Catoids as a Basis for Algebras of Programs

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I’ve worked on algebras of programs for some years (semirings, Kleene algebras, quantales, relation algebras, . . . )

developed variants such as modal/concurrent Kleene algebras and studied their models/properties

formalised algebra/models with proof assistants and built program verification tools based on them

formalising models felt like playing variations on a theme

but which theme?
Kleene's Quest
Kleene Algebra

regular expressions \( t ::= 0 \mid 1 \mid a \in \Sigma \mid t + t \mid tt \mid t^* \)

languages \( X \subseteq \Sigma^* \)

interpretation map \( L : \text{RegExp}_\Sigma \rightarrow \mathcal{P}\Sigma^* \) defines regular languages

task: axiomatise congruence \( s \approx t \iff L(s) = L(t) \)

find algebra \( \text{KA} \) with signature \((\oplus, \cdot, 0, 1, *)\)

prove \( \text{KA} \vdash s = t \iff L(s) = L(t) \)
Conway’s Visions
Kleene Algebra Axioms

\((K, +, \cdot, 0, 1, *)\)

\(x + (y + z) = (x + y) + z\)
\(x + y = y + x\)
\(x + 0 = x\)
\(x + x = x\)
\(x(yz) = (xy)z\)
\(x1 = x\)
\(1x = x\)
\(x(y + z) = xy + xz\)
\((x + y)z = xz + yz\)
\(x0 = 0\)
\(0x = 0\)
\(1 + xx^* = x^*\)
\(z + xy \leq y \Rightarrow x^*z \leq y\)
\(1 + x^*x = x^*\)
\(z + yx \leq y \Rightarrow zx^* \leq y\)

where \(x \leq y \iff x + y = y\)

and indeed \(KA \models s = t \iff L(s) = L(t)\)
Language Kleene Algebras

soundness proof constructs language $\mathcal{KA}$ over free monoid $\Sigma^*$

$$(\mathcal{P}\Sigma^*, \cup, \cdot, \emptyset, \{\varepsilon\}, *)$$

$AB = \{vw \mid v \in A, w \in B\}$

$A^* = \bigcup_{i \geq 0} A^i$ for $A^0 = 1$, $A^{i+1} = AA^i$

or just $\mathcal{KA} \mathcal{PM}$ for any monoid $M$

regular languages are then sub-KAs generated by $\Sigma$
weighted languages $f : \Sigma^* \to K$ form convolution KAs

$$(K^{\Sigma^*}, +, *, 0, id, *)$$

$$(f + g)(w) = f(w) + g(w)$$
$$0(w) = 0$$
$$(f * g)(w) = \sum_{w = u \cdot v} f(u) \cdot g(v)$$
$$id(w) = \delta_\varepsilon(w)$$
$$f^*(\varepsilon) = f(\varepsilon)^*$$
$$f^*(w) = f^*(\varepsilon) \cdot \sum_{w = u \cdot v \atop u \neq 1} f(u) \cdot f^*(v) \text{ for } x \neq 1$$

standard languages take weights in KA 2
Matrix Kleene Algebras

completeness proof formalises automata as $K$-valued matrices

\[ \begin{align*}
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a + b & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{align*} \]

KAs are closed under matrix formation: for $m, n : l \times l \rightarrow K$

\[
(m + n)_{ij} = f_{ij} + g_{ij} \quad (m \cdot n)_{ij} = \sum_k f_{ik} \cdot g_{kj} \quad 0_{ij} = 0 \quad id_{ij} = \delta_{ij}
\]

the star is somewhat tricky
\[ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad M^* = \begin{pmatrix} f^* & f^* bd^* \\ d^* cf^* & d^* + d^* cf^* bd^* \end{pmatrix} \text{ for } f = a + bd^* c \]

partition larger matrices into submatrices with squares along diagonal
binary relations are 2-valued matrices $X \times X \to 2$

and thus KAs

$$(\mathcal{P}(X \times X), \cup, ;, \emptyset, \Delta, *)$$

$$(RS)_{ab} \iff \exists c. R_{ac} \land S_{cb}$$

$$\Delta_{ab} \iff a = b$$

$$(R^*)_{ab} \iff \exists k \geq 0. (R^k)_{ab}$$

but we can’t write $$(RS)_{a,b} = \sum_c R_{a,c} \land R_{c,b}$$ — sums may be infinite!
Quantales

