VALID INEQUALITIES FOR MIXED INTEGER LINEAR PROGRAMS

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Gérard Cornuéjols
LIF, Faculté des Sciences de Luminy, Marseille, France and
Tepper School of Business, Carnegie Mellon Univ, Pittsburgh, PA
gc0v@andrew.cmu.edu

1. Lift and Project Cuts for Mixed 0,1 Programs

Let $S = \{x \in \{0,1\}^n \times \mathbb{R}^p_+ \ : \ Ax \geq b\}$.
Here $Ax \geq b$ includes $x_j \geq 0$ for all $j = 1, \ldots, n+p$, and $x_j \leq 1$ for $j = 1, \ldots, n$.
Balas, Ceria and Cornuéjols study the following "lift-and-project" procedure:

Step 0: Select $j \in \{1, \ldots, n\}$.
Step 1: Generate the nonlinear system $x_j(Ax - b) \geq 0$ and $(1 - x_j)(Ax - b) \geq 0$.
Step 2: Linearize the system by substituting y_i for $x_i x_j$, $i \neq j$, and x_j for x_j^2. Call this polyhedron M_j.
Step 3: Project M_j onto the x-space, call the resulting polyhedron P_j.

Theorem 1: $P_j = \text{Conv } \{(Ax \geq b, x_j = 0) \cup (Ax \geq b, x_j = 1)\}$

Proof: Call P^* the set in the RHS. To show $P_j \subseteq P^*$, we take $\alpha x \geq \beta$ valid for P^*. Since it's valid for $Ax \geq b, x_j = 0$ we can find a λ such that $\alpha x + \lambda x_j \geq \beta$ is valid for P. Similarly, we can find μ such that $\alpha x + \mu(1 - x_j) \geq \beta$ is valid for P.
So, $(1 - x_j)(\alpha x + \lambda x_j - \beta) \geq 0$ and $x_j(\alpha x + \mu(1 - x_j) - \beta) \geq 0$ are valid for the nonlinear system of Step 1, and their sum is too.

$$\alpha x + (\lambda + \mu)(x_j - x_j^2) - \beta \geq 0$$

Step 2 replaces x_j^2 by x_j, this gives $\alpha x \geq b$ valid for M_j, and thus for P_j.

To show $P^* \subseteq P_j$, let $\overline{x}$ be a point in $Ax \geq b, x_j = 0$ or in $Ax \geq b, x_j = 1$. Define $\overline{y}_i = \overline{x}_i \overline{x}_j$ for $i \neq j$. Then $(\overline{x}, \overline{y}) \in M_j$ since $\overline{x}_j^2 = \overline{x}_j$. So, $\overline{x} \in P_j$. By convexity of P_j it follows that $P^* \subseteq P_j$. □

Theorem 2: $P_n(P_{n-1}(\cdots P_2(P_1)\ldots)) = \text{Conv } S$

Proof: by induction. Let $S_t = \{x \in \{0,1\}^t \times \mathbb{R}_+^{n-t+p} : Ax \geq b\}$. We want to show $P_t(P_{t-1}(\cdots P_2(P_1)\ldots)) = \text{Conv } S_t$. This is true for $t = 1$ by Theorem 1 so consider $t \geq 2$. Suppose that this is true for $t - 1$. By IH we have equality to

$$P_t(P_{t-1}(\cdots P_2(P_1)\ldots)) = P_t(\text{Conv } S_{t-1})$$

so by Theorem 1,

$$= \text{Conv } ((\text{Conv } (S_{t-1}) \cap x_t = 0) \cup (\text{Conv } (S_{t-1}) \cap x_t = 1))$$

For any set S that lies entirely on one side of a hyperplane H, the following equality holds

$$\text{Conv } (S) \cap H = \text{Conv } (S \cap H)$$

To prove this, one can use the definition of the convex hull (we leave it as an exercise). Therefore

$$P_t(P_{t-1}(\cdots P_2(P_1)\ldots)) = \text{Conv }((\text{Conv }(S_{t-1} \cap x_t = 0)) \cup (\text{Conv }(S_{t-1} \cap x_t = 1)))$$
$$= \text{Conv }((S_{t-1} \cap x_t = 0) \cup (S_{t-1} \cap x_t = 1)) = \text{Conv } S_t$$

