Higher dimensional automata
between topology and concurrency

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GETCO 22
Goals

- Introduce precubical sets alias higher dimensional automata.
- Topological executions: directed path spaces.
- Combinatorial executions: track complexes.
- How these models are related?
Directed spaces

Idea

Model computer programs by topological spaces.

- points of space = states of a program,
- distinguished paths = (partial) executions.

Definition (Grandis)

A d-space is a pair $\langle X, \bar{P}(X) \rangle$, where

- $X$ is a topological space,
- $\bar{P}(X) \subseteq P(X)$ is a family of d-paths ($I = [0, 1]$),
- $\forall x \in X \ const_x \in \bar{P}(X)$,
- $\alpha, \beta \in \bar{P}(X)$, $\alpha(1) = \beta(0) \implies \alpha \ast \beta \in \bar{P}(X)$.
- $\alpha \in \bar{P}(X)$, $f : I \rightarrow I$ increasing $\implies \alpha \circ f \in \bar{P}(X)$.

A map $f : X \rightarrow Y$ between d-spaces is a d-map if $\alpha \in \bar{P}(X) \implies f(\alpha) \in \bar{P}(Y)$. 
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- \(\alpha, \beta \in \tilde{P}(X), \ \alpha(1) = \beta(0) \implies \alpha \ast \beta \in \tilde{P}(X)\).
- \(\alpha \in \tilde{P}(X), \ \text{f : I \to I} \text{ increasing} \implies \alpha \circ \text{f} \in \tilde{P}(X)\).

A map \(f : X \to Y\) between d-spaces is a d-map if \(\alpha \in \tilde{P}(X) \implies f(\alpha) \in \tilde{P}(Y)\).
Directed spaces: examples

Example

**Directed interval:** $\vec{I} = (I, \{\alpha : I \to I \text{ increasing}\})$.

The category $\mathbf{dTop}$ of d-spaces and d-maps is complete and cocomplete. We obtain more examples:

Example

- **Directed cube:** $\vec{I^n} = (I^n, \{\alpha : I \to I^n \text{ all coordinates increasing}\})$
- **Directed circle:** $\vec{S^1} = \vec{I}/0 \sim 1 = (S^1, \{\text{counterclockwise paths}\})$
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Precubical sets

Definition

A **precubical set** $K$ consist of

- a sequence of sets $(K[n])_{n \geq 0}$ (**$n$-cells** or **$n$-cubes**),
- a collection of maps $\delta_i^\varepsilon : K[n] \to K[n-1]$, $1 \leq i \leq n$, $\varepsilon = 0, 1$ (**face maps**),
- $\delta_i^\varepsilon \circ \delta_j^\eta = \delta_j^{\eta-1} \circ \delta_i^\varepsilon$ for $i < j$ (**precubical identities**).

A **precubical map** $f : K \to L$ is a sequence of compatible functions $f[n] : K[n] \to L[n]$.

Definition

The **geometric realization** of a precubical set $K$:

$$|K| = \prod_{n \geq 0} K[n] \times \vec{I}^n / \sim$$

$$(\delta_i^\varepsilon(c), (x_1, \ldots, x_{n-1})) \sim (c, (x_1, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_{n-1})).$$
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$$(\delta_i^\varepsilon(c), (x_1, \ldots, x_{n-1})) \sim (c, (x_1, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_{n-1})).$$
Examples of precubical sets

Example

The *standard $n$-cube* $\square^n$:

- $\square^n[k] = \{(a_1, \ldots, a_n) \mid a_i \in \{0, *, 1\}, \text{ exactly } k \text{ stars among } a_i's\}$.
- $\delta^\varepsilon_i$ converts $i$-th star into $\varepsilon$.

The geometric realization of $\square^n$ is $\vec{l}^n$.

A *Euclidean complex* is a precubical subset of a standard cube.

Example

The *final precubical set* $Z$:

- $Z[n]$ has exactly one element for every $n$,
- face maps are defined the only possible way.
Directed paths spaces on precubical sets

Question

Let $K$ be a precubical set, $v, w \in K[0]$ its vertices. What is the (homotopy type of) the space $\vec{P}(K)_v^w$ of directed paths in $|K|$ from $v$ to $w$?