A quantale \((Q, \leq, \cdot, 1)\) consists of a complete lattice \((Q, \leq)\) and a monoid \((Q, \cdot, 1)\) such that

\[
x(\bigvee Y) = \bigvee \{xy \mid y \in Y\} \quad (\bigvee X)y = \{xy \mid x \in X\}
\]

Quantales are KAs with \(x^* = \bigvee_{i \geq 0} x^i\)

Examples: \((\mathbb{R}_+^\infty, \geq, \max, 0)\) (Lawvere quantale) or \(([0, 1], \leq, \cdot, 1)\)

We can now construct quantale \(Q^{X \times X}\) of \(Q\)-valued relations and convolution quantale \(Q^M\) for any monoid \(M\)
Path Quantales

automata are digraphs $s, t : E \rightarrow V$

paths are sequences $\pi : v_1 \rightarrow v_n = (v_1, e_1, v_2, \ldots, v_{n-1}, e_{n-1}, v_n)$

we compose them on matching ends:

we define $AB = \{\pi \pi' \mid \pi \in A, \pi' \in B, t(\pi) = s(\pi')\}$ and $id = \{(v) \mid V\}$

this yields path KA/quantale . . . and we can add weights to edges

more generally, $Q^C$ forms a category quantale for any (small) category $C$
categories. A category is a set $C$ of arrows with two functions 
$s, t : C \rightarrow C$, called “source” and target”, and a partially defined binary 
operation $\#$, called composition, all subject to the following axioms, for 
all $x, y,$ and $z$ in $C$:

The operation $x \# y$ is defined iff $sx = ty$ and then

\begin{align*}
s(x \# y) &= sy, & t(x \# y) &= tx; \\
x \# sx &= x, & tx \# x &= x; \\
(x \# y) \# z &= x \# (y \# z) \text{ if either side is defined};
\end{align*}

(1) \hspace{1cm} (2) \hspace{1cm} (3)

\begin{align*}
ssx &= sx = tsx; \\
ttx &= tx = stx.
\end{align*}

(4)

Then $x$ is an identity iff $x = sx$ or, equivalently, iff $x = tx$. 
Shuffle Quantales

shuffle $\Sigma^* \times \Sigma^* \rightarrow \mathcal{P}\Sigma^*$ is defined, for $a, b \in \Sigma$ and $v, w \in \Sigma^*$ as

$$v \| \varepsilon = \{v\} = \varepsilon \| v \quad (av)\| (bw) = a(v\| (bw)) \cup b((av)\| w)$$

we extend to $\| : \mathcal{P}\Sigma^* \times \mathcal{P}\Sigma^* \rightarrow \mathcal{P}\Sigma^*$

$$A\| B = \bigcup \{v\| w \mid v \in A, \ w \in B\}$$

we can construct shuffle $\mathcal{K}A$/quantale — and convolution algebras with

$$(f\| g)(w) = \sum_{w \in u\| v} f(u) \cdot g(v)$$

words under $\|$ don’t form category!
Catoids

A catoid \((X, \circ, s, t)\) equips set \(X\) with multioperation \(\circ : X \times X \to \mathcal{P}X\) and source/target maps \(s, t : X \to X\) that satisfy

\[
\bigcup \{x \circ v \mid v \in y \circ z\} = \bigcup \{u \circ z \mid u \in x \circ y\}
\]

\[
x \circ y \neq \emptyset \Rightarrow t(x) = s(y) \quad s(x) \circ x = \{x\} \quad x \circ t(x) = \{x\}
\]

If we extend to \(\circ : \mathcal{P}X \times \mathcal{P}X \to \mathcal{P}X\)

\[
A \circ B = \bigcup_{x \in A, y \in B} x \circ y,
\]

the first axiom becomes

\[
x \circ (y \circ z) = (x \circ y) \circ z
\]
a catoid morphism $f : X \to Y$ satisfies

$$f(x \circ_x y) \subseteq f(x) \circ_Y f(y) \quad f \circ s_X = s_Y \circ f \quad f \circ t_X = t_Y \circ f$$

it is bounded if $f(x) \in u \circ_Y v$ implies $x \in y \circ_X z$, $u = f(y)$, $v = f(z)$ for some $y, z \in X$

a catoid is functional if $x, x' \in y \circ z \Rightarrow x = x'$

and local if $t(x) = s(y) \Rightarrow x \circ y \neq \emptyset$

a single-set category is a local functional catoid

$X_s = \{x \mid s(x) = x\} = X_t$ determines objects of (small) category
all structures considered so far are catoids