□

Cut generation LP:

$$M_j = \{x \in \mathbb{R}_+^{n+p}, y \in \mathbb{R}_+^{n+p} \;:\; Ay - bx_j \geq 0 \;\;,\;\; Ax - Ay + bx_j \geq b \;\;,\;\; y_j = x_j\}$$

The first two constraints come from linearizing the inequalities of Step 1. We don't really need y_j. Let A_j be A without the j-th column. By modifying the coefficient matrix of x appropriately, we can rewrite M_j as

$$M_j = \{x \in \mathbb{R}_+^{n+p}, y \in \mathbb{R}_+^{n+p-1} \;:\; \tilde{A}_j x - A_j y \geq b \quad \text{and} \quad \tilde{B}_j x + A_j y \geq 0\}$$

We want to project out the y variables. This is done using the cone $Q = \{(u,v) : -uA_j + vA_j = 0, u \geq 0, v \geq 0\}$. Namely the set P_j can be written like this

$$P_j = \{x \in \mathbb{R}_+^{n+p} \;:\; (u\tilde{A}_j + v\tilde{B}_j)x \geq ub \text{ for all } (u,v) \in Q\}$$

Given a fractional solution $\overline{x}$, we want $\alpha x \geq \beta$ valid for P_j which is a cut, i.e. $\alpha\overline{x} < \beta$. Thus $\alpha = u\tilde{A}_j + v\overline{B}_j$ and $\beta = ub$ for $(u,v) \in Q$. Now we have our cut generation LP to get a deepest cut.

$$\max \beta - \alpha\overline{x} \text{ subject to } \alpha = u\tilde{A}_j + v\tilde{B}_j \text{ and } \beta = ub, \;\; -uA_j + vA_j = 0, u \geq 0, v \geq 0.$$

This along with a normalization constraint to truncate the cone will do. For example, we could add the constraint $\sum u_i + \sum v_i = 1$.

Another (equivalent) way of describing P_j is by using Theorem 1, which shows that P_j is the convex hull of the union of two polyhedra. Again, such a description involves additional variables which can then be projected out. We state a general result about union of polyhedra, which is of independent interest. Assume we have bounded and nonempty polyhedra. We want the union

$$\bigvee_{i=1}^{k} A_i x \leq b^i \qquad (1)$$

Denote the following conditions by (2)

$$\begin{gathered} A_1 x^1 \leq b^1 y_1 \\ \vdots \\ A_k x^k \leq b^k y_k \\ x^1 + x^2 + \cdots + x^k = x \\ y_1 + \cdots + y_k = 1 \\ y_i \in \{0,1\} \text{ for } i = 1, 2, \ldots, k \end{gathered}$$

Proposition x satisfies (1) if and only if there exists $x^1, \ldots, x^k, y_1, \ldots, y_k$ such that $(x, x^1, \ldots, x^k, y_1, \ldots, y_k)$ satisfies (2).

Proof: To prove $\Rightarrow$ is obvious, if x satisfies $A_1 x \leq b^1$ take $x_1 = x, y_1 = 1$, others 0.
$\Leftarrow$, say $y_1 = 1$ WLOG, then $y_2, \ldots, y_k = 0$. Because $A_i x \leq b^i$ is bounded, the only solution to $A_i x^i \leq 0$ is $x^i = 0$ for $i = 2, \ldots, k$. Thus $x = x^1$ and therefore x satisfies $A^1 x \leq b^1$, i.e.

x satisfies (1). □

Remark: To show that the formulation (2) is correct, we can relax the assumption "all $\{x : A_i x \leq b^i\}$ are bounded" by "all $\{x : A_i x \leq b^i\}$ have the same recession cone", i.e. the set $\{x : A_i x \leq 0\}$ is the same for all i.

Theorem: (Balas 1979) The convex hull of the solutions of (2) is obtained by replacing $y_i \in \{0,1\}$ by $0 \leq y_i \leq 1$ in the last line of the formulation.