Results for Euclidean complexes

- $\vec{P}(\vec{n})_0^1$ is contractible,
- $\vec{P}(\partial \vec{n})_0^1 \simeq S^{n-2}$,
- The length decomposition: $\vec{P}(K)_v^w = \bigsqcup_{n \geq 0} \vec{P}(K; n)_v^w$.
- Prodsimplicial models for Euclidean complexes [Raussen 2010, 2012].
- Every finite CW-complex is homotopy equivalent to $\vec{P}(K)_0^1$ for $K \subseteq \square^n$ [Z, 2016].
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Observation

Every d-path $\alpha \in \vec{P}(K)$ has a presentation

$$\alpha = [c_1; \beta_1]^t_1 \ast [c_2; \beta_1]^t_2 \ast \cdots \ast [c_n; \beta_n]$$

where $c_k \in K[n_k]$, $\beta_k \in \vec{P}_{[t_{k-1}, t_k]}(\vec{n}_k)$, $0 < t_1 < \cdots < t_{n-1} < 1$. 
A $d$-path $\alpha \in \tilde{P}(K)$ is **tame** if there exists a presentation

$$\alpha = [c_1; \beta_1] \ast \cdots \ast [c_n; \beta_n]$$

such that $\beta_k(t_{k-1}) = (0, \ldots, 0)$ and $\beta_k(t_k) = (1, \ldots, 1)$ for all $k$. 

Both paths are tame. 

Tame in $\square^3$ but not in $\partial \square^3$. 

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Higher dimensional automata
Theorem [Z 2020]

For every precubical set $K$, the inclusion

$$\bar{P}_{tame}(K)_v^w \subseteq \bar{P}(K)_v^w$$

is a homotopy equivalence.
Cube chains

**Definition**

A *cube chain* in $K$ from $v$ to $w$ is a sequence of cubes $(c_1, \ldots, c_n)$, such that

$$\delta^0_{all}(c_1) = v, \quad \delta^1_{all}(c_k) = \delta^0_{all}(c_{k+1}), \quad \delta^1_{all}(c_n) = w.$$  

Every tame path "lies" in a cube chain:
Cube chains on Euclidean complexes

**Proposition**

If $K$ is a Euclidean complex, then:

- The set $\text{Ch}(K)_w^v$ of cube chains from $v$ to $w$ is a poset with respect to inclusion.
- The set of paths $\vec{P}(K; C)$ lying in cube chain $C$ is contractible.
- $\bigcap_{j=1}^k \vec{P}(K; C_j)$ is contractible if there exists $C'$ such that $C' \leq C_j$ for all $j$.
- Otherwise, $\bigcap_{j=1}^k \vec{P}(K; C_j)$ is empty.

Thus, $\{\vec{P}(K; C) \mid C \in \text{Ch}(K)_w^v\}$ is a good cover of $\vec{P}_{tame}(K)_w^v$.

**Theorem [Z 2018]**

If $K$ is a Euclidean complex, then Nerve Lemma implies:

$$\vec{P}(K)_w^v \simeq \vec{P}_{tame}(K)_w^v \simeq |\text{Ch}(K)_w^v|.$$
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A model for directed paths on Euclidean complexes

Proposition

The following posets are isomorphic:

- $Ch(□^n)_0^1$,
- The poset of ordered partitions of $\{1, \ldots, n\}$.
- The face lattice of $(n-1)$-dimensional permutahedron

$$\Pi^{n-1} = \text{conv}\{(\sigma(1), \ldots, \sigma(n)) \mid \sigma \in \text{Perm}(\{1, \ldots, n\})\}.$$  

Example

If $K \subseteq □^n$, then $|Ch(K)_0^1|$ is a subcomplex of the permutahedron, for example

$|Ch(\partial □^3)_0^1| =$

\begin{center}
\begin{tikzpicture}
  \draw (0,0) -- (1,1) -- (0,2) -- (-1,1) -- (0,0);
  \draw (0,0) -- (-1,0) -- (-1,-1) -- (0,-1) -- (0,0);

delete{\draw (0,0) -- (0,1) -- (1,0) -- (0,0);
}\end{tikzpicture}
\end{center}
A model for directed paths on Euclidean complexes

Proposition

The following posets are isomorphic:

