Directed Homotopy Type Theory

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Outline

1. Homotopy theory via type theory
2. Desiderata for directed homotopy type theory
3. Directed homotopy type theory
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1. Homotopy theory via type theory
2. Desiderata for directed homotopy type theory
3. Directed homotopy type theory
MORAL: Not only can types represent spaces, but homotopy type theory is the right setting in which to do homotopy theory. This is because everything we can say or do respects equality / identity / homotopy (terms of $\text{Id}_A(a,b)$).
Types $\rightarrow$ $\infty$-groupoids

\[
\begin{align*}
A & : \text{Type} & a,b & : A \\
\text{Id}_A(a,b) & : \text{Type} \\
\end{align*}
\]

(\text{Id-form})

\[
\begin{align*}
A & : \text{Type} & a & : A \\
\text{ra} & : \text{Id}_A(a,a) \\
\end{align*}
\]

(\text{Id-intro})

→ We get a tower

\[
\text{Id}_A(a,b), \text{Id}_{\text{Id}_A(a,b)}(p,q), \text{Id}_{\text{Id}_{\text{Id}_A(a,b)}}(p,q)(\alpha,\beta) \ldots
\]

with canonical terms

\[
\text{ra} : \text{Id}_A(a,a), \text{rp} : \text{Id}_{\text{Id}_A(a,b)}(p,p), \text{rd} : \text{Id}_{\text{Id}_{\text{Id}_A(a,b)}}(p,q)(\alpha,\alpha) \ldots
\]
**Pitstop in B**

- Get one type $B$ with canonical terms $0, 1$

- $x : B \vdash D(x) : Type$
  - $z : D(0)$
  - $v : D(1)$

- $x : B \vdash \text{ind}_{z,v}(x) : D(x)$
  - $z \in \text{ind}_{z,v}(0) : D(0)$
  - $v \in \text{ind}_{z,v}(1) : D(1)$

**Pitstop in N**

- Get one type $N$ with canonical terms $0, s0, ss0, ...$

- $x : N \vdash D(x) : Type$
  - $z : D(0)$
  - $y : D(x) \vdash \sigma(y) : D(sx)$

- $x : N \vdash \text{ind}_{z,r}(x) : D(x)$
  - $z \in \text{ind}_{z,r}(0) : D(0)$
  - $r \in \text{ind}_{z,r}(s) : D(s)$

- $x : N \vdash \sigma(\text{ind}_{z,r}(x)) \equiv \text{ind}_{z,r}(sx) : D(sx)$

**Behavior determined at canonical terms**
Types $\rightarrow \infty$-groupoids

$\begin{align*}
\forall x, y : A, \ z : \text{Id}_A(x, y) \vdash \text{D}(z) : \text{Type} \\
x : A \vdash \text{p}(x) : \text{D}(\text{r}_x) \\
\forall x, y : A, \ z : \text{Id}_A(x, y) \vdash \text{indp}(z) : \text{D}(z) \\
x : A \vdash \text{p}(x) = \text{indp}(\text{p}(x)) : \text{D}(\text{r}_x)
\end{align*}$

(Pf. Every identity has an inverse.)

(Pf. Every reflexivity identity has an inverse (itself).)

(Pf. There is a composition of any $p : \text{Id}_A(x, y)$ and $q : \text{Id}_A(y, z)$.)

(Pf. There is a composition of any $p : \text{Id}_A(x, y)$ and $r_q$ (namely $p$).)

Cor. Every type has the structure of an $\infty$-groupoid.
**Fibrations & transport**

**Prop (transport).** Given any \( x : A \vdash D(x) : \text{Type} \) and \( p : \text{Id}_A(x, y) \), there is a function \( P_x : D(x) \to D(y) \).

In fact, \( P_x \) is an equivalence and there is an identity \( \text{Id}_{D(x)}(d, P_x(d)) \) for any \( d : D(x) \).

**Pr.** If \( p \) is \( r_x \), let \( P_x \) be the identity.

**Cf.**

![Diagram](attachment:diagram.png)

\[ D \cong D \]
\[ D \times \text{Path}(A) \to A \]

Looks like a Hurewicz fibration in \( \text{Top} \).
Fibrations & transport

(Let \( \mathcal{C} \) be a finitely complete category.)

Thin \((N)\): Identity types can be interpreted in any weak factorization system in \( \mathcal{C} \) that

1. is generated by a path object and
2. is symmetric.