Proof: Let (3) denote the set obtained by replacing $y_i \in \{0,1\}$ by $0 \leq y_i \leq 1$ in (2). Clearly the convex hull of the solutions of (2) is contained in (3). Now we show the converse. Consider a solution $z = (x, x^1, \ldots, x^k, y_1, \ldots, y_k)$ of (3). Write z as the convex combination $\sum_{i:y_i \neq 0} y_i z^i$, where $z^i = (\frac{x^i}{y_i}, 0, \ldots, 0, \frac{x^i}{y_i}, 0, \ldots, 0, 1, 0, \ldots, 0)$. It is easy to verify that z^i is a feasible solution of (2), proving the theorem. □

This theorem has important consequences: It shows that one can optimize over the union of k polyhedra by solving a linear program.

Back to lift-and-project. One can obtain a stronger relaxation (Sherali-Adams) by skipping Step 0 and considering the nonlinear constraints $x_j(Ax - b) \geq 0$ and $(1 - x_j)(Ax - b) \geq 0$ for all $j = 1, \ldots, n$ in Step 1. Then, in Step 2, variables y_{ij} are introduced for all $i = 1, \ldots, n+p$ and $j = 1, \ldots, n$ with $i \neq j$. An even stronger relaxation can be obtained as follows:

Lovász-Schrijver Relaxation:
Step 1: Generate the nonlinear system $x_j(Ax - b) \geq 0$ and $(1 - x_j)(Ax - b) \geq 0$ for all $j = 1, \ldots, n$.
Step 2: Linearize the system by substituting y_{ij} for $x_i x_j$, for all $i = 1, \ldots, n+p$, $j = 1, \ldots, n$ such that $j \neq i$, and x_j for x_j^2 for all $j = 1, \ldots, n$. Denote by Y the symmetric $(n+1) \times (n+1)$ matrix with the vector $(1, x_1, \ldots, x_n)$ in row 0, in column 0 and in the diagonal, and entry y_{ij} in row i and column j for $i, j = 1, \ldots, n$ and $i \neq j$. Call M the convex set in $\mathbb{R}_+^{n+p}$ of all (x, y) that satisfy the above linear inequalities and such that Y is a positive semidefinite matrix.
Step 3: Project M onto the x-space, call N the resulting convex set.

Obviously $N \subseteq \cap_{j=1}^n P_j$. A major interest in the Lovász and Schrijver procedure is due to the fact that one can optimize a linear function over M in polynomial time.

2. Gomory Mixed integer cut:

Let

$$S = \{x \in \mathbb{Z}_+^n, y \in \mathbb{R}_+^p \ : \ \sum_{j\in N} a_j x_j + \sum_{j \in J} g_j y_j = b\}$$

Let $b = \lfloor b \rfloor + f_0$ where $0 < f_0 < 1$.
Let $a_j = \lfloor a_j \rfloor + f_j$ where $0 \le f_j < 1$.

$$\sum_{f_j \le f_0} f_j x_j + \sum_{f_j > f_0} (f_j - 1) x_j + \sum_{j \in J} g_j y_j \ = \ k + f_0$$

k is some integer so $k \le -1$ or $k \ge 0$. So, we get the disjunction

$$\sum_{f_j \le f_0} \frac{f_j}{f_0} x_j - \sum_{f_j > f_0} \frac{1 - f_j}{f_0} x_j + \sum_{j \in J} \frac{g_j}{f_0} y_j \ \ge \ 1$$

OR

$$-\sum_{f_j \le f_0} \frac{f_j}{1 - f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j - \sum_{j \in J} \frac{g_j}{1 - f_0} y_j \ \ge \ 1$$

This is of the form $a^1 x \ge 1$ or $a^2 x \ge 1$ which implies $\sum_j \max(a_j^1, a_j^2) x \ge 1$ for $x \ge 0$. What is the maximum? It is easy since one coefficient is positive and one negative for each variable.

$$\sum_{f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j + \sum_{g_j > 0} \frac{g_j}{f_0} y_j - \sum_{g_j < 0} \frac{g_j}{1 - f_0} y_j \ \ge \ 1$$

This is valid for S, it is the *Gomory mixed integer cut* (GMI cut).

Let us compare the GMI cut applied to the pure integer program (g_j's$= 0$) with another cut introduced by Gomory, the *Gomory fractional cut*

$$\sum_{f_j \le f_0} f_j x_j + \sum_{f_j > f_0} f_j x_j \ \ge \ f_0$$

This is to be compared with the Gomory Mixed Integer Cut:

$$\sum_{f_j \le f_0} f_j x_j + \frac{f_0}{1 - f_0} \sum_{f_j > f_0} (1 - f_j) x_j \ \ge \ f_0$$

So, we're comparing $f_0/(1-f_0) * (1 - f_j)$ with f_j, the comparison is always $<$ when $f_j > f_0$, so the GMI cut dominates the fractional cut.