- $Ch(□^n)_{10}$,
- The poset of ordered partitions of $\{1, \ldots, n\}$.
- The face lattice of $(n - 1)$-dimensional permutahedron

$$\Pi^{n-1} = \text{conv}\{ (\sigma(1), \ldots, \sigma(n)) \mid \sigma \in \text{Perm}(\{1, \ldots, n\}) \}.$$  

Example

If $K \subseteq □^n$, then $|Ch(K)_{10}|$ is a subcomplex of the permutahedron, for example

$$|Ch(\partial □^3)_{10}| = \begin{array}{c}
\text{Diagram}
\end{array}$$
Algorithm

If \( K \subseteq \square^n \) (or \( K \subseteq [0, n_1] \times [0, n_2] \times \cdots \times [0, n_d] \)), then there is an efficient algorithm for calculating \( H_*(\vec{P}(K)_v^w) \) via discrete Morse theory.

Theorem (Raussen-Meshulam, Z)

Calculation of homology of \( \vec{P}(K)_v^w \) for \( K \) being the \( k \)-skeleton of

\[
[0, n_1] \times [0, n_2] \times \cdots \times [0, n_d].
\]

This is a “no \((k + 1)\)-equal” configuration spaces of sequences of points on \( \mathbb{R} \).
Cube chain complex: general case

Definition

The **wedge cube** is a precubical set $\Box^\vee n = \Box^{n_1}_{1\sim0} \vee \cdots \vee \Box^{n_k}_{1\sim0}$. For example,

$\Box^\vee(2,1,3,2) = \Box^2 \vee \Box^1 \vee \Box^3 \vee \Box^1 = \perp \rightarrow \top$

The **cube chain** in $K$ is a (bipointed) precubical map $c : (\Box^\vee n, \perp, \top) \rightarrow (K, v, w)$.

Problem

If $K$ is not a Euclidean complex, then a d-path may have different “tame” presentations using the same cube chain $c$. As a consequence, the map

$c_* : \vec{P}(\Box^\vee n)_{\top} \rightarrow \vec{P}_{tame}(K)^w$

induced by $c$ is not necessarily injective.
Cube chain complex: general case

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Cube chain category

Definition

The *cube chain category* $\text{Ch}(K)^w_v$ of $K$:

- objects = cube chains $c : (\square^m, \bot, \top) \rightarrow (K, v, w)$,
- morphisms = commutative diagrams of bipointed precubical maps

\[
\begin{array}{ccc}
\square^m & \rightarrow & K \\
\downarrow & & \\
\square^n & \rightarrow &
\end{array}
\]

Theorem [Z 2020]

For every precubical set $K$ there are homotopy equivalences

\[|\text{Ch}(K)^w_v| \simeq \tilde{P}_\text{tame}(K)^w_v \simeq \tilde{P}(K)^w_v.\]
**Cube chain category**

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Definition

The *cube wedge* category \( \mathcal{P} \):
- objects: cube wedges \( \square^n = \square^{n_1} \lor \cdots \lor \square^{n_k} \),
- morphisms: precubical maps preserving the initial and final vertices.

Properties

- For every bipointed precubical set \( K \) there is a forgetful functor
  \[
  \text{Ch}(K)_w : (c : \square^n \to K) \mapsto \square^n \in \mathcal{P}.
  \]
- The cube chains on \( K \) form a presheaf on \( \mathcal{P} \) (a functor \( \mathcal{P}^{\text{op}} \to \text{Set} \)):
  \[
  \text{Ch}(K)(\square^n) = \square^\text{Set}_*(\square^n, K).
  \]
- \( \mathcal{P} \cong \text{Ch}(Z)_* \), where \( Z \) is the final precubical set.
The final precubical set

**Theorem [Paliga-Z, 2022]**

Let $Z$ be a final precubical set. Then

$$|\mathcal{P}| \cong \tilde{P}(Z)_* \cong \coprod_{n \geq 0} \tilde{P}(Z; n)_* \cong \coprod_{n \geq 0} \text{Conf}(n, \mathbb{R}^2).$$

As a consequence, $\tilde{P}(Z; n)_* = K(B_n, 1)$ ($B_n$ denotes the braid group on $n$ strands).

**Applications**

Every precubical set $K$ has a unique (bipointed) precubical map $K \rightarrow Z$, which induces:

- a representation $\pi_1(\tilde{P}(K; n)_w) \rightarrow B_n$,
- “characteric classes” in $H^*(\tilde{P}(K; n)_v)$ induced by elements of $H^*(B_n)$.

