dual of maximizing $c^{\mathsf{T}}x$ over $Ax \leq b$ has a half-integer optimum solution, if it is finite. Similarly, $Ax \leq b$ is called *totally dual quarter-integral* if A and b are rational and for each $c \in \mathbb{Z}^n$, the dual of maximizing $c^{\mathsf{T}}x$ over $Ax \leq b$ has a quarter-integer optimum solution y, if it is finite.

5.18. Hilbert bases and minimal TDI systems

For any $X \subseteq \mathbb{R}^n$ we denote

(5.76) $$\text{lattice}X := \{\lambda_1x_1 + \cdots + \lambda_kx_k \mid k \geq 0, \lambda_1,\ldots,\lambda_k \in \mathbb{Z}, x_1,\ldots,x_k \in X\}.$$

A subset L of $\mathbb{R}^n$ is called a *lattice* if $L = \text{lattice}X$ for some base X of $\mathbb{R}^n$. So for general X, lattice$X$ need not be a lattice.

The *dual lattice* of X is, by definition:

(5.77) $$\{x \in \mathbb{R}^n \mid y^{\mathsf{T}}x \in \mathbb{Z} \text{ for each } y \in X\}.$$

Again, this need not be a lattice in the proper sense.

A set X of vectors is called a *Hilbert base* if each vector in lattice$X\cap$cone$X$ is a nonnegative integer combination of vectors in X. The Hilbert base is called *integer* if it consists of integer vectors only.

One may show:

(5.78) Each rational polyhedral cone C is generated by an integer Hilbert base. If C is pointed, there exists a unique inclusionwise minimal integer Hilbert base generating C.

(Theorem 16.4 in Schrijver [1986b].)

There is a close relation between Hilbert bases and total dual integrality:

Theorem 5.32. *A rational system $Ax \leq b$ is TDI if and only if for each face F of $P := \{x \mid Ax \leq b\}$, the rows of A which are active in F form a Hilbert base.*

(Theorem 22.5 in Schrijver [1986b].)

(5.78) and Theorem 5.32 imply (Giles and Pulleyblank [1979], Schrijver [1981b]):

Theorem 5.33. *Each rational polyhedron P is determined by a TDI system $Ax \leq b$ with A integer. If moreover P is full-dimensional, there exists a unique minimal such system.*

(Theorem 22.6 in Schrijver [1986b].)

5.19. The integer rounding and decomposition properties

A system $Ax \leq b$ is said to have the *integer rounding property* if $Ax \leq b$ is rational and

(5.79) $$\min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}, y \text{ integer}\} \\ = \lceil \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}\} \rceil$$

for each integer vector c for which $\min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}A = c^{\mathsf{T}}\}$ is finite. So any TDI system has the integer rounding property.

A polyhedron P is said to have the *integer decomposition property* if for each natural number k, each integer vector in $k \cdot P$ is the sum of k integer vectors in P.

Baum and Trotter [1978] showed that an integer matrix A is totally unimodular if and only if the polyhedron $\{x \mid x \geq \mathbf{0}, Ax \leq b\}$ has the integer decomposition property for each integer vector b. In another paper, Baum and Trotter [1981] observed the following relation between the integer rounding and the integer decomposition property:

(5.80) Let A be a nonnegative integer matrix. Then the system $x \geq \mathbf{0}, Ax \geq \mathbf{1}$ has the integer rounding property if and only if the blocking polyhedron $B(P)$ of $P := \{x \mid x \geq \mathbf{0}, Ax \geq \mathbf{1}\}$ has the integer decomposition property and all minimal integer vectors in $B(P)$ are transposes of rows of A (minimal with respect to $\leq$).

Similarly,

(5.81) Let A be a nonnegative integer matrix. Then the system $x \geq \mathbf{0}, Ax \leq \mathbf{1}$ has the integer rounding property if and only if the antiblocking polyhedron $A(P)$ of $P := \{x \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\}$ has the integer decomposition property and all maximal integer vectors in $A(P)$ are transposes of rows of A (maximal with respect to $\leq$).

(Theorem 22.19 in Schrijver [1986b].)

5.20. Box-total dual integrality

A system $Ax \leq b$ is called *box-totally dual integral*, or just *box-TDI*, if the system $d \leq x \leq c, Ax \leq b$ is totally dual integral for each choice of vectors $d, c \in \mathbb{R}^n$. By Theorem 5.22,

(5.82) if $Ax \leq b$ is box-totally dual integral, then the polyhedron $\{x \mid Ax \leq b\}$ is box-integer.

We will need the following two results.