relations are constructed from the pair groupoid on $X \times X$

shuffle languages form the shuffle catoid with $||$ total and $s(w) = \varepsilon = t(w)$ for all $w \in \Sigma^*$

there are many other interesting examples
Jónsson-Tarski Duality

in boolean algebras with operators

\( n \)-ary modalities in \( B \) are dual to \( n + 1 \)-ary relations in \( X \)

we view \( \cdot : \mathcal{P}X \times \mathcal{P}X \rightarrow \mathcal{P}X \) as binary modality
and \( \odot : X \times X \rightarrow \mathcal{P}X \) as ternary relation
for powerset structures this duality is almost trivial

\[ x \in y \odot z \iff \{x\} \subseteq \{y\} \cdot \{z\} \]

atoms in powerset structure \( Q \) define relational structure \( Q_+ \)

relational structure \( X \) yields powerset structure \( X^+ \) with

\[ AB = \bigcup \{ y \odot z \mid y \in A, \ z \in B \} \]

Jónsson/Tarski have shown that \( (Q_+)^+ \cong Q \) and \( (X^+)^+ \cong X \)

in fact, the categories of powerset and relational structures are dually equivalent

Jónsson-Tarski duality yields modal correspondences translating identities between \( X \) and \( Q \)
more generally we can prove 2-out-of-3 correspondences in convolution algebras

\[(f \ast g)(x) = \bigvee_{x \in y \otimes z} f(y) \cdot g(z)\]

\[\text{id}_{X_s}(x) = \begin{cases} 1 & \text{if } x \in X_s \\ 0 & \text{otherwise} \end{cases}\]

\[(\bigvee F)(x) = \bigvee \{f(x) \mid f \in F\}\]

\[0(x) = 0\]
Basic Correspondences

theorem:
1. if $X$ is catoid and $Q$ quantale, then $Q^X$ is quantale
2. if $Q^X$ is quantale and $Q$ supported quantale, then $X$ is catoid
3. if $Q^X$ is quantale and $X$ supported catoid, then $Q$ is quantale

“supported” means structures have enough elements for a construction (e.g., $0 \neq 1$ or some composable elements)

we get KA if $X$ is finitely decomposable: $\{(y, z) \mid x \in y \circ z\}$ finite f.a. $x$
\[ (f \ast (g \ast h))(x) = \bigvee_{x \in u \circ y} f(u) \cdot \left( \bigvee_{y \in v \circ w} g(v) \cdot h(w) \right) \]
\[ = \bigvee_{x \in u \circ (v \circ w)} f(u) \cdot (g(v) \cdot h(w)) \]
\[ = \bigvee_{x \in (u \circ v) \circ w} (f(u) \cdot g(v)) \cdot h(w) \]
\[ = \bigvee_{x \in y \circ w} \left( \bigvee_{y \in u \circ w} f(u) \cdot g(v) \right) \cdot h(w) \]
\[ = ((f \ast g) \ast h)(x) \]

\[ x \in u \circ (v \circ w) \iff (\delta_u \ast (\delta_v \ast \delta_w))(x) = 1 \]
\[ \iff ((\delta_u \ast \delta_v) \ast \delta_w)(x) = 1 \]
\[ \iff x \in (u \circ v) \circ w \]
Catoids and Modal Quantales

A domain quantale equips a quantale with \( \text{dom} : Q \rightarrow Q \) satisfying

\[
\text{dom}(x)x = x \quad \text{dom}(x + y) = \text{dom}(x) + \text{dom}(y) \\
\text{dom}(0) = 0 \quad \text{dom}(x) \leq 1 \quad \text{dom}(x\text{dom}(y)) = \text{dom}(xy)
\]

A codomain quantale \((Q, \text{cod})\) is a domain quantale \((Q^{\text{op}}, \text{dom})\)

A modal quantale is a domain and codomain quantale such that

\[
\text{dom} \circ \text{cod} = \text{cod} \quad \text{cod} \circ \text{dom} = \text{dom}
\]