What these invariants measure?
Towards concurrency (with U. Fahrenberg, C. Johansen, G. Struth)

Definition

- A **transition system** is a directed graph with edges labeled with letters of an alphabet $\Sigma$.
- An **automaton** is a transition system with distinguished “**start**” and “**accept**” vertices.
- Automata recognize **languages**: sets of words given by paths from start to accept states.
- Letters (“events”) of words are totally ordered: no two events cannot be active simultaneously.

Definition (Pratt-van Glabbeek)

A **higher dimensional automaton** is a precubical set $X$ with

- a **labeling** $\lambda : X[1] \rightarrow \Sigma$,
- **start states** $X_{\perp} \subseteq X[0]$,
- **accept states** $X^\top \subseteq X[0]$,
- $\lambda(\delta_i^0(q)) = \lambda(\delta^1(q))$ for $q \in X[2]$, $i = 1, 2$. 
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**Definition**

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Looking for better definitions

**Definition**

- A *presheaf* over a category $C$ is a contravariant functor $F : C^{op} \to \text{Set}$.
- An *element category* $El(F)$ of a presheaf $F$:
  - objects = pairs $(c, x)$ such that $c \in C$, $x \in F(x)$.
  - morphisms $(c, x) \to (c', x') = \{ \alpha \in C(x, x') \mid F(\alpha)(x') = x \}$.
- The canonical projection: $El(F) \ni (c, x) \mapsto c \in C$.

**Example**

Directed graphs are presheaves over $G = \emptyset \xrightarrow{d^0} 1 \xleftarrow{d^1} \emptyset$.

**Example**

Transition systems are presheaves over $G_\Sigma$,

- $Ob(G_\Sigma) = \Sigma \cup \{\emptyset\}$,
- $G_\Sigma(\emptyset, a) = \{ d_a^0, d_a^1 \}$ ($a \in \Sigma$)
- $G_\Sigma(a, a) = \{ id_a \}$, $G_\Sigma(\emptyset, \emptyset) = \{ id_{\emptyset} \}$
- no other morphisms
Definition

- A **presheaf** over a category $\mathcal{C}$ is a contravariant functor $F : \mathcal{C}^{\text{op}} \to \text{Set}$.
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- $\mathcal{G}_\Sigma(a, a) = \{\text{id}_a\}$, $\mathcal{G}_\Sigma(\emptyset, \emptyset) = \{\text{id}_\emptyset\}$
- no other morphisms
Example

\[ \Sigma = \{ \bullet, \bullet \} \]

Transition system \( G \)

\[ \begin{align*}
\text{El}(G) & \\
G_\Sigma &
\end{align*} \]
Lo-sets and lo-maps

Orders

We use two strict transitive relations: $<$ and $\rightarrow$.

- $p < q$ means that “$p$ happens before $q$” (precedence),
- $p \rightarrow q$ means that “$p$ has smaller id than $q$” (event order).

Definition

- An **lo-set** is a triple $U = (U, \rightarrow, \lambda)$, where
  - $U$ is a finite set,
  - $\rightarrow$ is a (strict) total order on $U$,
  - $\lambda : U \rightarrow \Sigma$ is a labeling.
- An **lo-map** is an order- and label-preserving map $f : U \rightarrow V$ (it is always injective).
- Every lo-map $U \rightarrow V$ has the form $(A \subseteq V)$
  \[ \partial_A : U \cong V \setminus A \subseteq V. \]
Lo-sets and lo-maps

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Precube maps

Definition

- A **precube map** from $U$ to $V$ is a triple $(f, A, B)$, where $f : U \to V$ is an lo-map and

  \[ V = f(U) \cup A \cup B \]

- Every precube map has the form $(A, B \subseteq V, A \cap B = \emptyset)$

  \[ d_{A,B} = (\partial_{A \cup B}, A, B) : U \to V. \]

- Composition of precube maps $d_{A,B} : U \to V$ and $d_{C,D} : V \to W$

  \[ d_{C,D} \circ d_{A,B} = d_{\partial_{A \cup B}(A) \cup C, \partial_{A \cup B}(B) \cup D}. \]

- Notation: $d^0_A = d_{A,\emptyset}$, $d^1_B = d_{\emptyset,B}$. 
Definition of HDA — precube categories