Application: Let $P := \{(x, y) \in \mathbb{R}_+^{n+p} \ : \ Ax + Gy \le b\}$ be a rational polyhedron and let $S := \{x \in \mathbb{Z}_+^n, y \in \mathbb{R}_+^p \ : \ Ax + Gy \le b\}$. Add slack variables $Ax + Gy + s = b$. Define $S' := \{x \in \mathbb{Z}_+^n, (y, s) \in \mathbb{Z}_+^{p+m} \ : \ Ax + Gy + s = b\}$. For any $\lambda \in \mathbb{R}^m$, the equation $\lambda(Ax + Gy + s) = \lambda b$ can be used to generate a GMI cut valid for S'. Eliminating $s = b - Ax - Gy$ from this inequality, we get a valid inequality for S, in the space $\mathbb{R}^{n+p}$ of the variables x, y. Let us also call these inequalities *GMI cuts*. Define the *Gomory mixed integer closure* of P to be obtained from P by adding all the GMI cuts.

3. Split cuts

Let $P := \{(x, y) \in \mathbb{R}^{n+p} \ : \ Ax + Gy \leq b\}$ where A, G, b have rational entries, and let $S := P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$.
For $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$, define

$$\Pi_1 := P \cap \{(x, y) \ : \ \pi x \leq \pi_0\}$$

$$\Pi_2 := P \cap \{(x, y) \ : \ \pi x \geq \pi_0 + 1\}$$

Clearly $S \subseteq \Pi_1 \cup \Pi_2$.
Therefore any inequality $cx + hy \leq c_0$ that is valid for $\Pi_1 \cup \Pi_2$ is also valid for S. An inequality $cx + hy \leq c_0$ is called a *split inequality* if there exists $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $cx + hy \leq c_0$ is valid for $\Pi_1 \cup \Pi_2$.
The intersection of all split cuts, denoted by P^1, is called the *split closure* of P.

Theorem: (Cook, Kannan and Schrijver 1990) If P is a rational polyhedron, the split closure of P is a rational polyhedron.

For $k \geq 2$, P^k denotes the split closure of P^{k-1} and it is called the k^{th} split closure of P. It follows from the above theorem that P^k is a polyhedron. Unlike for the pure integer case, there is in general no finite r such that $P^r = Conv(S)$, as shown by the following example.

Example: Let $S := \{(x_1, x_2, y) \in \mathbb{Z}^2 \times \mathbb{R} \ : \ x_1 \geq y, \ x_2 \geq y, \ x_1 + x_2 + 2y \leq 2\}$. Starting from $P := \{(x_1, x_2, y) \in \mathbb{R}^3 \ : \ x_1 \geq y, \ x_2 \geq y, \ x_1 + x_2 + 2y \leq 2\}$, we claim that there is no finite r such that $P^r = Conv(S)$.
To see this, note that P is a simplex with vertices $O = (0, 0, 0)$, $A = (2, 0, 0)$, $B = (0, 2, 0)$ and $C = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. S is contained in the plane $y = 0$. More generally, consider a simplex P with vertices O, A, B and $C = (\frac{1}{2}, \frac{1}{2}, t)$ with $t > 0$. Let $C_1 = C$, let C_2 be the point on the edge from C to A with coordinate $x_1 = 1$ and C_3 the point on the edge from C to B with coordinate $x_2 = 1$. Observe that no split disjunction removes all three points C_1, C_2, C_3. Let Q_i be the intersection of all split cuts that do not cut off C_i. All split cuts belong to at least one of these three sets, thus $P^1 = Q_1 \cap Q_2 \cap Q_3$. Let S_i be the simplex with vertices O, A, B, C_i. Clearly, $S_i \subseteq Q_i$. Thus $S_1 \cap S_2 \cap S_3 \subseteq P^1$. It is easy to verify that $(\frac{1}{2}, \frac{1}{2}, \frac{t}{3}) \in S_i$. Thus $(\frac{1}{2}, \frac{1}{2}, \frac{t}{3}) \in P^1$. By induction, $(\frac{1}{2}, \frac{1}{2}, \frac{t}{3^k}) \in P^k$.

