LINEAR PROGRAMMING

1. Introduction.

A linear programming problem may be defined as the problem of *maximizing or minimizing a linear function subject to linear constraints.* The constraints may be equalities or inequalities. Here is a simple example.

Find numbers x_1 and x_2 that maximize the sum $x_1 + x_2$ subject to the constraints $x_1 \ge 0$, $x_2 \ge 0$, and

$$\begin{aligned} x_1 + 2x_2 &\le 4 \\ 4x_1 + 2x_2 &\le 12 \\ -x_1 + x_2 &\le 1 \end{aligned}$$

In this problem there are two unknowns, and five constraints. All the constraints are inequalities and they are all linear in the sense that each involves an inequality in some linear function of the variables. The first two constraints, $x_1 \ge 0$ and $x_2 \ge 0$, are special. These are called *nonnegativity constraints* and are often found in linear programming problems. The other constraints are then called the *main constraints.* The function to be maximized (or minimized) is called the *objective function.* Here, the objective function is $x_1 + x_2$.

Since there are only two variables, we can solve this problem by graphing the set of points in the plane that satisfies all the constraints (called the constraint set) and then finding which point of this set maximizes the value of the objective function. Each inequality constraint is satisfied by a half-plane of points, and the constraint set is the intersection of all the half-planes. In the present example, the constraint set is the five-sided figure shaded in Figure 1.

We seek the point (x_1, x_2), that achieves the maximum of $x_1 + x_2$ as (x_1, x_2) ranges over this constraint set. The function $x_1 + x_2$ is constant on lines with slope -1, for example the line $x_1 + x_2 = 1$, and as we move this line further from the origin up and to the right, the value of $x_1 + x_2$ increases. Therefore, we seek the line of slope -1 that is farthest from the origin and still touches the constraint set. This occurs at the intersection of the lines $x_1 + 2x_2 = 4$ and $4x_1 + 2x_2 = 12$, namely, $(x_1, x_2) = (8/3, 2/3)$. The value of the objective function there is $(8/3) + (2/3) = 10/3$.

Exercises 1 and 2 can be solved as above by graphing the feasible set.

It is easy to see in general that the objective function, being linear, always takes on its maximum (or minimum) value at a corner point of the constraint set, provided the

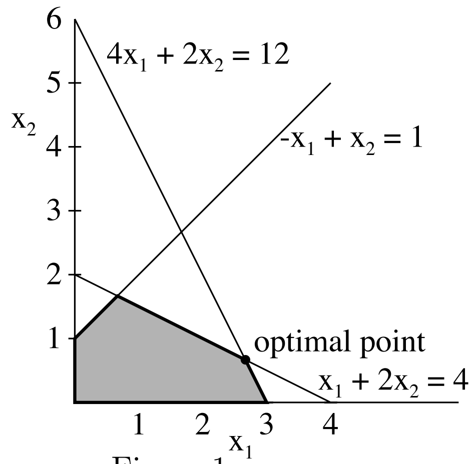

Figure 1.

constraint set is bounded. Occasionally, the maximum occurs along an entire edge or face of the constraint set, but then the maximum occurs at a corner point as well.

Not all linear programming problems are so easily solved. There may be many variables and many constraints. Some variables may be constrained to be nonnegative and others unconstrained. Some of the main constraints may be equalities and others inequalities. However, two classes of problems, called here the *standard maximum problem* and the *standard minimum problem*, play a special role. In these problems, all variables are constrained to be nonnegative, and all main constraints are inequalities.

We are given an m-vector, $\boldsymbol{b} = (b_1, \ldots, b_m)^{\mathsf{T}}$, an n-vector, $\boldsymbol{c} = (c_1, \ldots, c_n)^{\mathsf{T}}$, and an $m \times n$ matrix,

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

of real numbers.