Example (composition of precube maps)

\[ U \xrightarrow{d_{q,0}} V \xrightarrow{d_{s,u}} W \]

\[ U \xrightarrow{d_{sv,u}} W \]

\[ a \quad p \quad t \quad u \quad v \quad w \]

\[ b \quad q \quad r \quad v \quad w \]

\[ a \quad t \quad u \quad v \quad w \]

\[ b \quad u \quad v \quad w \]

\[ 0 \quad 0 \quad 0 \quad 0 \quad 1 \]

\[ 0 \quad 0 \quad 0 \quad 1 \quad 1 \]

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Higher dimensional automata
Precubical sets

Definition

- The **precube category** \( \Box \) has lo-sets as objects and precube maps as morphisms.
- We do not distinguish the precube category and its skeleton (or the quotient by isomorphisms).
- Morphisms \( d^0_A := d_{A,\emptyset} : U \cong V \setminus A \subseteq V \) are **forth-morphisms**.
- Morphisms \( d^1_B := d_{\emptyset,B} : U \cong V \setminus B \subseteq V \) are **back-morphisms**.

Definition

A **precubical set** \( X \) is a presheaf over \( \Box \), ie, a functor \( X : \Box^{op} \to \text{Set} \). Namely:

- For every \( U = (a_1 \to \cdots \to a_n) \in \Box \) there is a set \( X[U] \).
- For \( A, B \subseteq U \in \Box \), \( A \cap B = \emptyset \), there is a map

\[
\delta_{A,B} = X[d_{A,B}] : X[U] \to X[U \setminus (A \cup B)].
\]

- \( \delta_{A,B} \circ \delta_{C,D} = \delta_{A \cup C, B \cup D} : X[U] \to X[U \setminus (A \cup B \cup C \cup D)] \).
Let
- $X$ be a precubical set ($X \in \square \text{Set}$),
- $U = (u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_n) \in \square$,
- $x \in X[U]$.

**Geometry**
- $x$ is a cube with “directions” $u_1, \ldots, u_n$.
- $\delta^1_{u_k}(x)$ is the upper face of $x$ in direction $u_k$.
- $\delta^0_{u_k}(x)$ is the lower face of $x$ in direction $u_k$.
- $\delta_{A,B}(x)$ is an iterated face of $x$: lower in directions $a \in A$ and upper in directions $b \in B$.

**Concurrency**
- $x$ is a state with active events $u_1, \ldots, u_n$.
- $\delta^1_{u_k}(x)$ is the state after terminating $u_k$.
- $\delta^0_{u_k}(x)$ is the state before starting $u_k$.
- $\delta_{A,B}(x)$ is the state obtained from $x$ after terminating events $a \in A$ and “unstarting” events $b \in B$.
Higher dimensional automata

Definition

- The cell category $\textbf{Cell}(X)$ of a precubical set $X$ is its category of elements.
- $\text{ev} : \textbf{Cell}(X) \to \square$ is the canonical projection.

Definition

An higher dimensional automaton (HDA) is a precubical set $X$ with

- the set $\text{start cells}$ $\text{start cells} \subseteq \text{Cell}(X)$,
- the set $\text{accept cells} \subseteq \text{Cell}(X)$.

A HDA is simple if it has one start and one accept cell. Precubical sets are regarded as HDA with no start/accept cells.

Definition

The standard $U$-cube $\square^U$ (for $U \in \square$) is the presheaf represented by $U$: $\square^U[V] = \square(V, U)$, with $(\square^U)_\perp = \{d^0_U \in \square(\emptyset, U)\}$, $(\square^U)_\top = \{d^1_U \in \square(\emptyset, U)\}$. 
Higher dimensional automata

Definition

- The cell category $\text{Cell}(X)$ of a precubical set $X$ is its category of elements.
- $\text{ev} : \text{Cell}(X) \to □$ is the canonical projection.

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A path in a HDA $X$ is a sequence

$$\alpha = (x_0, \varphi_1, x_1, \varphi_2, \ldots, \varphi_n, x_n)$$

such that $x_k \in \text{Cell}(X)$ and either

- $\varphi_k = \delta^0_A$ and $\delta^0_A(x_k) = x_{k-1}$ for $A \subseteq \text{ev}(x_k)$ (up-step, notation: $x_{k-1} \uparrow^A x_k$) or
- $\varphi_k = \delta^1_B$ and $\delta^1_B(x_{k-1}) = x_k$ for $B \subseteq \text{ev}(x_{k-1})$ (down-step, notation: $x_{k-1} \downarrow^B x_k$).