In relation quantale \( \text{dom}(R)_{aa} \leftrightarrow \exists b. R_{ab} \) and \( \text{cod}(R)_{aa} \leftrightarrow \exists b. R_{ba} \)
domain elements \( Q_{dom} = \{ x \mid \text{dom}(x) = x \} \) form distributive lattice and boolean algebra if \( Q \) is boolean

we define modal operators for \( x \in Q \) and \( p \in Q_{dom} \)

\[
\langle x \rangle p = \text{dom}(xp) \quad \langle x \rangle p = \text{cod}(px)
\]

\[
\langle x \rangle p = \bigvee \{ q \mid \langle x \rangle q \leq p \} \quad [x|p = \bigvee \{ q \mid \langle x \rangle q \leq p \}
\]

this yields dynamic logics/algebras, predicate transformer algebras, boolean algebras with operators

in relation quantale

\[
(|R\rangle P)_{aa} \iff \exists b. \ R_{ab} \land P_{bb} \quad (|R\rangle P)_{aa} \iff \forall b. \ R_{ab} \Rightarrow P_{bb}
\]
Modal Quantales and Program Correctness
we use relations over program store to verify programs

\[ x \in Q \text{ as programs, } + \text{ as nondeterministic choice, } \cdot \text{ as sequential composition, } (\neg)^* \text{ as finite iteration} \]

in boolean quantale, for \( x \in Q, p \in Q_{dom} \)

\[
\text{if } p \text{ then } x \text{ else } y = px + \overline{p}y \quad \text{while } p \text{ do } x = (px)^* \overline{p}
\]

\(|x|p\) calculates wlp of program \( x \) from postcondition \( q \)

program \( x \) is (partially) correct if \( p \leq |x|q \)
Local Catoids and Modal Quantales

Theorem: we have 2-out-of-3 correspondences

Local catoid $X$ \quad Modal quantale $Q$

$\text{dom}(f) = \bigvee_{x \in X} \text{dom}(f(x))\delta_{s(x)}$

$\text{cod}(f) = \bigvee_{x \in X} \text{cod}(f(x))\delta_{t(x)}$
for $Q = 2$

1. if $X$ is local catoid, then $(\mathcal{PX}, \subseteq, \odot, X_s, \mathcal{Ps}, \mathcal{Pt})$ is modal quantale

2. if $\mathcal{PX}$ is modal quantale, then $X$ is local catoid

we derive $s(xs(y)) = s(xy)$ and $s \circ r = r$ in $X$ and lift to $\text{dom}$-axioms in $\mathcal{PX}$ (other $\text{dom}$-axioms don’t depend on identities in $X$)

$$\text{dom}(A \odot \text{dom}(B)) = \bigcup \{s(x \odot s(y)) \mid x \in A, y \in B, t(x) = s(s(y))\}$$

$$= \bigcup \{s(x \odot y) \mid x \in A, y \in B, t(x) = s(y)\}$$

$$= \text{dom}(A \odot B)$$

we can recover the catoid axioms from the atom structure in $\mathcal{PX}$

$$s(x \odot s(y)) = \text{dom}({x} \odot \text{dom}({y}))$$

$$= \text{dom}({x} \odot {y})$$

$$= s(x \odot y)$$
Models of Modal Quantaless

if you want to build a modal convolution quantale, look for a catoid

the lifting is then generic

locality axiom $\text{dom}(\text{dom}(y)) = \text{dom}(xy)$ is precisely the composition pattern of categories

absorption axiom $\text{dom}(x)x = x$ corresponds to left unit axiom of catoids

every category gives rise to modal quantale
word concatenation interacts with shuffle via interchange law

\[(v\parallel v') \cdot (w\parallel w') \subseteq (v \cdot w)\parallel(v' \cdot w')\]

we can lift it to \((A\parallel A') \cdot (B\parallel B') \subseteq (A \cdot B)\parallel(A' \parallel B')\)

an interchange catoid \((X, \circ_0, s_0, t_0, \circ_1, s_1, t_1)\) consists of two catoids that interact via \((x \circ_1 x') \circ_0 (y \circ_1 y') \subseteq (x \circ_0 y) \circ_1 (x' \circ_0 y')\)

an interchange quantale \((Q, \leq, \cdot_0, 1_0, \cdot_1, 1_1)\) consists of two quantales that interact via \((x \cdot_1 x') \cdot_0 (y \cdot_0 y') \leq (x \cdot_0 y) \cdot_1 (x' \cdot_0 y')\)
theorem: we have 2-out-of-3 correspondences

\[ Q^X \]

int. quantale \( Q \)

int. quantale \( Q \)

int. catoid \( X \)

it suffices to consider correspondences for interchange
Interleaving Concurrency

Correspondences yield (weighted) shuffle languages with interchange laws

$\parallel$ is commutative, there’s a general 2-out-of-3 for commutativity

The shuffle catoid has one single unit $\varepsilon$

In interchange catoids/quantales with one single unit there’s a collapse à la Eckmann-Hilton, small interchange laws are derivable

