The equivalence classes are called the *k-edge-connected components* of G. So the 1-edge connected components of G coincide with the components of G, and can be found in linear time by Corollary 6.6a. Also for $k = 2$, the k-edge-connected components can be found in linear time (Karzanov [1970]; we follow the proof of Tarjan [1972]):

Theorem 15.12. *Given an undirected graph $G = (V, E)$, its 2-edge-connected components can be found in linear time.*

Proof. We may assume that G is connected, since by Corollary 6.6a, the components of G can be found in linear time.

Choose $s \in V$ arbitrarily, and consider a depth-first search tree T starting at s. Orient each edge in T away from s. For each remaining edge $e = uv$, there is a directed path in T that connects u and v. Let the path run from u to v. Then orient e from v to u. This gives the orientation D of G.

Then any edge not in T belongs to a directed circuit in D. Moreover, any edge in T that is not a cut edge, belongs to a directed circuit in D. Then the 2-edge-connected components of G coincide with the strong components of D. By Theorem 6.6, these components can be found in linear time. ∎

More on finding 2-edge-connected components can be found in Gabow [2000a].

15.4. Gomory-Hu trees

In previous sections of this chapter we have considered the problem of determining a minimum cut in a graph, where the minimum is taken over all pairs s, t. The *all-pairs minimum-size cut problem* asks for a minimum $s - t$ cut for all pairs of vertices s, t. Clearly, this can be solved in time $O(n^2\tau)$, where τ is the time needed for finding a minimum $s - t$ cut for any given s, t.

Gomory and Hu [1961] showed that for *undirected* graphs it can be done faster, and that there is a concise structure, the Gomory-Hu tree, to represent all minimum cuts. Similarly for the capacitated case.

Fix an undirected graph $G = (V, E)$ and a capacity function $c : E \to \mathbb{R}_+$. A *Gomory-Hu tree* (*for* G and c) is a tree $T = (V, F)$ such that for each edge $e = st$ of T, $\delta(U)$ is a minimum-capacity $s-t$ cut of G, where U is any of the two components of $T - e$. (Note that it is not required that T is a subgraph of G.)

Gomory and Hu [1961] showed that for each G, c there indeed exists a Gomory-Hu tree, and that it can be found by $n - 1$ minimum-cut computations.

For distinct $s, t \in V$, define $r(s, t)$ as the minimum capacity of an $s - t$ cut. The following triangle inequality holds:

$$(15.13) \qquad r(u, w) \geq \min\{r(u, v), r(v, w)\}$$

for all distinct $u, v, w \in G$. Now a Gomory-Hu tree indeed describes concisely minimum-capacity $s - t$ cuts for all s, t:

Theorem 15.13. *Let $T = (V, F)$ be a Gomory-Hu tree. Consider any $s, t \in V$, the $s-t$ path P in T, an edge $e = uv$ on P with $r(u, v)$ minimum, and any component K of $T-e$. Then $r(s, t) = r(u, v)$ and $\delta(K)$ is a minimum-capacity $s-t$ cut.*

Proof. Inductively, (15.13) gives $r(s, t) \geq r(u, v)$. Moreover, $\delta(K)$ is an $s-t$ cut, and hence $r(s, t) \leq c(\delta(K)) = r(u, v)$. ∎

To show that a Gomory-Hu tree does exist, we first prove:

Lemma 15.14α. *Let $s, t \in V$, let $\delta(U)$ be a minimum-capacity $s-t$ cut in G, and let $u, v \in U$ with $u \neq v$. Then there exists a minimum-capacity $u-v$ cut $\delta(W)$ with $W \subseteq U$.*

Proof. Consider a minimum-capacity $u-v$ cut $\delta(X)$. By symmetry we may assume that $s \in U$ (otherwise interchange s and t), $t \notin U$, $s \in X$ (otherwise replace X by $V \setminus X$), $u \in X$ (otherwise interchange u and v), and $v \notin X$. So one of the diagrams of Figure 15.2 applies.