### Definition

**Equivalence** of paths $\alpha, \beta \in P(X)$ ($\alpha \sim \beta$) is the equivalence relation spanned by

- $(x \uparrow^A y \uparrow^C z) \sim (x \uparrow^{A\cup C} z)$
- $(x \downarrow^B y \downarrow^D z) \sim (x \downarrow^{B\cup D} z)$
- $\alpha \sim \beta \implies \gamma \ast \alpha \ast \delta \sim \gamma \ast \beta \ast \delta$.

**Subsumption** of paths $\alpha, \beta \in P(X)$ ($\alpha \sqsubseteq \beta$) is the transitive relation spanned by

- $(y \downarrow^B w \uparrow^A z) \sqsubseteq (y \uparrow^A x \downarrow^B z)$
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Higher dimensional automata
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Paths: example

\[ \alpha = (x_0 \xrightarrow{ac} x_1 \xleftarrow{a} x_2 \xrightarrow{b} x_3 \xleftarrow{c} x_4) \]

\[ \beta = (x_0 \xrightarrow{a} y \xrightarrow{c} x_1 \xleftarrow{a} x_2 \xrightarrow{b} x_3 \xleftarrow{c} x_4) \sim \alpha \]

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**Paths as functors**

**Definition**

A **Directed category** is a category $\mathcal{C}$ with wide subcategories $\mathcal{C}_0 \subseteq \mathcal{C} \supseteq \mathcal{C}_1$. Morphisms of $\mathcal{C}_0$ are **forth-morphisms**, morphisms of $\mathcal{C}_1$, **back-morphisms**. A functor is **directed** if it preserves forth- and back-morphism.

**Examples**

- Category $\square$: $d_A^0$ are forth-morphisms, $d_B^1$ are back-morphisms.
- The category of cells $\text{Cell}(X)$ is directed: a morphism $(x, U) \xrightarrow{\varphi} (y, V)$ is a forth/back-morphism if $\varphi \in \square(U, V)$ is such. Further, $ev : \text{Cell}(X) \to \square$ is a directed functor.
- **Linear categories** ($\rightarrow$ are forth-morphisms, $\leftarrow$ are back-morphisms)

$$\downarrow = 0 \rightarrow 1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow 5 \rightarrow 6 \rightarrow 7 \leftarrow \cdots \rightarrow n = \top$$
Paths as functors

**Definition**

*Directed category* is a category $C$ with wide subcategories $C_0 \subseteq C \supseteq C_1$.

Morphisms of $C_0$ are *forth-morphisms*, morphisms of $C_1$, *back-morphisms*.

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- Category $□$: $d_A^0$ are forth-morphisms, $d_B^1$ are back-morphisms.
- The category of cells $\text{Cell}(X)$ is directed: a morphism $(x, U) \xrightarrow{\varphi} (y, V)$ is a forth/back-morphism if $\varphi \in □(U, V)$ is such.
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- *Linear categories* ($\rightarrow$ are forth-morphisms, $\leftarrow$ are back-morphisms)

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\downarrow = 0 \rightarrow 1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \leftarrow \cdots \rightarrow n = \top
\]
Labels of paths

**Definition**

A path on HDA $X$ is a directed functor $\alpha : L \to \text{Cell}(X)$ from a linear category $L$.

**Definition**

The label of a path $\alpha : L \to \text{Cell}(X)$ is a simple HDA

$$\lambda(\alpha) = \text{colim} \left( L \xrightarrow{\alpha} \text{Cell}(X) \xrightarrow{\text{ev}} \square \xrightarrow{\text{Yoneda}} \square \text{Set} \right) \in \square \text{Set}$$

with $\lambda(\alpha)_{\perp} = \alpha(\perp)$, $\lambda(\alpha)^{\top} = \alpha(\top)$.

**Remark**

Not every simple HDA may be a label of a path.
Labels of paths

**Definition**

A *path* on HDA $X$ is a directed functor $\alpha : \mathcal{L} \to \text{Cell}(X)$ from a linear category $\mathcal{L}$.