The Standard Maximum Problem: Find an n-vector, $\boldsymbol{x} = (x_1, \ldots, x_n)^{\mathsf{T}}$, to maximize

$$\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} = c_1x_1 + \cdots + c_nx_n$$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \end{aligned} \qquad (\text{or } \boldsymbol{Ax} \le \boldsymbol{b})$$

and

$$x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0 \qquad (\text{or } \boldsymbol{x} \ge \boldsymbol{0}).$$

The Standard Minimum Problem: Find an m-vector, $\boldsymbol{y} = (y_1, \ldots, y_m)$, to minimize
$$\boldsymbol{y}^\mathsf{T}\boldsymbol{b} = y_1 b_1 + \cdots + y_m b_m$$
subject to the constraints
$$\begin{aligned} y_1 a_{11} + y_2 a_{21} + \cdots + y_m a_{m1} &\geq c_1 \\ y_1 a_{12} + y_2 a_{22} + \cdots + y_m a_{m2} &\geq c_2 \\ &\vdots \\ y_1 a_{1n} + y_2 a_{2n} + \cdots + y_m a_{mn} &\geq c_n \end{aligned} \qquad (\text{or } \boldsymbol{y}^\mathsf{T}\boldsymbol{A} \geq \boldsymbol{c}^\mathsf{T})$$
and
$$y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0 \qquad (\text{or } \boldsymbol{y} \geq \mathbf{0}).$$

Note that the main constraints are written as $\leq$ for the standard maximum problem and $\geq$ for the standard minimum problem. The introductory example is a standard maximum problem.

We now present examples of four general linear programming problems. Each of these problems has been extensively studied.

Example 1. The Diet Problem. There are m different types of food, $F_1, \ldots, F_m$, that supply varying quantities of the n nutrients, $N_1, \ldots, N_n$, that are essential to good health. Let c_j be the minimum daily requirement of nutrient, N_j. Let b_i be the price per unit of food, F_i. Let a_{ij} be the amount of nutrient N_j contained in one unit of food F_i. The problem is to supply the required nutrients at minimum cost.

Let y_i be the number of units of food F_i to be purchased per day. The cost per day of such a diet is
$$b_1 y_1 + b_2 y_2 + \cdots + b_m y_m. \tag{1}$$
The amount of nutrient N_j contained in this diet is
$$a_{1j} y_1 + a_{2j} y_2 + \cdots + a_{mj} y_m$$
for $j = 1, \ldots, n$. We do not consider such a diet unless all the minimum daily requirements are met, that is, unless
$$a_{1j} y_1 + a_{2j} y_2 + \cdots + a_{mj} y_m \geq c_j \qquad \text{for } j = 1, \ldots, n. \tag{2}$$
Of course, we cannot purchase a negative amount of food, so we automatically have the constraints
$$y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0. \tag{3}$$
Our problem is: minimize (1) subject to (2) and (3). This is exactly the standard minimum problem.

Example 2. The Transportation Problem. There are I ports, or production plants, $P_1, \ldots, P_I$, that supply a certain commodity, and there are J markets, $M_1, \ldots, M_J$, to which this commodity must be shipped. Port P_i possesses an amount s_i of the commodity ($i = 1, 2, \ldots, I$), and market M_j must receive the amount r_j of the commodity ($j = 1, \ldots, J$). Let b_{ij} be the cost of transporting one unit of the commodity from port P_i to market M_j. The problem is to meet the market requirements at minimum transportation cost.

Let y_{ij} be the quantity of the commodity shipped from port P_i to market M_j. The total transportation cost is

$$\sum_{i=1}^{I} \sum_{j=1}^{J} y_{ij} b_{ij}. \tag{4}$$

The amount sent from port P_i is $\sum_{j=1}^{J} y_{ij}$ and since the amount available at port P_i is s_i, we must have

$$\sum_{j=1}^{J} y_{ij} \le s_i \qquad \text{for } i = 1, \ldots, I. \tag{5}$$

The amount sent to market M_j is $\sum_{i=1}^{I} y_{ij}$, and since the amount required there is r_j, we must have

$$\sum_{i=1}^{I} y_{ij} \ge r_j \qquad \text{for } j = 1, \ldots, J. \tag{6}$$

It is assumed that we cannot send a negative amount from P_I to M_j, we have

$$y_{ij} \ge 0 \qquad \text{for } i = 1, \ldots, I \text{ and } j = 1, \ldots, J. \tag{7}$$

Our problem is: minimize (4) subject to (5), (6) and (7).

Let us put this problem in the form of a standard minimum problem. The number of y variables is IJ, so $m = IJ$. But what is n? It is the total number of main constraints. There are $n = I + J$ of them, but some of the constraints are $\ge$ constraints, and some of them are $\le$ constraints. In the standard minimum problem, all constraints are $\ge$. This can be obtained by multiplying the constraints (5) by -1:

$$\sum_{j=1}^{J} (-1) y_{ij} \ge -s_j \qquad \text{for } i = 1, \ldots, I. \tag{5$'$}$$

The problem "minimize (4) subject to (5′), (6) and (7)" is now in standard form. In Exercise 3, you are asked to write out the matrix $\boldsymbol{A}$ for this problem.

