Capturing Changes in Combinatorial Dynamical Systems via Persistent Homology

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Overview & Outline

• Motivating Example and Persistence
• Combinatorial Dynamical Systems & Conley Index
• Conley Index Persistence
• Conley-Morse Graph Persistence
Motivating Example: Hopf Bifurcation

\[ x' = -y + x(\lambda - x^2 - y^2) \]
\[ y' = x + y(\lambda - x^2 - y^2) \]
Motivating Example: Hopf Bifurcation

\[ \lambda \ll 0 \]
Motivating Example: Hopf Bifurcation

\[ \lambda = 0 \]
Motivating Example: Hopf Bifurcation

\[ \lambda = 1 \]
Motivating Example: Hopf Bifurcation

$\lambda = 2.5$
Motivating Example: Hopf Bifurcation

\[
\lambda = 5
\]
Motivating Example: Hopf Bifurcation

$\lambda = 10$
Motivating Example: Hopf Bifurcation

\[ \lambda = 15 \]
Motivating Example: Hopf Bifurcation

\[
\lambda = 17.5
\]
Motivating Example: Hopf Bifurcation

Note: attractor from $\lambda = -\infty$ to $\lambda = 16$

Repeller from $\lambda = 0$ to $\lambda = \infty$

Can we use computational topology to automatically detect these features?
Motivating Example: Hopf Bifurcation

Note: attractor from $\lambda = -\infty$ to $\lambda = 16$

Repeller from $\lambda = 0$ to $\lambda = \infty$

Can we use computational topology to automatically detect these features?

Yes, by using persistence
Persistent Homology

Summarizes changing homology of a filtration [ELZ00]

\[ K_1 \subseteq K_2 \subseteq \ldots \subseteq K_n = K \]
Persistence Example

\[ K = \]

\[ K_0 \subset K_1 \subset K_2 \subset K_3 \subset K_4 \subset K_5 \subset K_6 \]
Zigzag Persistence

\[ K_1 \subseteq K_2 \supseteq K_3 \subseteq \ldots \supseteq K_n \]
“Level Set” Persistence

[CDM09] [DW07]
Level Set Barcode
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Multivectors

Let $K$ denote a simplicial complex and $\leq$ denote the face relation.

**Definition:** A **multivector** $V$ is a convex subset of $K$ with respect to $\leq$.

**Definition:** A **multivector field** $\mathcal{V}$ is a partition of $K$ into multivectors.

\[ \mathcal{V} = \{\{b\}, \{c, bc\}, \{a, ab, ac, abc\}\} \]
Multivector Fields
Let $\sigma \in K$. Then $\text{cl}(\sigma) = \{\tau \in K \mid \tau \leq \sigma\}$.

$[\sigma]_{\mathcal{V}}$ denotes the vector in $\mathcal{V}$ containing $\sigma$.

Dynamics generator $F_{\mathcal{V}} : K \rightarrow K$ defined as:

$$F_{\mathcal{V}}(\sigma) = [\sigma]_{\mathcal{V}} \cup \text{cl}(\sigma)$$
Multivector Fields as a Dynamical System

\[ F_\mathcal{V} (\sigma) = [\sigma]_\mathcal{V} \cup \text{cl} (\sigma) \]
Paths

**Definition:** A path is a finite sequence of simplices $\sigma_1, \sigma_2, \ldots, \sigma_n$ such that $\sigma_{i+1} \in F_{\gamma}(\sigma_i)$.
Solutions

**Definition:** A solution is a bi-infinite sequence of simplices...

\[ \ldots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots \] such that \( \sigma_{i+1} \in F_{\mathcal{V}}(\sigma_i) \)


**Definition:** A solution is a bi-infinite sequence of simplices
\[ \ldots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots \] such that \( \sigma_{i+1} \in F_{\mathcal{V}}(\sigma_i) \)

But as \( F_{\mathcal{V}}(\sigma) = [\sigma]_{\mathcal{V}} \cup \text{cl}(\sigma) \), every simplex gives a solution!
Critical Multivectors

Definition: Let $A \subseteq K$. The mouth of $A$ is defined as
$$\text{mo}(A) := \text{cl}(A) \setminus A$$

Definition: A multivector $[\sigma]_\mathcal{V}$ is critical if there exists a $k$ such that
$$H_k(\text{cl}([\sigma]_\mathcal{V}), \text{mo}([\sigma]_\mathcal{V}))$$ is nontrivial.
Critical Multivectors