However, for mixed 0,1 programs, Theorem 2 of Section 1 implies that $P^n = Conv(S)$ (Indeed, the lift-and-project polytope P_1 contains the split closure of P by Theorem 1 of Section 1. Similiarly, $P_2(P_1))$ contains the 2^{nd} split closure, etc).

Example: Cornuéjols and Li observed that the n^{th} split closure is needed for 0,1 programs, i.e. there are examples where $P^k \neq Conv(S)$ for all $k < n$. They use the following well-known polytope studied by Chvátal, Cook, and Hartmann:

$$P_{CCH} \equiv \{x \in [0, 1]^n | \sum_{j \in J} x_j + \sum_{j \notin J} (1 - x_j) \geq \frac{1}{2}, \text{ for all } J \subseteq \{1, 2, \cdots, n\}\}$$

Let F_j be the set of all vectors $x \in R^n$ such that j components of x are $\frac{1}{2}$ and each of the remaining $n - j$ components are equal to 0 or 1. The polytope P_{CCH} is the convex hull of F_1.

Lemma: If a polyhedron $P \subseteq R^n$ contains F_j, then its split closure P^1 contains F_{j+1}.
Proof: It suffices to show that, for every $(\pi, \pi_0) \in Z^{n+1}$, the polyhedron $\Pi = Conv((P \cap \{x| \ \pi x \le \pi_0\}) \cup (P \cap \{x| \ \pi x \ge \pi_0 + 1\}))$ contains F_{j+1}. Let $v \in F_{j+1}$ and assume w.l.o.g. that the first $j+1$ elements of v are equal to $\frac{1}{2}$. If $\pi v \in Z$, then clearly $v \in \Pi$. If $\pi v \notin Z$, then at least one of the first $j+1$ components of π is nonzero. Assume w.l.o.g. that $\pi_1 > 0$. Let $v_1, v_2 \in F_j$ be equal to v except for the first component which is 0 and 1 respectively. Notice that $v = \frac{v_1+v_2}{2}$. Clearly, each of the intervals $[\pi v_1, \pi v]$ and $[\pi v, \pi v_2]$ contains an integer. Since πx is a continuous function, there are points $\tilde{v}_1$ on the line segment $Conv(v, v_1)$ and $\tilde{v}_2$ on the line segment $Conv(v, v_2)$ with $\pi \tilde{v}_1 \in Z$ and $\pi \tilde{v}_2 \in Z$. This means that $\tilde{v}_1$ and $\tilde{v}_2$ are in Π. Since $v \in Conv(\tilde{v}_1, \tilde{v}_2)$, this implies $v \in \Pi$. □
Starting from $P = P_{CCH}$ and applying the lemma recursively, it follows that the $(n-1)$st split closure of P_{CCH} contains F_n, which is nonempty. Since $Conv(P_{CCH} \cap \{0,1\}^n)$ is empty, the n^{th} split closure is needed to obtain $Conv(P_{CCH} \cap \{0,1\}^n)$. □

Nemhauser and Wolsey proved that the split closure and the GMI closure are identical. To simplify the proof, we will assume that P is bounded. The following lemma will be useful.

Lemma: Assume P is bounded and nonempty. Let $cx + hy \le c_0$ be a split cut. Then there exist $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ and $\alpha, \beta \in \mathbb{R}_+$ such that

$$cx + hy - \alpha(\pi x - \pi_0) \le c_0 \text{ and}$$
$$cx + hy + \beta(\pi x - (\pi_0 + 1)) \le c_0$$

are both valid for P.