$x \cdot_0 y \leq x \cdot_1 y \quad x \cdot_0 (y \cdot_1 z) \leq (x \cdot_0 y) \cdot_1 z \quad (x \cdot_1 y) \cdot_0 z \leq x \cdot_1 (y \cdot_0 z)$

And commutative variants in catoid/quantale
Non-Interleaving Concurrency

pomsets are a standard model of non-interleaving concurrency

\[
a \rightarrow b \rightarrow c \rightarrow a
\]

they are composed using serial/parallel composition

\[
a \cdot f = a \]

\[
a \parallel c \rightarrow d = c \rightarrow d
\]

operations \( \cdot \) and \( \parallel \) share the empty pomset \( \varepsilon \) as their unit
pomset $Q$ subsumes pomset $P$, $P \preceq Q$, if there exists pomset morphism $Q \to P$ that is bijective on points

$\preceq$ is partial order on pomsets

we get interchange catoid $(\text{Pom}(\Sigma), \cdot, \sqsubseteq, \varepsilon)$ with $x \sqsubseteq y = \{ z \mid z \preceq x \parallel y \}$

it lifts to a powerset interchange quantale, the downclosed languages form subquantale

this generalises to convolution quantales (under technical restrictions)
Models of Conurrent Quantales

construction of interchange/concurrent quantales motivated this approach

correspondences for interchange catoids/quantales simplified discussions about potential models
Single-Set \( n \)-Categories

Similarly a 2-category can be considered to be a single set \( X \) considered as the set of 2-cells (e.g., of natural transformations). Then the previous 1-cells (the arrows) and the 0-cells (the objects) are just regarded as special "degenerate" 2-cells. On the set \( X \) of 2-cells there are then two category structures, the "horizontal" structure \((\#_0, s_0, t_0)\) and the "vertical" structure \((\#_1, s_1, t_1)\). Each satisfies the axioms above for a category structure and in addition

(i) Every identity for the 0-structure is an identity for the 1-structure;
(ii) The two category structures commute with each other.

Here, the condition (ii) means, of course, that

\[
 s_0 s_1 = s_1 s_0 , \quad s_0 t_1 = t_1 s_0 , \quad t_0 s_1 = s_1 t_0 , \quad t_0 t_1 = t_1 t_0 \quad (7)
\]

and that, for \( \alpha, \beta = 0, 1 \) or \( 1, 0 \), and for all \( x, y, u, \) and \( v \)

\[
 (x \#_\alpha y) \#_\beta (u \#_\alpha v) \#_\alpha (y \#_\beta v) , \quad (8)
\]

\[
 t_\alpha (x \#_\beta y) = (t_\alpha x) \#_\beta (t_\alpha y) ,
\]

\[
 s_\alpha (x \#_\beta y) = (s_\alpha x) \#_\beta (s_\alpha y) ,
\]

whenever both sides are defined.

Since \( s_0 x \) and \( t_0 x \) are identities for the 0-structure, they are also identities for the 1-structure by condition (i) above. Hence,

\[
 s_1 s_0 = s_0 , \quad t_1 s_0 = s_0 , \quad s_1 t_0 = t_0 , \quad t_1 t_0 = t_0 . \quad (9)
\]
With this preparation, we can now readily define a 3-category or more generally an $n$-category for any natural number $n$. The latter is a set $X$ with $n$ different category structures $(\#_i, s_i, t_i)$, for $i = 0, \ldots, n - 1$, which commute with each other and are such that an identity for structure $i$ is also an identity for structures $j$ whenever $j > i$. Put differently, each pair $\#_i$ and $\#_j$ for $j > i$ constitute a 2-category. This readily leads to a definition of the useful notion of an $\omega$-category: $i = 0, 1, 2, \ldots$. 
**$n$-Catoids**