Critical:

Regular:
**Essential Solutions**

**Definition:** Let \( \cdots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \cdots \) denote a solution. If for each \( \sigma_i \) where \([\sigma_i]_V\) is noncritical, there exists a \( j > i \) and \( j' < i \) where \([\sigma_i]_V \neq [\sigma_j]_V\) and \([\sigma_i]_V \neq [\sigma_{j'}]_V\), then \( \cdots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \cdots \) is an essential solution.
Definition: Let $A \subseteq K$. The invariant part of $A$, denoted $\text{Inv}(A)$, is the set of simplices in $A$ which appear in an essential solution in $A$.

If $A = \text{Inv}(A)$, then $A$ is an invariant set.
Isolated Invariant Sets

**Definition:** Let \( A \subseteq N \subseteq K \), where \( A \) is an invariant set and \( N \) is closed (i.e. \( N = \text{cl}(N) \)). If every path in \( N \) with endpoints in \( A \) is contained in \( A \), then \( A \) is an isolated invariant set, and \( N \) is an isolating neighborhood for \( A \).
**Index Pairs**

**Definition:** Let $A$ be an isolated invariant set, and $E$ and $P$ closed sets such that $E \subseteq P$. If:

1. $F_V(E) \cap P \subseteq E$,
2. $F_V(P \setminus E) \subseteq P$, and
3. $A = \text{Inv}(P \setminus E)$

Then $(P, E)$ is an index pair for $A$. 
Conley Index

**Theorem [LKMW2019]:** Let $\overline{A}$ denote an isolated invariant set. The pair $(\text{cl}(A), \text{mo}(A))$ is an index pair for $\overline{A}$. 
Index Pairs are Not Unique
Conley Index

**Definition:** Let \((P, E)\) be an index pair for \(A\). Then the \(k\)-dimensional Conley Index is given by \(H_k(P, E)\).

**Theorem [LKMW 2019]:** The \(k\)-dimensional Conley Index for \(A\) is well defined.
Conley Indices

\[ H_2(R \cup Y, R) = \mathbb{Z}_2 \]
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Conley Index Persistence

First attempt: for each $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$, compute an isolated invariant set, $A_1, A_2, \ldots, A_n$ and corresponding index pairs.

$$(\text{cl}(A_1), \text{mo}(A_1)), (\text{cl}(A_2), \text{mo}(A_2)), \ldots, (\text{cl}(A_n), \text{mo}(A_n))$$

Gives a relative zigzag filtration:

$$\ldots \subseteq (\text{cl}(A_i), \text{mo}(A_i)) \supseteq (\text{cl}(A_i) \cap \text{cl}(A_{i+1}), \text{mo}(A_i) \cap \text{mo}(A_{i+1})) \subseteq (\text{cl}(A_{i+1}), \text{mo}(A_{i+1})) \supseteq \ldots$$

Problem: $(\text{cl}(A_i) \cap \text{cl}(A_{i+1}), \text{mo}(A_i) \cap \text{mo}(A_{i+1}))$ generally not an index pair.
Intersection Example
Index Pairs in an Isolating Neighborhood

Let \( E \subset P \subset N \) for closed \( P, E, N \), and \( A \subset N \). If:

1. \( F_N(P) \cap N \subset P \),
2. \( F_N(E) \cap N \subset E \),
3. \( F_N(P \setminus E) \subset N \), and
4. \( A = \text{Inv}(P \setminus E) \)

then \( (P, E) \) is an index pair in \( N \).
Push Forward

Let $A \subseteq K$ denote an arbitrary set in some closed $N$. Then the push forward of $A$ in $N$ is $A$ together with all simplices in $N$ which are reachable from paths originating in $A$ and contained in $N$. 