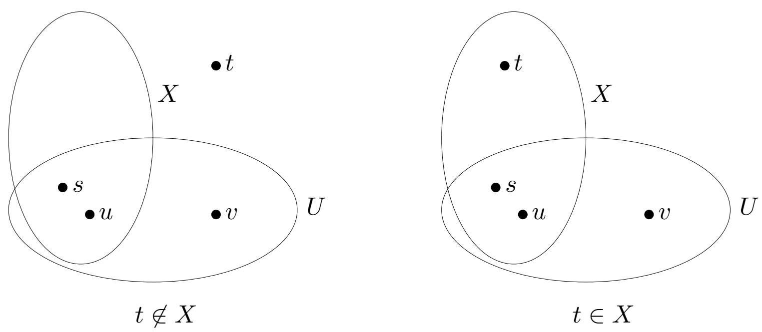

Figure 15.2

In particular, $\delta(U \cap X)$ and $\delta(U \setminus X)$ are $u-v$ cuts. If $t \notin X$, then $\delta(U \cup X)$ is an $s-t$ cut. As

$$(15.14) \qquad c(\delta(U \cap X)) + c(\delta(U \cup X)) \leq c(\delta(U)) + c(\delta(X))$$

and

$$(15.15) \qquad c(\delta(U \cup X)) \geq c(\delta(U)),$$

we have $c(\delta(U \cap X)) \leq c(\delta(X))$. So $\delta(U \cap X)$ is a minimum-capacity $u-v$ cut.

If $t \in X$, then $\delta(X \setminus U)$ is an $s-t$ cut. As

$$(15.16) \qquad c(\delta(U \setminus X)) + c(\delta(X \setminus U)) \leq c(\delta(U)) + c(\delta(X))$$

and

(15.17) $\quad c(\delta(X \setminus U)) \geq c(\delta(U))$,

we have $c(\delta(U \setminus X)) \leq c(\delta(X))$. So $\delta(U \setminus X)$ is a minimum-capacity $u - v$ cut. ∎

This lemma is used in proving the existence of Gomory-Hu trees:

Theorem 15.14. *For each graph $G = (V, E)$ and each capacity function $c : E \to \mathbb{R}_+$ there exists a Gomory-Hu tree.*

Proof. Define a *Gomory-Hu tree for* a set $R \subseteq V$ to be a pair of a tree (R, T) and a partition $(C_r \mid r \in R)$ of V such that:

(15.18) (i) $r \in C_r$ for each $r \in R$,
(ii) $\delta(U)$ is a minimum-capacity $s-t$ cut for each edge $e = st \in T$, where $U := \bigcup_{u \in K} C_u$ and K is a component of $T - e$.

We show by induction on $|R|$ that for each nonempty $R \subseteq V$ there exists a Gomory-Hu tree for R. Then for $R = V$ we have a Gomory-Hu tree.

If $|R| = 1$, (15.18) is trivial, so assume $|R| \geq 2$. Let $\delta(W)$ be a minimum-capacity cut separating at least one pair of vertices in R. Contract $V \setminus W$ to one vertex, v' say, giving graph G'. Let $R' := R \cap W$. By induction, G' has a Gomory-Hu tree (R', T'), $(C'_r \mid r \in R')$ for R'.

Similarly, contract W to one vertex, v'' say, giving graph G''. Let $R'' := R \setminus W$. By induction, G'' has a Gomory-Hu tree (R'', T''), $(C''_r \mid r \in R'')$ for R''.

Now let $r' \in R'$ be such that $v' \in C'_{r'}$. Similarly, let $r'' \in R''$ be such that $v'' \in C''_{r''}$. Let $T := T' \cup T'' \cup \{r'r''\}$, Let $C_{r'} := C'_{r'} \setminus \{v'\}$ and let $C_r := C'_r$ for all other $r \in R'$. Similarly, let $C_{r''} := C''_{r''} \setminus \{v''\}$ and let $C_r := C''_r$ for all other $r \in R''$.

