Chapter 18

Linear programming methods and the bipartite matching polytope

The weighted matching problem for bipartite graphs discussed in the previous chapter is related to the 'matching polytope' and the 'perfect matching polytope', and can be handled with linear programming methods by the total unimodularity of the incidence matrix of a bipartite graph.
In this chapter, graphs can be assumed to be simple.

18.1. The matching and the perfect matching polytope

Let $G = (V, E)$ be a graph. The *perfect matching polytope* $P_{\text{perfect matching}}(G)$ of G is defined as the convex hull of the incidence vectors of perfect matchings in G. So $P_{\text{perfect matching}}(G)$ is a polytope in $\mathbb{R}^E$.

The perfect matching polytope is a polyhedron, and hence can be described by linear inequalities. The following are clearly valid inequalities:

(18.1) (i) $x_e \geq 0$ for each edge e,
(ii) $x(\delta(v)) = 1$ for each vertex v.

These inequalities are generally not enough (for instance, not for K_3). However, as Birkhoff [1946] showed, for bipartite graphs they are enough:

Theorem 18.1. *If G is bipartite, the perfect matching polytope of G is determined by* (18.1).

Proof. Let x be a vertex of the polytope determined by (18.1). Let F be the set of edges e with $x_e > 0$. Suppose that F contains a circuit C. As C has even length, $EC = M \cup N$ for two disjoint matchings M and N. Then for ε close enough to 0, both $x + \varepsilon(\chi^M - \chi^N)$ and $x - \varepsilon(\chi^M - \chi^N)$ satisfy (18.1), contradicting the fact that x is a vertex of the polytope. So (V, F) is a forest, and hence by (18.1), F is a perfect matching. ∎

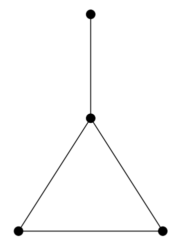

Figure 18.1

The implication cannot be reversed, as is shown by the graph in Figure 18.1.

Theorem 18.1 was shown by Birkhoff in the terminology of doubly stochastic matrices. A matrix A is called *doubly stochastic* if A is nonnegative and each row sum and each column sum equals 1. A *permutation matrix* is an integer doubly stochastic matrix (so it is $\{0,1\}$-valued, and has precisely one 1 in each row and in each column). Then:

Corollary 18.1a (Birkhoff's theorem)**.** *Each doubly stochastic matrix is a convex combination of permutation matrices.*

Proof. Directly from Theorem 18.1, by taking $G = K_{n,n}$. ∎

Theorem 18.1 also implies a characterization of the matching polytope for bipartite graphs. For any graph $G = (V, E)$, the *matching polytope* $P_{\text{matching}}(G)$ of G is the convex hull of the incidence vectors of matchings in G. So again it is a polytope in $\mathbb{R}^E$. The following are valid inequalities for the matching polytope:

(18.2)
(i) $x_e \geq 0$ for each edge e,
(ii) $x(\delta(v)) \leq 1$ for each vertex v.

Then:

Corollary 18.1b. *The matching polytope of G is determined by* (18.2) *if and only if G is bipartite.*

Proof. To see necessity, suppose that G is not bipartite, and let C be an odd circuit in G. Define $x_e := \frac{1}{2}$ if $e \in C$ and $x_e := 0$ otherwise. Then x satisfies (18.2) but does not belong to the matching polytope of G.

To see sufficiency, let G be bipartite and let x satisfy (18.2). Let G' and x' be a copy of G and x, and add edges vv', where v' is the copy of $v \in V$. Define $y(vv') := 1 - x(\delta(v))$. Then x, x', y satisfy (18.1) with respect to the new graph, and hence by Theorem 18.1, it is a convex combination of incidence vectors of perfect matchings in the new graph. Hence x is a convex combination of incidence vectors of matchings in G. ∎

Notes. Birkhoff derived Corollary 18.1a from Hall's marriage theorem (Theorem 22.1), which is equivalent to Kőnig's matching theorem. (Also Dulmage and Halperin [1955] derived Birkhoff's theorem from Kőnig's matching theorem.) Other proofs were given by von Neumann [1951,1953], Dantzig [1952], Hoffman and Wielandt [1953], Koopmans and Beckmann [1955,1957], Hammersley and Mauldon [1956] (a polyhedral proof based on total unimodularity), Tompkins [1956], Mirsky [1958], and Vogel [1961]. A survey was given by Mirsky [1962]. More can be found in Johnson, Dulmage, and Mendelsohn [1960], Nishi [1979], and Brualdi [1982].