Theorem 5.34. *If $Ax \leq b$ is box-TDI in n dimensions and $w \in \mathbb{R}^n$, then $Ax \leq b - Aw$ is box-TDI.*

Proof. Directly from the definition of box-total dual integrality. ∎

Theorem 5.35. *Let $Ax \leq b$ be a system of linear inequalities, with A an $m \times n$ matrix. Suppose that for each $c \in \mathbb{R}^n$, $\max\{c^{\mathsf{T}}x \mid Ax \leq b\}$ has (if finite) an optimum dual solution $y \in \mathbb{R}^m_+$ such that the rows of A corresponding to positive components of y form a totally unimodular submatrix of A. Then $Ax \leq b$ is box-TDI.*

Proof. Choose $d, c \in \mathbb{R}^n$, with $d \leq c$, and choose $c \in \mathbb{Z}^n$. Consider the dual of maximizing $c^{\mathsf{T}}x$ over $Ax \leq b$, $d \leq x \leq c$:

(5.83) $$\min\{y^{\mathsf{T}}b + z_1^{\mathsf{T}}c - z_2^{\mathsf{T}}d \mid y \in \mathbb{R}^m_+, z_1, z_2 \in \mathbb{R}^n_+, y^{\mathsf{T}}A + z_1^{\mathsf{T}} - z_2^{\mathsf{T}} = c^{\mathsf{T}}\}.$$

Let y, z_1, z_2 attain this optimum. Define $c' := c - z_1 + z_2$. By assumption, $\min\{{y'}^{\mathsf{T}}b \mid y' \in \mathbb{R}^m_+, {y'}^{\mathsf{T}}A = {c'}^{\mathsf{T}}\}$ has an optimum solution such that the rows of A corresponding to positive components of y' form a totally unimodular matrix. Now y', z_1, z_2 is an optimum solution of (5.83). Also, the rows in $Ax \leq b$, $d \leq x \leq c$ corresponding to positive components of y', z_1, z_2 form a totally unimodular matrix. Hence by Corollary 5.20b, (5.83) has an integer optimum solution. ∎

5.21. The integer hull and cutting planes

Let P be a rational polyhedron. The *integer hull* P_{I} of P is the convex hull of the integer vectors in P:

(5.84) $$P_{\mathrm{I}} = \text{conv.hull}(P \cap \mathbb{Z}^n).$$

It can be shown that P_{I} is a rational polyhedron again.

Consider any rational affine halfspace $H = \{x \mid c^{\mathsf{T}}x \leq \delta\}$, where c is a nonzero integer vector such that the g.c.d. of its components is equal to 1 and where $\delta \in \mathbb{Q}$. Then it is easy to show that

(5.85) $$H_{\mathrm{I}} = \{x \mid c^{\mathsf{T}}x \leq \lfloor\delta\rfloor\}.$$

The inequality $c^{\mathsf{T}}x \leq \lfloor\delta\rfloor$ (or, more correctly, the hyperplane $\{x \mid c^{\mathsf{T}}x = \lfloor\delta\rfloor\}$) is called a *cutting plane*.

Define for any rational polyhedron P:

(5.86) $$P' := \bigcap_{H \supseteq P} H_{\mathrm{I}},$$

where H ranges over all rational affine halfspaces H containing P. Then P' is a rational polyhedron contained in P. Since $P \subseteq H$ implies $P_{\mathrm{I}} \subseteq H_{\mathrm{I}}$, we know

(5.87) $$P_{\mathrm{I}} \subseteq P' \subseteq P.$$

For $k \in \mathbb{Z}_+$, define $P^{(k)}$ inductively by:

(5.88) $$P^{(0)} := P \text{ and } P^{(k+1)} := (P^{(k)})'.$$

Then (Gomory [1958,1960], Chvátal [1973a], Schrijver [1980b]):

Theorem 5.36. *For each rational polyhedron there exists a $k \in \mathbb{Z}_+$ with $P_{\mathrm{I}} = P^{(k)}$.*

(For a proof, see Theorem 23.2 in Schrijver [1986b].)

5.21a. Background literature

Most background on polyhedra and linear and integer programming needed for this book can be found in Schrijver [1986b].

More background can be found in Dantzig [1963] (linear programming), Grünbaum [1967] (polytopes), Hu [1969] (integer programming), Garfinkel and Nemhauser [1972a] (integer programming), Brøndsted [1983] (polytopes), Chvátal [1983] (linear programming), Lovász [1986] (ellipsoid method), Grötschel, Lovász, and Schrijver [1988] (ellipsoid method), Nemhauser and Wolsey [1988] (integer programming), Padberg [1995] (linear programming), Ziegler [1995] (polytopes), and Wolsey [1998] (integer programming).