Proof: By definition of a split cut, there exist $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $cx + hy \le c_0$ is valid for $\Pi_1 \cup \Pi_2$. Consider $\Pi_1 = \{(x,y) \ : \ Ax + Gy \le b, \ \pi x \le \pi_0\}$.
If $\Pi_1 = \emptyset$, then choose $\alpha \ge \frac{cx^t + hy^t - c_0}{\pi x^t - \pi_0}$ for all the extreme points (x^t, y^t) of P. This implies that $cx + hy - \alpha(\pi x - \pi_0) \le c_0$ is valid for P. Now consider the case where $\Pi_1 \neq \emptyset$. Then, by Farkas's lemma, $Dz \le d$ implies $\gamma z \le \gamma_0$ if and only if there exists $v \ge 0$ such that $vD = \gamma$ and $\gamma_0 \ge vd$. [This is also a consequence of LP duality: $\max\{\gamma z \ : \ Dz \le d\} = \min\{vd \ : \ vD = \gamma, \ v \ge 0\} \ \le \gamma_0$.] Therefore there exist $u \ge 0, v \ge 0$ such that

$$cx + hy = u(Ax + Gy) + \alpha \pi x$$
$$\text{and} \quad c_0 \ge ub + \alpha \pi_0$$

Since $u(Ax + Gy) \le ub$ is valid for P, it follows that $cx + hy - \alpha(\pi x - \pi_0) \le c_0$ is also valid for P.
A similar argument applied to Π_2 shows that $cx + hy + \beta(\pi x - (\pi_0 + 1)) \le c_0$ is valid for P for some $\beta \ge 0$. □

Theorem: Let $P := \{(x,y) \in \mathbb{R}_+^{n+p} \ : \ Ax + Gy \le b\}$ be a bounded rational polyhedron and let $S := P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$. The split closure of P is identical to the Gomory mixed integer closure of P.

Proof: We may assume that the constraints $x \ge 0$ and $y \ge 0$ are part of $Ax + Gy \le b$ in the description of P.

Consider first a GMI cut. Its derivation was obtained by arguing that $k = a_0 - \sum_{f_j \le f_0} a_j x_j - \sum_{f_j > f_0} (a_j + 1) x_j$ is an integer, and either $k \le -1$ or $k \ge 0$. This is a disjunction of the form $\pi x \le \pi_0$ or $\pi x \ge \pi_0 + 1$ with $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$. Thus the derivation of the GMI cut implies that it is a split inequality.

Conversely, let $cx + hy \le c_0$ be a split cut. By the previous lemma, there exists $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ and $\alpha, \beta \in \mathbb{R}_+$ such that

$$(1)\ \ cx + hy - \alpha(\pi x - \pi_0) \le c_0 \text{ and}$$

$$(2)\ \ cx + hy + \beta(\pi x - (\pi_0 + 1)) \le c_0$$

are both valid for P. We can assume $\alpha > 0$ and $\beta > 0$ since, otherwise, $cx + hy \le c_0$ is valid for P and therefore also to its Gomory mixed integer closure. We now apply the Gomory mixed integer procedure to (1) and (2). Introduce slack variables s_1 and s_2 in (1) and (2) respectively and subtract (1) from (2).

$$(\alpha + \beta)\pi x + s_2 - s_1 = (\alpha + \beta)\pi_0 + \beta$$

Dividing by $\alpha + \beta$ we get

$$\pi x + \frac{s_2}{\alpha + \beta} - \frac{s_1}{\alpha + \beta} = \pi_0 + \frac{\beta}{\alpha + \beta}$$

From this equation, we can derive a GMI cut. Note that $f_0 = \frac{\beta}{\alpha + \beta}$ and that the continuous variable s_2 has a positive coefficient while s_1 has a negative coefficient. So the GMI cut is

$$\frac{\frac{1}{\alpha+\beta}}{\frac{\beta}{\alpha+\beta}} s_2 + \frac{\frac{1}{\alpha+\beta}}{1 - \frac{\beta}{\alpha+\beta}} s_1 \ge 1$$

which simplifies to

$$\frac{1}{\alpha} s_1 + \frac{1}{\beta} s_2 \ge 1.$$

We now replace s_1 and s_2 as defined by the equations (1) and (2) to get the GMI cut in the space of the x, y vaviables. The resulting inequality is

$$cx + hy \le c_0.$$

Therefore $cx + hy \le c_0$ is a GMI cut. □

4. Intersection Cuts

Intersection cuts were introduced by Balas. They are split cuts obtained from a basis of the linear programming relaxation. For convenience, we assume that the constraints are in equality form.