A (globular) $n$-catoid $(X, \circ_i, s_i, t_i)_{0 \leq i < n}$ consists of $n$-catoids $(X, \circ_i, s_i, t_i)$ that interact, for all $0 \leq i < j < n$, via

\[
\begin{align*}
    s_i \circ s_j &= s_j \circ s_i & s_i \circ t_j &= t_j \circ s_i & t_i \circ s_j &= s_j \circ t_i & t_i \circ t_j &= t_j \circ t_i \\
    (w \circ_j x) \circ_i (y \circ_j z) &\subseteq (w \circ_i y) \circ_j (x \circ_i z) \\
    s_j(x \circ_i y) &= s_j(x) \circ_i s_j(y) & t_j(x \circ_i y) &= t_j(x) \circ_i t_j(y) \\
    s_i(x \circ_j y) &\subseteq s_i(x) \circ_j s_i(y) & t_i(x \circ_j y) &\subseteq t_i(x) \circ_j t_i(y) \\
    s_j \circ s_i &= s_i & s_j \circ t_i &= t_i & t_j \circ s_i &= s_i & t_j \circ t_i &= t_i
\end{align*}
\]

A single-set $n$-category is a local functional $n$-catoid.
\[ s_1(x \odot_0 y) = s_1(x) \odot_0 s_1(y) \text{ and } t_1(x \odot_0 y) = t_1(x) \odot_0 t_1(y) \]
\[ s_0(x \odot_1 y) \subseteq s_0(x) \odot_1 s_0(y) \text{ and } t_0(x \odot_1 y) \subseteq t_0(x) \odot_1 t_0(y) \]
\((w \odot_1 x) \odot_0 (y \odot_1 z) \subseteq (w \odot_0 y) \odot_1 (x \odot_0 z)\)
Reduced $n$-Catoid Axioms

the following axioms are irredundant and subsume the previous ones

$$(w \odot_j x) \odot_i (y \odot_j z) \subseteq (w \odot_i y) \odot_j (x \odot_i z)$$

$$s_j(x \odot_i y) = s_j(x) \odot_i s_j(y) \quad t_j(x \odot_i y) = t_j(x) \odot_i t_j(y)$$

this streamlines correspondence proofs
a *(globular)* $n$-quantale $(Q, \leq, \cdot_i, 1_i, \text{dom}_i, \text{cod}_i)_{0 \leq i < n}$ consists of $n$ modal quantales $(Q, \leq, \cdot_i, 1_i, \text{dom}_i, \text{cod}_i)$ that interact, for all $0 \leq i < j < n$, via

\[
(w \cdot_j x) \cdot_i (y \cdot_j z) \leq (w \cdot_i y) \cdot_j (x \cdot_i z)
\]

\[
\text{dom}_j(x \cdot_i y) = \text{dom}_j(x) \cdot_i \text{dom}_j(y) \quad \text{cod}_j(x \cdot_i y) = \text{cod}_j(x) \cdot_i \text{cod}_j(y)
\]

\[
\text{dom}_i(x \cdot_j y) \leq \text{dom}_i(x) \cdot_j \text{dom}_i(y) \quad \text{cod}_i(x \cdot_j y) \leq \text{cod}_i(x) \cdot_j \text{cod}_i(y)
\]

\[
\text{dom}_j(\text{dom}_i(x)) = \text{dom}_i(x)
\]
$n$-Catoids and $n$-Quantales

Theorem: we have 2-out-of-3 correspondences

$n$-quantale $Q^X$

Local $n$-catoid $X$ $n$-quantale $Q$

Relative to previous correspondences it remains to check the globular ones
Higher Rewriting

(modal) Kleene algebras allow proving facts from abstract rewriting (Church-Rosser theorem, Newman’s lemma, …)

\(n\)-Kleene algebras allow proving analogous fact from higher rewriting (using free \((n, p)\)-categories constructed using polygraphs/computads)

our correspondences justify the axioms of \(n\)-Kleene algebra firmly in terms of (free) \(n\)-categories

we can justify those of \((n, p)\)-Kleene algebras by integrating (single-set) groupoids

Jónnson-Tarski knew about correspondence between groupoids and relation algebras

single-set approach makes approach easily accessible to proof assistants and even SMT-solvers
Conclusion

catoids simplify the construction of models for algebras of programs
they often tell where axioms in algebras of programs come from
they provide a particular way of dealing with partiality
(in algebra or category theory)
they might allow formalizing higher categories using automated theorem provers/SMT solvers . . . but this is speculation
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Papers

C. Calk, P. Malbos, G. Struth, D. Pous. Catoids and Globular Convolution Quantales (manuscript)