Example 3. The Activity Analysis Problem. There are n activities, $A_1, \ldots, A_n$, that a company may employ, using the available supply of m resources, $R_1, \ldots, R_m$ (labor hours, steel, etc.). Let b_i be the available supply of resource R_i. Let a_{ij} be the amount of resource R_i used in operating activity A_j at unit intensity. Let c_j be the net value to the company of operating activity A_j at unit intensity. The problem is to choose the intensities which the various activities are to be operated to maximize the value of the output to the company subject to the given resources.

Let x_j be the intensity at which A_j is to be operated. The value of such an activity allocation is

$$\sum_{j=1}^{n} c_j x_j. \tag{8}$$

The amount of resource R_i used in this activity allocation must be no greater than the supply, b_i; that is,

$$\sum_{j=1} a_{ij} x_j \le b_i \qquad \text{for } i = 1, \ldots, m. \tag{9}$$

It is assumed that we cannot operate an activity at negative intensity; that is,

$$x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0. \tag{10}$$

Our problem is: maximize (8) subject to (9) and (10). This is exactly the standard maximum problem.

Example 4. The Optimal Assignment Problem. There are I persons available for J jobs. The value of person i working 1 day at job j is a_{ij}, for $i = 1, \ldots, I$, and $j = 1, \ldots, J$. The problem is to choose an assignment of persons to jobs to maximize the total value.

An assignment is a choice of numbers, x_{ij}, for $i = 1, \ldots, I$, and $j = 1, \ldots, J$, where x_{ij} represents the proportion of person i's time that is to be spent on job j. Thus,

$$\sum_{j=1}^{J} x_{ij} \le 1 \qquad \text{for } i = 1, \ldots, I \tag{11}$$

$$\sum_{i=1}^{I} x_{ij} \le 1 \qquad \text{for } j = 1, \ldots, J \tag{12}$$

and

$$x_{ij} \ge 0 \qquad \text{for } i = 1, \ldots, I \text{ and } j = 1, \ldots, J. \tag{13}$$

Equation (11) reflects the fact that a person cannot spend more than 100% of his time working, (12) means that only one person is allowed on a job at a time, and (13) says that no one can work a negative amount of time on any job. Subject to (11), (12) and (13), we wish to maximize the total value,

$$\sum_{i=1}^{I} \sum_{j=1}^{J} a_{ij} x_{ij}. \tag{14}$$

This is a standard maximum problem with $m = I + J$ and $n = IJ$.

Terminology.

The function to be maximized or minimized is called the **objective function**.

A vector, $\boldsymbol{x}$ for the standard maximum problem or $\boldsymbol{y}$ for the standard minimum problem, is said to be **feasible** if it satisfies the corresponding constraints.

The set of feasible vectors is called the **constraint set**.

A linear programming problem is said to be **feasible** if the constraint set is not empty; otherwise it is said to be **infeasible**.

A feasible maximum (resp. minimum) problem is said to be **unbounded** if the objective function can assume arbitrarily large positive (resp. negative) values at feasible vectors; otherwise, it is said to be **bounded**. Thus there are three possibilities for a linear programming problem. It may be bounded feasible, it may be unbounded feasible, and it may be infeasible.

The **value** of a bounded feasible maximum (resp, minimum) problem is the maximum (resp. minimum) value of the objective function as the variables range over the constraint set.

A feasible vector at which the objective function achieves the value is called **optimal**.

All Linear Programming Problems Can be Converted to Standard Form. A linear programming problem was defined as maximizing or minimizing a linear function subject to linear constraints. All such problems can be converted into the form of a standard maximum problem by the following techniques.

A minimum problem can be changed to a maximum problem by multiplying the objective function by -1. Similarly, constraints of the form $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$ can be changed into the form $\sum_{j=1}^{n}(-a_{ij})x_j \leq -b_i$. Two other problems arise.