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The *label* of a path $\alpha : \mathcal{L} \to \text{Cell}(X)$ is a simple HDA

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with $\lambda(\alpha)_{\bot} = \alpha(\bot)$, $\lambda(\alpha)^{\top} = \alpha(\top)$.

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Labels of paths

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A path on HDA $X$ is a directed functor $\alpha : \mathcal{L} \rightarrow \text{Cell}(X)$ from a linear category $\mathcal{L}$.

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with $\lambda(\alpha)_{\bot} = \alpha(\bot)$, $\lambda(\alpha)^{\top} = \alpha(\top)$.

Remark
Not every simple HDA may be a label of a path.
Tracks

Definition

- A **track object** is a simple HDA having the form

\[ T = \text{colim} \left( \mathcal{L} \xrightarrow{\omega} \square \xrightarrow{\text{Yoneda}} \square \text{Set} \right), \]

\[ T_\bot = \omega(\bot \mathcal{L}), \quad T^\top = \omega(\top \mathcal{L}) \]

- A **track** in HDA \(X\) is a precubical map \(\alpha : T \to X\) from a track object \(T\).
- The **label** of a track \(\alpha\) is \(T\) itself.

Proposition

There is a natural label-preserving bijection between

- Tracks on \(X\).
- Equivalence classes of paths on \(X\).

Subsumption of paths corresponds to inclusion of tracks.
Tracks

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There is a natural label-preserving bijection between

- Tracks on $X$.
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Subsumption of paths corresponds to inclusion of tracks.
The category of track objects

Definition

The 2-category of tracks objects $\text{TrO}$:

- Objects are lo-sets ($\text{Ob}(\text{TrO}) = \text{Ob}(\boxtimes)$)
- Morphisms from $U$ to $V$ are (isomorphisms classes of) tracks objects $T$ such that $\text{ev}(T_\bot) = U$ and $\text{ev}(T^\top) = V$.
- Composition of $T \in \text{TrO}(U, V)$ and $T' \in \mathcal{T}(V, W)$ is

$$T \ast T' = \text{colim}\left( T \xleftarrow{T} \Box V \xrightarrow{\bot} T' \right).$$

- 2-morphisms $T \Rightarrow T'$ are HDA-maps (subsumptions).
- 2-composition is the composition of HDA-maps.
Definition

The track complex $\text{Tr}(X)$ of a precubical set $X$ is a 2-category:

- Objects are cells of $X$ ($\text{Ob}(\text{Tr}(X)) = \text{Ob}(\text{Cell}(X)))$
- Morphisms from $x$ to $y$ are tracks $\alpha : T \to X$ from $x$ to $y$ (ie, $\alpha(T_{\bot}) = x$, $\alpha(T_{\top}) = y$).
- Composition of $\alpha : T \to X$ and $\beta : T' \to X$ is the concatenation
  \[ \alpha \ast \beta : T \ast T' \to X. \]
- 2-morphisms $T \Rightarrow T'$ are HDA-maps (subsumptions).
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Proposition

The forgetful functor $\text{Tr}(X) \to \text{TrO}$ is a “presheaf” on $\text{TrO}$.
The category of tracks

**Definition**

The *track complex* \( \text{Tr}(X) \) of a precubical set \( X \) is a 2-category:

- Objects are cells of \( X \) \( (\text{Ob}(\text{Tr}(X)) = \text{Ob}(\text{Cell}(X))) \)
- Morphisms from \( x \) to \( y \) are tracks \( \alpha : T \to X \) from \( x \) to \( y \) (ie, \( \alpha(T_{\perp}) = x \), \( \alpha(T^\top) = y \)).
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- 2-composition is the composition of HDA-maps.

**Proposition**

The forgetful functor \( \text{Tr}(X) \to \text{TrO} \) is a “presheaf” on \( \text{TrO} \).
An ipomset is a tuple \((P, \lambda, <, \rightarrow, S, T)\), where

- \(P\) is a finite set,
- \(\lambda : P \rightarrow \Sigma\) is a labelling,
- \(<\) is a partial order on \(P\) (precedence order),
- \(\rightarrow\) is a partial order on \(P\) (event order),
- \(S \subseteq P\) is a subset of \(<\)-minimal elements of \(P\) (source interface),
- \(T \subseteq P\) is a subset of \(<\)-maximal elements of \(P\) (target interface).