18.2. Totally unimodular matrices from bipartite graphs

In this section we show that the results on matchings discussed above can also be derived from linear programming duality with total unimodularity (Hoffman [1956b]).

Let A be the $V \times E$ incidence matrix of a graph $G = (V, E)$. The matrix A generally is not totally unimodular. E.g., if G is the complete graph K_3 on three vertices, then the determinant of A is equal to $+2$ or -2.

However, the following can be proved (necessity can also be derived directly from the total unimodularity of the incidence matrix of a directed graph (Theorem 13.9) — we give a direct proof):

Theorem 18.2. *A graph $G = (V, E)$ is bipartite if and only if its incidence matrix A is totally unimodular.*

Proof. *Sufficiency.* Assume that A is totally unimodular and G is not bipartite. Then G has a circuit of odd length, t say. The submatrix of A induced by the vertices and edges in C is a $t \times t$ matrix with exactly two ones in each row and each column. As t is odd, the determinant of this matrix is ± 2, contradicting the total unimodularity of A.

Necessity. Let G be bipartite. We show that A is totally unimodular. Let B be a square submatrix of A, of order $t \times t$ say. We show that $\det B$ equals 0 or ± 1 by induction on t. If $t = 1$, the statement is trivial. So let $t > 1$. We distinguish three cases.

Case 1: B has a column with only 0*'s.* Then $\det B$=0.

Case 2: B has a column with exactly one 1. In that case we can write (possibly after permuting rows or columns):

$$B = \begin{pmatrix} 1 & b^{\mathsf{T}} \\ \mathbf{0} & B' \end{pmatrix}, \tag{18.3}$$

for some matrix B' and vector b, where $\mathbf{0}$ denotes the all-zero vector in $\mathbb{R}^{t-1}$. By the induction hypothesis, $\det B' \in \{0, \pm 1\}$. Hence, by (18.3), $\det B \in \{0, \pm 1\}$.

Case 3. Each column of B contains exactly two 1*'s.* Then, since G is bipartite, we can write (possibly after permuting rows):

$$B = \begin{pmatrix} B' \\ B'' \end{pmatrix}, \tag{18.4}$$

in such a way that each column of B' contains exactly one 1 and each column of B'' contains exactly one 1. So adding up all rows in B' gives the all-one vector, and also adding up all rows in B'' gives the all-one vector. The rows of B therefore are linearly dependent, and hence $\det B=0$. ∎

18.3. Consequences of total unimodularity

Let $G = (V, E)$ be a bipartite graph and let A be its $V \times E$ incidence matrix. Consider Kőnig's matching theorem (Theorem 16.2): the maximum size of a matching in G is equal to the minimum size of a vertex cover in G. This can be derived from the total unimodularity of A as follows. By Corollary 5.20a, both optima in the LP-duality equation

$$\max\{\mathbf{1}^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\} = \min\{y^{\mathsf{T}}\mathbf{1} \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \geq \mathbf{1}^{\mathsf{T}}\} \tag{18.5}$$

have integer optimum solutions x^* and y^*. Now x^* necessarily is the incidence vector of a matching and y^* is the incidence vector of a vertex cover. So we have Kőnig's matching theorem.

One can also derive the weighted version of Kőnig's matching theorem, Egerváry's theorem (Theorem 17.1): for any weight function $w : E \to \mathbb{Z}_+$, the maximum weight of a matching in G is equal to the minimum value of $\sum_{v \in V} y_v$, where y ranges over all $y : V \to \mathbb{Z}_+$ with $y_u + y_v \geq w_e$ for each edge $e = uv$ of G. To derive this, consider the LP-duality equation

$$\max\{w^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\} = \min\{y^{\mathsf{T}}\mathbf{1} \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \geq w^{\mathsf{T}}\}. \tag{18.6}$$

By the total unimodularity of A, these optima are attained by integer x^* and y^*, and we have the theorem.

The min-max relation for minimum-weight *perfect* matching (Theorem 17.5) follows similarly.

One can also derive the characterizations of the matching polytope and perfect matching polytope of a bipartite graph (Theorem 18.1 and Corollary 18.1b) from the total unimodularity of the incidence matrix of a bipartite graph. This amounts to the fact that the polyhedra

$$\{x \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\} \tag{18.7}$$

and

$$\{x \mid x \geq \mathbf{0}, Ax = \mathbf{1}\} \tag{18.8}$$

are integer polyhedra, by the total unimodularity of A.