Now (R, T) and the C_r form a Gomory-Hu tree for R. Indeed, for any $e \in T$ with $e \neq r'r''$, (15.18) follows from Lemma 15.14α. If $e = r'r''$, then $U = W$ and $\delta(W)$ is a minimum-capacity $r' - r''$ cut (as it is minimum-capacity over all cuts separating at least one pair of vertices in R). ∎

The method can be sharpened to give the following algorithmic result:

Theorem 15.15. *A Gomory-Hu tree can be found by $n - 1$ applications of a minimum-capacity cut algorithm.*

Proof. In the proof of Theorem 15.14, it suffices to take for $\delta(W)$ just a minimum-capacity $s - t$ cut for at least one pair $s, t \in R$. Then $\delta(W)$ is also a minimum-capacity $r' - r''$ cut. For suppose that there exists an $r' - r''$ cut $\delta(X)$ of smaller capacity. We may assume that $s \in W$ and $t \notin W$. As $\delta(W)$ is a minimum-capacity $s - t$ cut, $\delta(X)$ is not an $s - t$ cut. So it should separate

s and r' or t and r''. By symmetry, we may assume that it separates s and r'. Then it also as is a $u-v$ cut for some edge uv on the $s-r'$ path in T'. Let uv determine cut $\delta(U)$. This cut is an $s-t$ cut, and hence $c(\delta(U)) \geq c(\delta(W))$. On the other hand, $c(\delta(U)) \leq c(\delta(X))$, as $\delta(U)$ is a minimum-capacity $u-v$ cut. This contradicts our assumption that $c(\delta(X)) < c(\delta(W))$. ∎

This implies for the running time:

Corollary 15.15a. *A Gomory-Hu tree can be found in time $O(n\tau)$ time, if for any $s, t \in V$ a minimum-capacity $s-t$ cut can be found in time τ.*

Proof. Directly from Theorem 15.15. ∎

Notes. The method gives an $O(m^2)$ method to find a Gomory-Hu tree for the capacity function $c = \mathbf{1}$, since $O(m^2) = O(\sum_v d(v)m)$, and for each new vertex v a minimum cut can be found in time $O(d(v)m)$. Hao and Orlin [1992,1994] gave an $O(n^3)$-time method to find, for given graph $G = (V, E)$ and $s \in V$, all minimum-size $s-t$ cuts for all $t \neq s$ (with push-relabel). Shiloach [1979b] gave an $O(n^2m)$ algorithm to find a maximum number of edge-disjoint paths between all pairs of vertices in an undirected graph. Ahuja, Magnanti, and Orlin [1993] showed that the best directed all-pairs cut algorithm takes $\Omega(n^2)$ max-flow iterations.

For planar graphs, Hartvigsen and Mardon [1994] gave an $(n^2 \log n + m)$ algorithm to find a Gomory-Hu tree (they observed that this bound can be derived also from Frederickson [1987b]). This improves a result of Shiloach [1980a], who gave an $O(n^2(\log n)^2)$-time algorithm to find minimum-size cuts between all pairs of vertices in a planar graph.

Theorem 15.13 implies that a Gomory-Hu tree for a graph $G = (V, E)$ is a maximum-weight spanning tree in the complete graph on V, for weight function $r(u, v)$. However, not every maximum-weight spanning tree is a Gomory-Hu tree (for $G = K_{1,2}$, $c = \mathbf{1}$, only G itself is a Gomory-Hu tree, but all spanning trees on $VK_{1,2}$ have the same weight).

More on Gomory-Hu trees can be found in Elmaghraby [1964], Hu and Shing [1983], Agarwal, Mittal, and Sharma [1984], Granot and Hassin [1986], Hassin [1988], Chen [1990], Gusfield [1990], Hartvigsen and Margot [1995], Talluri [1996], Goldberg and Tsioutsiouliklis [1999,2001], and Hartvigsen [2001b]. Generalizations were given by Cheng and Hu [1990,1991,1992] and Hartvigsen [1995] (to matroids).

15.4a. Minimum-requirement spanning tree

Hu [1974] gave the following additional application of Gomory-Hu trees. Let $G = (V, E)$ be an undirected graph and let $r : E \to \mathbb{R}_+$ be a ‘requirement’ function (say, the number of telephone calls to be made between the end vertices of e).

We want to find a tree T on V minimizing

$$\sum_{e \in E} r(e)\text{dist}_T(e), \tag{15.19}$$