(1) *Some constraints may be equalities.* An equality constraint $\sum_{j=1}^{n} a_{ij}x_j = b_i$ may be removed, by solving this constraint for some x_j for which $a_{ij} \neq 0$ and substituting this solution into the other constraints and into the objective function wherever x_j appears. This removes one constraint and one variable from the problem.

(2) *Some variable may not be restricted to be nonnegative.* An unrestricted variable, x_j, may be replaced by the difference of two nonnegative variables, $x_j = u_j - v_j$, where $u_j \geq 0$ and $v_j \geq 0$. This adds one variable and two nonnegativity constraints to the problem.

Any theory derived for problems in standard form is therefore applicable to general problems. However, from a computational point of view, the enlargement of the number of variables and constraints in (2) is undesirable and, as will be seen later, can be avoided.

Exercises.

1. Consider the linear programming problem: Find y_1 and y_2 to minimize $y_1 + y_2$ subject to the constraints,

$$\begin{aligned} y_1 + 2y_2 &\geq 3 \\ 2y_1 + y_2 &\geq 5 \\ y_2 &\geq 0. \end{aligned}$$

Graph the constraint set and solve.

2. Find x_1 and x_2 to maximize $ax_1 + x_2$ subject to the constraints in the numerical example of Figure 1. Find the value as a function of a.

3. Write out the matrix $\boldsymbol{A}$ for the transportation problem in standard form.

4. Put the following linear programming problem into standard form. Find x_1, x_2, x_3, x_4 to maximize $x_1 + 2x_2 + 3x_3 + 4x_4 + 5$ subject to the constraints,

$$\begin{array}{rcrcrcrcr} 4x_1 &+& 3x_2 &+& 2x_3 &+& x_4 &\leq& 10 \\ x_1 & & &-& x_3 &+& 2x_4 &=& 2 \\ x_1 &+& x_2 &+& x_3 &+& x_4 &\geq& 1, \end{array}$$

and

$$x_1 \geq 0, x_3 \geq 0, x_4 \geq 0.$$

2. Duality.

To every linear program there is a dual linear program with which it is intimately connected. We first state this duality for the standard programs. As in Section 1, $\boldsymbol{c}$ and $\boldsymbol{x}$ are n-vectors, $\boldsymbol{b}$ and $\boldsymbol{y}$ are m-vectors, and $\boldsymbol{A}$ is an $m \times n$ matrix. We assume $m \geq 1$ and $n \geq 1$.

Definition. *The dual of the standard maximum problem*

$$\begin{array}{l} \text{maximize } \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\ \text{subject to the constraints } \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \text{ and } \boldsymbol{x} \geq 0 \end{array} \tag{1}$$

is defined to be the standard minimum problem

$$\begin{array}{l} \text{minimize } \boldsymbol{y}^{\mathsf{T}}\boldsymbol{b} \\ \text{subject to the constraints } \boldsymbol{y}^{\mathsf{T}}\boldsymbol{A} \geq \boldsymbol{c}^{\mathsf{T}} \text{ and } \boldsymbol{y} \geq 0 \end{array} \tag{2}$$

Let us reconsider the numerical example of the previous section: Find x_1 and x_2 to maximize $x_1 + x_2$ subject to the constraints $x_1 \geq 0$, $x_2 \geq 0$, and

$$\begin{array}{rcrcl} x_1 & + & 2x_2 & \leq & 4 \\ 4x_1 & + & 2x_2 & \leq & 12 \\ -x_1 & + & x_2 & \leq & 1. \end{array} \tag{3}$$

The dual of this standard maximum problem is therefore the standard minimum problem: Find y_1, y_2, and y_3 to minimize $4y_1 + 12y_2 + y_3$ subject to the constraints $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$, and

$$\begin{array}{rcrcrcl} y_1 & + & 4y_2 & - & y_3 & \geq & 1 \\ 2y_1 & + & 2y_2 & + & y_3 & \geq & 1. \end{array} \tag{4}$$

If the standard minimum problem (2) is transformed into a standard maximum problem (by multiplying $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ by -1), its dual by the definition above is a standard minimum problem which, when transformed to a standard maximum problem (again by changing the signs of all coefficients) becomes exactly (1). Therefore, *the dual of the standard minimum problem (2) is the standard maximum problem (1). The problems (1) and (2) are said to be duals.*

