time $O(n^{3/4}m\log L)$ and $O(nm\log_{2+\frac{m}{n}} L)$. Goldberg and Tarjan [1990] gave an $O(nm\log(nL))$ algorithm for minimum-cost flow with unit capacities. More complexity results follow from the table in Section 12.5a. Disjoint $s-t$ cuts were considered by Wagner [1990] and Talluri and Wagner [1994].

13.3. Network matrices

Let $D=(V,A)$ be a digraph and let $T=(V,A')$ be a directed tree. Let C be the $A'\times A$ matrix defined as follows. Take $a'\in A'$ and $a=(u,v)\in A$ and let P be the undirected $u-v$ path in T. Define

$$C_{a',a} := \begin{cases} +1 \text{ if } a' \text{ occurs in forward direction in } P, \\ -1 \text{ if } a' \text{ occurs in backward direction in } P, \\ \ \ 0 \text{ if } a' \text{ does not occur in } P. \end{cases} \tag{13.44}$$

Matrix C is called a *network matrix, generated by* $T=(V,A')$ and $D=(V,A)$.

Theorem 13.19. *Any submatrix of a network matrix is again a network matrix.*

Proof. Deleting column indexed by $a\in A$ corresponds to deleting a from $D=(V,A)$. Deleting the row indexed by $a'=(u,v)\in A'$ corresponds to contracting a' in the tree $T=(V,A')$ and identifying u and v in D. ∎

The following theorem is implicit in Tutte [1965a]:

Theorem 13.20. *A network matrix is totally unimodular.*

Proof. By Theorem 13.19, it suffices to show that any square network matrix C has determinant 0, 1, or -1. We prove this by induction on the size of C, the case of 1×1 matrices being trivial. We use notation as above.

Assume that $\det C\neq 0$. Let u be an end vertex of T and let a' be the arc in T incident with u. By reversing orientations, we can assume that each arc in A and A' incident with u, has u as tail. Then, by definition of C, the row indexed by a' contains only 0's and 1's.

Consider two 1's in row a'. That is, consider two columns indexed by arcs $a_1=(u,v_1)$ and $a_2=(u,v_2)$ in A. Subtracting column a_1 from column a_2, has the effect of resetting a_2 to (v_1,v_2). So after that, column a_2 has a 0 in position a'. Since this subtraction does not change the determinant, we can assume that there is exactly one arc in A incident with u; that is, row a' has exactly one nonzero. Then by expanding the determinant by row a', we obtain inductively that $\det C=\pm 1$. ∎

The incidence matrix of a digraph $D=(V,A)$ is a network matrix: add a new vertex u to D giving digraph $D'=(V\cup\{u\},A)$. Let T be the directed