Elements \(p, q \in P\) are parallel \((p \parallel q)\) if \(p \neq q\), \(p \nless q\) and \(q \nless p\).

We require that

- If \(p \parallel q\), then \(p \rightarrow q\) or \(q \rightarrow p\).

An ipomset is interval if \((P, \prec)\) is an interval order.
Ipomsets: an example

- colors = labels
- precedence
- event order
- S source interface
- T target interface
Serial composition of ipomsets

**Definition**

A *serial composition* of ipomsets $P$, $Q$ such that $T_P \simeq S_Q$ is

$$P \ast Q = (P \cup Q) / T_P \sim S_Q$$

- $r <_{P \ast Q} s$ if $r <_P s$ or $r <_Q s$ or $r \in P \setminus T_P$, $s \in Q \setminus S_Q$,
- $\rightarrow_{P \ast Q}$ is the transitive closure of $<_P \cup <_Q$,
- $S_{P \ast Q} = S_P$, $T_{P \ast Q} = T_Q$.
Subsumption of ipomsets

Definition

A subsumption of ipomsets \( (P \sqsubseteq Q) \) is a bijective map \( f : P \to Q \) that
- preserves labels \( (\lambda(f(p)) = \lambda(p)) \),
- reflects precedence \( (f(p) < f(p') \implies p < p') \),
- preserves essential event order \( (p \parallel p' \land p \dashrightarrow p' \implies f(p) \dashrightarrow f(p')) \),
- preserves interfaces \( (f(S_P) = S_Q, f(T_P) = T_Q) \).
Ipomset category

Definition

The 2-category of ipomsets **iPoms**:  
- Objects are lo-sets ($\text{Ob}(\text{iPoms}) = \text{Ob}(\Box)$)  
- Morphisms from $U$ to $V$ are (isomorphism classes of) ipomsets $P$ such that $S_P \simeq U$ and $T_P \simeq V$.  
- Composition of $P \in \text{iPoms}(U, V)$ and $Q \in \text{iPoms}(V, W)$ is  
  $$P \ast Q \in \text{iPoms}(U, W).$$  
- 2-morphisms $P \Rightarrow Q$ are subsumptions $f : P \sqsubseteq Q$.  
- 2-composition is the composition of subsumptions.

Let $\text{iiPoms} \subseteq \text{iPoms}$ be the full subcategory of interval ipomsets.

Theorem

There is a natural 2-equivalence $\text{iiPoms} \ni P \mapsto \Box^P \in \text{TrO}$.
The 2-category of ipomsets \textbf{iPoms}:

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- Composition of \(P \in \text{iPoms}(U, V)\) and \(Q \in \text{iPoms}(V, W)\) is \(P \star Q \in \text{iPoms}(U, W)\).
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**Theorem**

There is a natural 2-equivalence \(\text{iiPoms} \ni P \mapsto \Box^P \in \text{TrO}\).
Conclusions

- The track complex $\text{Tr}(X)$ admits a functor $\text{ev} : \text{Tr}(X) \rightarrow \text{iiPoms}$ that makes it a “presheaf” over $\text{iiPoms}$.

- The cube chain category $\mathcal{P}$ is a full subcategory of $\text{iiPoms}$ consisting of serial compositions of discrete ipomsets:

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- “Taming” theorem for track complexes: every track complex is determined uniquely by its values on $\mathcal{P} \subseteq \text{iiPoms}$ (it is a “sheaf”).
Appendix: languages of HDA and Kleene theorem

Definition

Let $X$ be a HDA.

- A track $\alpha : T \rightarrow X$ is **accepting** if $\alpha(T_\bot) \in X_\bot$ and $\alpha(T^\top) \in X^\top$.
- The **language** of $X$ is $\text{Lang}(X) = \{ P \in \text{iiPoms} \mid \text{HDA}(\Box_P, X) \neq \emptyset \}$.

Definition

A language $L \subseteq \text{iiPoms}$ is **regular** if $L = \text{Lang}(X)$ for a finite HDA $X$.

Kleene theorem for HDA

The family of regular languages is the concurrent Kleene algebra generated from singleton languages by unions, serial compositions, parallel compositions and Kleene plus.