The general standard maximum problem and the dual standard minimum problem may be simultaneously exhibited in the display:

$$\begin{array}{c|cccc|c} & x_1 & x_2 & \cdots & x_n & \\ \hline y_1 & a_{11} & a_{12} & \cdots & a_{1n} & \leq b_1 \\ y_2 & a_{21} & a_{22} & \cdots & a_{2n} & \leq b_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ y_m & a_{m1} & a_{m2} & \cdots & a_{mn} & \leq b_m \\ \hline & \geq c_1 & \geq c_2 & \cdots & \geq c_n & \end{array} \tag{5}$$

Our numerical example in this notation becomes

$$\begin{array}{c|cc|l} & x_1 & x_2 & \\ \hline y_1 & 1 & 2 & \leq 4 \\ y_2 & 4 & 2 & \leq 12 \\ y_3 & -1 & 1 & \leq 1 \\ \hline & \geq 1 & \geq 1 & \end{array} \tag{6}$$

The relation between a standard problem and its dual is seen in the following theorem and its corollaries.

Theorem 1. *If* $\boldsymbol{x}$ *is feasible for the standard maximum problem (1) and if* $\boldsymbol{y}$ *is feasible for its dual (2), then*

$$\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \leq \boldsymbol{y}^{\mathsf{T}}\boldsymbol{b}. \tag{7}$$

Proof.

$$\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \leq \boldsymbol{y}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{y}^{\mathsf{T}}\boldsymbol{b}.$$

The first inequality follows from $\boldsymbol{x} \geq \boldsymbol{0}$ and $\boldsymbol{c}^{\mathsf{T}} \leq \boldsymbol{y}^{\mathsf{T}}\boldsymbol{A}$. The second inequality follows from $\boldsymbol{y} \geq \boldsymbol{0}$ and $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$. ∎

Corollary 1. *If a standard problem and its dual are both feasible, then both are bounded feasible.*

Proof. If $\boldsymbol{y}$ is feasible for the minimum problem, then (7) shows that $\boldsymbol{y}^{\mathsf{T}}\boldsymbol{b}$ is an upper bound for the values of $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$ for $\boldsymbol{x}$ feasible for the maximum problem. Similarly for the converse. ∎

Corollary 2. *If there exists feasible* $\boldsymbol{x}^*$ *and* $\boldsymbol{y}^*$ *for a standard maximum problem (1) and its dual (2) such that* $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}^* = {\boldsymbol{y}^*}^{\mathsf{T}}\boldsymbol{b}$, *then both are optimal for their respective problems.*

Proof. If $\boldsymbol{x}$ is any feasible vector for (1), then $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \leq {\boldsymbol{y}^*}^{\mathsf{T}}\boldsymbol{b} = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}^*$. which shows that $\boldsymbol{x}^*$ is optimal. A symmetric argument works for $\boldsymbol{y}^*$. ∎

The following fundamental theorem completes the relationship between a standard problem and its dual. It states that the hypothesis of Corollary 2 are always satisfied if one of the problems is bounded feasible. The proof of this theorem is not as easy as the previous theorem and its corollaries. We postpone the proof until later when we give a constructive proof via the simplex method. (The simplex method is an algorithmic method for solving linear programming problems.) We shall also see later that this theorem contains the Minimax Theorem for finite games of Game Theory.

The Duality Theorem. *If a standard linear programming problem is bounded feasible, then so is its dual, their values are equal, and there exists optimal vectors for both problems.*

There are three possibilities for a linear program. It may be feasible bounded (f.b.), feasible unbounded (f.u.), or infeasible (i). For a program and its dual, there are therefore nine possibilities. Corollary 1 states that three of these cannot occur: If a problem and its dual are both feasible, then both must be bounded feasible. The first conclusion of the Duality Theorem states that two other possiblities cannot occur. If a program is feasible bounded, its dual cannot be infeasible. The x's in the accompanying diagram show the impossibilities. The remaining four possibilities can occur.