$$Ax = b,\ x \ge 0,\ x_j \text{ integer for } j \in N_I,$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $N_I \subseteq N := \{1, 2, \dots, n\}$. Wlog assume A is of full row rank. Let $P = \{x \ge 0 \ :\ Ax = b\}$. Let B index m linearly independent columns of A (B is a *basis*) and $J := N \setminus B$ index the non-basic variables. The *conic polyhedron* associated with B is given by:

$$(1)\qquad P(B) := \{x \in \mathbb{R}^n : Ax = b \text{ and } x_j \ge 0 \text{ for } j \in J\}.$$

The set $P(B)$ is the relaxation of P obtained by deleting the non-negativity constraints on the basic variables. Observe that $P(B)$ is a translate of a polyhedral cone. Specifically,

we may write $P(B) = C + \bar{x}$, where C is the polyhedral cone $C := \{x \in \mathbb{R}^n : Ax = 0$ and $x_j \geq 0$ for $j \in J\}$, and $\bar{x}$ solves the system $Ax = b$ and $x_j = 0$ for $j \in J$. The vector $\bar{x} \in \mathbb{R}^n$ is the *basic solution* corresponding to the basis B.
The extreme rays of the polyhedral cone C can be obtained by first solving the system $Ax = b$ in terms of the basic variables, which yields the *simplex tableau:*

$$\bar{x}_i = x_i + \sum_{j \in J} \bar{a}_{ij} x_j, \qquad i \in B. \tag{2}$$

The extreme rays of C can be obtained from the coefficients of the simplex tableau as follows. Given $j \in J$, define the vector r^j:

$$r^j_k := \begin{cases} -\bar{a}_{kj} & \text{if } k \in B, \\ 1 & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The conic polyhedron $P(B)$ can then be written as $P(B) = \bar{x} +$ cone $(\{r^j\}_{j \in J})$, where cone $(\{r^j\}_{j \in J})$ denotes the polyhedral cone generated by the vectors $\{r^j\}_{j \in J}$. Observe that, since there are $|J| = n - m$ non-basic variables, $P(B)$ has exactly $n - m$ extreme rays.

We now derive the intersection cut. Let $D(\pi, \pi_0)$ denote an arbitrary split disjunction $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$. Assume $\bar{x}$ violates the disjunction $D(\pi, \pi_0)$, and define $\epsilon(\pi, \pi_0) := \pi^T \bar{x} - \pi_0$ to be the amount by which $\bar{x}$ violates the first term of the disjunction. Since $\pi_0 < \pi^T \bar{x} < \pi_0 + 1$, we have $0 < \epsilon(\pi, \pi_0) < 1$. Also, for $j \in J$, define scalars:

$$\alpha_j(\pi, \pi_0) := \begin{cases} -\frac{\epsilon(\pi,\pi_0)}{\pi^T r^j} & \text{if } \pi^T r^j < 0, \\ \frac{1-\epsilon(\pi,\pi_0)}{\pi^T r^j} & \text{if } \pi^T r^j > 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{4}$$

The interpretation of the numbers $\alpha_j(\pi, \pi_0)$ for $j \in J$ is the following. Let $x^j(\alpha) := \bar{x} + \alpha r^j$, where $\alpha \in \mathbb{R}_+$, denote the half-line starting in $\bar{x}$ in the direction r^j. The value $\alpha_j(\pi, \pi_0)$ is the smallest value of $\alpha \in \mathbb{R}_+$ such that $x^j(\alpha)$ satisfies the disjunction $D(\pi, \pi_0)$. In other words, the point $x^j(\alpha_j(\pi, \pi_0))$ is the intersection of the half-line starting in $\bar{x}$ in direction r^j with the hyperplane $\pi^T x = \pi_0$ or the hyperplane $\pi^T x = \pi_0 + 1$. Note that $\alpha_j(\pi, \pi_0) = +\infty$ when the direction r^j is parallel to the hyperplane $\pi^T x = \pi_0$. Given the numbers $\alpha_j(\pi, \pi_0)$ for $j \in J$, the intersection cut associated with B and $D(\pi, \pi_0)$ is given by:

$$\sum_{j \in J} \frac{x_j}{\alpha_j(\pi, \pi_0)} \geq 1. \tag{5}$$

This inequality is valid for $P_I(B) := P(B) \cap \{x \geq 0 \ : \ x_j$ integer for $j \in N_I\}$ since it is a split cut. In fact, the intersection cut gives a complete description of the set of points in $P(B)$ that satisfy the disjunction $D(\pi, \pi_0)$. Andersen, Cornuéjols and Li showed that intersection cuts are sufficient for describing the split closure of P. Let $\mathcal{B}^*$ denote the set of all bases of A. We have:

$$\bigcap_{(\pi,\pi_0) \in \mathbb{Z}^{N_I+1}} \text{Conv}\,(P \cap (\{x \ : \ \pi x \leq \pi_0\} \cup \{x \ : \ \pi x \geq \pi_0 + 1\}))$$
$$= \bigcap_{B \in \mathcal{B}^*} \bigcap_{(\pi,\pi_0) \in \mathbb{Z}^{N_I+1}} \text{Conv}\,(P(B) \cap (\{x \ : \ \pi x \leq \pi_0\} \cup \{x \ : \ \pi x \geq \pi_0 + 1\})).$$

The following lemma shows that GMI cuts derived from rows of the simplex tableau can be obtained from (5) by choosing an appropriate disjunction $D(\pi, \pi_0)$:

Lemma 1 Let B be a basis of A, and let $\bar{x}$ be the corresponding basic solution. Also, let x_i be a basic integer constrained variable, and suppose $\bar{x}_i$ is fractional. The MIG cut obtained from the row of the simplex tableau, in which x_i is basic, is given by the inequality $\sum_{j\in J} \frac{x_j}{\alpha_j(\pi^i,\pi_0^i)} \geq 1$, where $\pi_0^i := \lfloor \bar{x}_i \rfloor$, and for $j \in N_I$:

$$\pi_j^i := \begin{cases} \lfloor \bar{a}_{ij} \rfloor & \text{if } j \in J \text{ and } f_j \leq f_0, \\ \lceil \bar{a}_{ij} \rceil & \text{if } j \in J \text{ and } f_j > f_0, \\ 1 & \text{if } j = i \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Proof: Let us compute $\alpha_j(\pi^i, \pi_0^i)$ for the above disjunction using formula (4), where $j \in J$. We have:

$$\epsilon(\pi, \pi_0) = (\pi^i)^T \bar{x} - \pi_0^i = \bar{x}_i - \lfloor \bar{x}_i \rfloor = f_0.$$

Using (3) and (6), we get

$$(\pi^i)^T r^j = \pi_i^i r_i^j - \pi_j^i r_j^j = \begin{cases} -f_j & \text{if } j \in N_I \text{ and } f_j \leq f_0, \\ 1 - f_j & \text{if } j \in N_I \text{ and } f_j > f_0, \\ -\bar{a}_{ij} & \text{if } j \in J \setminus N_I. \end{cases} \tag{7}$$

Now $\alpha_j(\pi^i, \pi_0^i)$ follows from formula (4). This yields the MIG cut as claimed. □

There is a closed form formula for the Euclidian distance cut off by an intersection cut derived from a split disjunction $D(\pi, \pi_0)$ and a basis B:

Lemma 2 Let B be a basis of A, let $\bar{x}$ be the corresponding basic solution, and let $D(\pi, \pi_0)$ be a split disjunction violated by $\bar{x}$. The distance $d(B, \pi, \pi_0)$ cut off by the split cut derived from B and $D(\pi, \pi_0)$ satisfies:

$$(d(B, \pi, \pi_0))^2 = \frac{1}{\sum_{j\in J} \frac{1}{(\alpha_j(\pi,\pi_0))^2}} \tag{8}$$

Proof: Let $\gamma^T x \geq 1$, where $\gamma \in \mathbb{R}^n$, denote the intersection cut (5) derived from B and the disjunction $D(\pi, \pi_0)$. Then $\gamma_j = 0$ for $j \in N \setminus J$, $\gamma_j = \frac{1}{\alpha_j(\pi,\pi_0)}$ for $j \in J$ and $\gamma^T \bar{x} = 0$. Since γ is a normal vector to the intersection cut (5), it follows that $d(B, \pi, \pi_0)$ satisfies $\gamma^T(\bar{x} + d(B, \pi, \pi_0) \frac{\gamma}{\|\gamma\|_2}) = 1$. Isolating $d(B, \pi, \pi_0)$ in this expression gives the formula. □