![Diagram showing the push forward concept]
Finding Index Pairs in N

Theorem [DMS20]: Let $A$ denote an isolated invariant set, and let $N$ denote an isolating neighborhood for $A$. The pair $(\text{pf(cl}(A)), \text{pf(mo}(A)))$ is an index pair in $N$ for $A$. 
Index Pairs in an Isolating Neighborhood

**Theorem (DMS20):** Index Pairs in $\mathcal{N}$ are index pairs.

**Definition:** Let $\mathcal{V}_1$, $\mathcal{V}_2$ denote multivector fields over $K$. The intersection of multivector fields is given by

$$\mathcal{V}_1 \cap \mathcal{V}_2 = \{ V_1 \cap V_2 \mid V_1 \in \mathcal{V}_1, \ V_2 \in \mathcal{V}_2 \}$$

**Theorem (DMS20):** Let $(P_1, E_1)$, $(P_2, E_2)$ denote index pairs in $\mathcal{N}$ under $\mathcal{V}_1, \mathcal{V}_2$. The pair $(P_1 \cap P_2, E_1 \cap E_2)$ is an index pair in $\mathcal{N}$ under $\mathcal{V}_1 \cap \mathcal{V}_2$ for $\text{Inv}((P_1 \cap P_2) \setminus (E_1 \cap E_2))$
Intersection Example

All simplices in N, Yellow union
Red is P, and Red is E

Dimension: 2
Conley Index Persistence: New Strategy

Fix $N$, and for each $V_1, V_2, \ldots, V_n$, compute the maximal invariant set in $N$, denoted $A_1, A_2, \ldots, A_n$, and corresponding index pairs.

$$(\text{cl}(A_1), \text{mo}(A_1)), (\text{cl}(A_2), \text{mo}(A_2)), \ldots, (\text{cl}(A_n), \text{mo}(A_n))$$

Gives a relative zigzag filtration:

$$(\text{pf}_N(\text{cl}A_i), \text{pf}_N(\text{mo}A_i)) \supseteq (\text{pf}_N(\text{cl}A_i) \cap \text{pf}_N(\text{cl}A_{i+1}), \text{pf}_N(\text{mo}A_i) \cap \text{pf}_N(\text{mo}A_{i+1})) \subseteq (\text{pf}_N(\text{cl}A_{i+1}), \text{pf}_N(\text{mo}A_{i+1}))$$
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Conley Index Persistence

Dimension: 0

Dimension: 2
Problem: Noise Resilience

All simplices are in N,
Yellow union Red = P,
and Red = E
Solution: Make E Smaller
Conley Index Persistence

**Proposition [DMS20]:** Let \((P, E)\) denote an index pair for \(A\) in \(\mathcal{N}\). If \(V \subseteq E\) is a regular multivector such that \(E' := E \setminus V\) is closed, then \((P, E')\) is an index pair in \(\mathcal{N}\) for \(A\).
Conley Index Persistence

**Proposition [DMS20]:** Let \((P, E)\) denote an index pair for \(\mathcal{A}\) in \(\mathcal{N}\). If \(V \subseteq E\) is a regular multivector such that \(E' := E \setminus V\) is closed, then \((P, E')\) is an index pair in \(\mathcal{N}\) for \(\mathcal{A}\).
Multivector Removal Strategy

Remove as many multivectors as possible, up to a fixed distance away from the isolated invariant set.
Multivector Removal Strategy

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Multivector Removal Strategy

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Algorithm

\textbf{MakeNoiseResilient}( P, E, A, \delta ):

\textbf{while} there exists a regular multivector \( V \subset E \) such that \( E \setminus V \) is closed and \( d(V, A) \leq \delta \):

\( E \leftarrow E \setminus V \)
Multivector Removal Strategy

**Theorem [DMS'20]:** This algorithm outputs index pairs
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Motivating Example
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Dimension: 0
Original Example

Dimension: 0

Dimension: 1
Conley-Morse Graph

A Morse decomposition graph equipped with information about the Conley Index
Conley-Morse Graph
Conley-Morse Graph Persistence

Two types of filtrations:
1. Graph Filtrations
2. Conley-Morse Filtrations
Conley-Morse Filtrations

1. Assume every isolated invariant set is isolated by the same $N$.
2. Fix index pair for each Morse set.
3. Find all “maximal” sequences of index pairs across Conley-Morse graphs with nontrivial intersection.
Conley-Morse Filtrations
Conley-Morse Barcodes
Conley-Morse Graph Barcodes

- Dimension: 0
  - Periodic attractor

- Dimension: 1

- Dimension: 2
  - Repelling fixed point AND periodic repeller

- Dimension: 1

- Dimension: 0
  - Attracting fixed point

- Dimension: 0
  - Graph connected component
Conclusion & Future Work

- In this presentation: devised method to capture changes in combinatorial dynamical systems. But...

- Stability?

- Inference?
References