Standard Maximum Problem

		f.b.	f.u.	i.
	f.b.		x	x
Dual	f.u.	x	x	
	i.	x		

(8)

As an example of the use of Corollary 2, consider the following maximum problem. Find x_1, x_2, x_2, x_4 to maximize $2x_1 + 4x_2 + x_3 + x_4$, subject to the constraints $x_j \geq 0$ for all j, and

$$\begin{aligned} x_1 + 3x_2 \qquad\quad + x_4 &\leq 4 \\ 2x_1 + x_2 \qquad\qquad\quad &\leq 3 \\ x_2 + 4x_3 + x_4 &\leq 3. \end{aligned} \tag{9}$$

The dual problem is found to be: find y_1, y_2, y_3 to minimize $4y_1 + 3y_2 + 3y_3$ subject to the constraints $y_i \geq 0$ for all i, and

$$\begin{aligned} y_1 + 2y_2 \qquad\quad &\geq 2 \\ 3y_1 + y_2 + y_3 &\geq 4 \\ 4y_3 &\geq 1 \\ y_1 \qquad + y_3 &\geq 1. \end{aligned} \tag{10}$$

The vector $(x_1, x_2, x_3, x_4) = (1, 1, 1/2, 0)$ satisfies the constraints of the maximum problem and the value of the objective function there is $13/2$. The vector $(y_1, y_2, y_3) = (11/10, 9/20, 1/4)$ satisfies the constraints of the minimum problem and has value there of $13/2$ also. Hence, both vectors are optimal for their respective problems.

As a corollary of the Duality Theorem we have

The Equilibrium Theorem. *Let $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ be feasible vectors for a standard maximum problem (1) and its dual (2) respectively. Then $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are optimal if, and only if,*

$$y_i^* = 0 \qquad \text{for all } i \text{ for which } \textstyle\sum_{j=1}^n a_{ij} x_j^* < b_i \tag{11}$$

and

$$x_j^* = 0 \qquad \text{for all } j \text{ for which } \textstyle\sum_{i=1}^m y_i^* a_{ij} > c_j \tag{12}$$

Proof. If: Equation (11) implies that $y_i^* = 0$ unless there is equality in $\sum_j a_{ij} x_j^* \leq b_i$. Hence

$$\sum_{i=1}^m y_i^* b_i = \sum_{i=1}^m y_i^* \sum_{j=1}^n a_{ij} x_j^* = \sum_{i=1}^m \sum_{j=1}^n y_i^* a_{ij} x_j^*. \tag{13}$$

Similarly Equation (12) implies

$$\sum_{i=1}^{m}\sum_{j=1}^{n} y_i^* a_{ij} x_j^* = \sum_{j=1}^{n} c_j x_j^*. \tag{14}$$

Corollary 2 now implies that $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are optimal.

Only if: As in the first line of the proof of Theorem 1,

$$\sum_{j=1}^{n} c_j x_j^* \le \sum_{i=1}^{m}\sum_{j=1}^{n} y_i^* a_{ij} x_j^* \le \sum_{i=1}^{m} y_i^* b_i. \tag{15}$$

By the Duality Theorem, if $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are optimal, the left side is equal to the right side so we get equality throughout. The equality of the first and second terms may be written as

$$\sum_{j=1}^{n} \left(c_j - \sum_{i=1}^{m} y_i^* a_{ij} \right) x_j^* = 0. \tag{16}$$

Since $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are feasible, each term in this sum is nonnegative. The sum can be zero only if each term is zero. Hence if $\sum_{i=1}^{m} y_i^* a_{ij} > c_j$, then $x_j^* = 0$. A symmetric argument shows that if $\sum_{j=1}^{n} a_{ij} x_j^* < b_i$, then $y_i^* = 0$. ∎

Equations (11) and (12) are sometimes called the **complementary slackness conditions**. They require that a strict inequality (a slackness) in a constraint in a standard problem implies that the complementary constraint in the dual be satisfied with equality.

As an example of the use of the Equilibrium Theorem, let us solve the dual to the introductory numerical example. Find y_1, y_2, y_3 to minimize $4y_1 + 12y_2 + y_3$ subject to $y_1 \ge 0$, $y_2 \ge 0$, $y_3 \ge 0$, and

$$\begin{aligned} y_1 + 4y_2 - y_3 &\ge 1 \\ 2y_1 + 2y_2 + y_3 &\ge 1. \end{aligned} \tag{17}$$

We have already solved the dual problem and found that $x_1^* > 0$ and $x_2^* > 0$. Hence, from (12) we know that the optimal $\boldsymbol{y}^*$ gives equality in both inequalities in (17) (2 equations in 3 unknowns). If we check the optimal $\boldsymbol{x}^*$ in the first three main constraints of the maximum problem, we find equality in the first two constraints, but a strict inequality in the third. From condition (11), we conclude that $y_3^* = 0$. Solving the two equations,

$$\begin{aligned} y_1 + 4y_2 &= 1 \\ 2y_1 + 2y_2 &= 1 \end{aligned}$$

we find $(y_1^*, y_2^*) = (1/3, 1/6)$. Since this vector is feasible, the "if" part of the Equilibrium Theorem implies it is optimal. As a check we may find the value, $4(1/3) + 12(1/6) = 10/3$, and see it is the same as for the maximum problem.

In summary, if you conjecture a solution to one problem, you may solve for a solution to the dual using the complementary slackness conditions, and then see if your conjecture is correct.

Interpretation of the dual. In addition to the help it provides in finding a solution, the dual problem offers advantages in the interpretation of the original, primal problem. In practical cases, the dual problem may be analyzed in terms of the primal problem.

As an example, consider the diet problem, a standard minimum problem of the form (2). Its dual is the standard maximum problem (1). First, let us find an interpretation of the dual variables, $x_1, x_2, \ldots, x_n$. In the dual constraint,

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \tag{18}$$

the variable b_i is measured as price per unit of food, F_i, and a_{ij} is measured as units of nutrient N_j per unit of food F_i. To make the two sides of the constraint comparable, x_j must be measured in of price per unit of food F_i. (This is known as a **dimensional analysis**.) Since c_j is the amount of N_j required per day, the objective function, $\sum_1^n c_j x_j$, represents the total price of the nutrients required each day. Someone is evidently trying to choose vector $\boldsymbol{x}$ of prices for the nutrients to maximize the total worth of the required nutrients per day, subject to the constraints that $\boldsymbol{x} \geq \mathbf{0}$, and that the total value of the nutrients in food F_i, namely, $\sum_{j=1}^n a_{ij} x_j$, is not greater than the actual cost, b_i, of that food.

We may imagine that an entrepreneur is offering to sell us the nutrients without the food, say in the form of vitamin or mineral pills. He offers to sell us the nutrient N_j at a price x_j per unit of N_j. If he wants to do business with us, he would choose the x_j so that price he charges for a nutrient mixture substitute of food F_i would be no greater than the original cost to us of food F_i. This is the constraint, (18). If this is true for all i, we may do business with him. So he will choose $\boldsymbol{x}$ to maximize his total income, $\sum_1^n c_j x_j$, subject to these constraints. (Actually we will not save money dealing with him since the duality theorem says that our minimum, $\sum_1^m y_i b_i$, is equal to his maximum, $\sum_1^n c_j x_j$.) The optimal price, x_j, is referred to as the **shadow price** of nutrient N_j. Although no such entrepreneur exists, the shadow prices reflect the actual values of the nutrients as shaped by the market prices of the foods, and our requirements of the nutrients.

Exercises.

1. Find the dual to the following standard minimum problem. Find y_1, y_2 and y_3 to minimize $y_1 + 2y_2 + y_3$, subject to the constraints, $y_i \geq 0$ for all i, and

$$\begin{array}{rcrcrcl} y_1 & - & 2y_2 & + & y_3 & \geq & 2 \\ -y_1 & + & y_2 & + & y_3 & \geq & 4 \\ 2y_1 & & & + & y_3 & \geq & 6 \\ y_1 & + & y_2 & + & y_3 & \geq & 2. \end{array}$$

2. Consider the problem of Exercise 1. Show that $(y_1, y_2, y_3) = (2/3, 0, 14/3)$ is optimal for this problem, and that $(x_1, x_2, x_3, x_4) = (0, 1/3, 2/3, 0)$ is optimal for the dual.

3. Consider the problem: Maximize $3x_1 + 2x_2 + x_3$ subject to $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, and

$$\begin{aligned} x_1 - x_2 + x_3 &\leq 4 \\ 2x_1 + x_2 + 3x_3 &\leq 6 \\ -x_1 \phantom{{}+x_2} + 2x_3 &\leq 3 \\ x_1 + x_2 + x_3 &\leq 8. \end{aligned}$$

(a) State the dual minimum problem.

(b) Suppose you suspect that the vector $(x_1, x_2, x_3) = (0, 6, 0)$ is optimal for the maximum problem. Use the Equilibrium Theorem to solve the dual problem, and then show that your suspicion is correct.

4. (a) State the dual to the transportation problem.

(b) Give an interpretation to the dual of the transportation problem.