1. There is a path object

\[
X \xrightarrow{r} \text{Path}(X) \xrightarrow{\varepsilon \times \varepsilon} X \times X
\]

functional in \( X \in \mathcal{C} \), and taking the mapping path factorization produces the wfs.

\( \text{Path}(X) \) plays the role of \( \Sigma_{x,y : X} \text{Id}_X(x,y) \) and \( r \) plays the role of reflexivity.

2. There is an involution

\[
X \xrightarrow{r} \text{Path}(X) \xrightarrow{\varepsilon \times \varepsilon} X \times X
\]

satisfying some properties...
Fibrations & transport: Examples

- In any category, take $\text{Path}(X)$ to be $X$ or $X \times X$. 
  (Dependent types correspond to all morphisms or isomorphisms.)

- In $\text{Cat}$, take $\text{Path}(X)$ to be $X^{\sim}$. 
  (Dependent types correspond to isofibrations.)

- In $\text{Top}$, take $\text{Path}(X)$ to be $TX$ (roughly $X^I$). 
  (Dependent types correspond to Hurewicz fibrations.)

- In $\text{Kan}$ complexes, take $\text{Path}(X)$ to be $X^\Delta^I$ 
  (Dependent types correspond to Kan fibrations.)

- which can be generalized to any Cisinski model category. 
  (Dependent types correspond to fibrations.)
Fibrations & transport: Examples

- In any category, take \( \text{Path}(X) \) to be \( X \) or \( X \times X \).
  (Dependent types correspond to all morphisms or isomorphisms.)

- In \( \text{Set} \), take \( \text{Path}(X) \) to be \( X^X \).
  (Dependent types correspond to isofibrations.)

- In \( \text{Top} \), take \( \text{Path}(X) \) to be \( TX \) (roughly \( X^X \)).
  (Dependent types correspond to Hurewicz fibrations.)

- In Kan complexes, take \( \text{Path}(X) \) to be \( X \times \Delta[3] \).
  (Dependent types correspond to Kan fibrations.)

- which can be generalized to any Cisinski model category.
  (Dependent types correspond to fibrations.)
Univalence

- The univalence axiom characterizes identities in Type:
  \[ \text{Id}_{\text{Type}}(A, B) \simeq (A \simeq B) \]

- We can use it to characterize identities in other types:
  \[ \text{Prop.} \quad \text{Id}_{A \to B}(f, g) \simeq \prod_{x : A} \text{Id}_{A}(fx, gx) \]
  \[ \text{Prop.} \quad \text{Id}_{\text{Set}}(S, T) \simeq (S \simeq T) \]
  \[ \text{Prop.} \quad \text{Id}_{\text{Group}}(G, H) \simeq (G \simeq H) \quad (\text{Logquand - Danielsson}) \]
  \[ \text{Prop.} \quad \text{Id}_{\text{Cat}}(\mathcal{C}, \mathcal{D}) \simeq (\mathcal{C} \simeq \mathcal{D}) \quad (\text{Ahrens-Kapulkin-Shulman}) \]

Univalence principles

Thm. (Ahrens-N-Shulman-Tsementzis) This pattern generalizes to encompass "any" algebraic structure.
HoTT

- Transport + univalence \rightarrow Everything we can say or do respects these notions of sameness.

Advantages of HoTT

1. Proofs can be verified by a computer.

2. It is the ‘theory’ of homotopy theory (in the sense of model theory), and so results are not just valid in \textit{Set}, but in all models.

3. We can study algebraic structures with homotopical tools. In particular, everything is invariant under the appropriate notion of equivalence.
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Directed transport

- Everything we can say or do should respect diverted identities, in a diverted way.

- **Des:** Given any \( x : A \to D(x) : \text{Type} \) and \( p : \text{hom}_A(x, y) \), there is a (noninvertible) \( p_* : D(x) \to D(y) \).

- Cf. in \( \text{Cat} \):

  - \( \Theta \xrightarrow{\theta} A \)
  - \( \Theta \xrightarrow{\theta} A \)

  These are (the retract closure of) the Grothendieck opfibrations.
Example from rewriting:

Consider \( n : N \to \text{Vect}(n) : \text{Type} \)

where \( N \) is a directed homotopy type with terms like \( 3+1, 4, \ldots \)

and directed paths like \( \text{plus\_one}_3 : \text{hom}_n(3+1, 4) \),

we need to be able to transport \( \text{Vect}(3+1) \to \text{Vect}(4) \)

along \( \text{plus\_one}_3 \).
Example

Reachability:
\[ x : F + R(x) := \text{from}((i_A, i_B), x) : \text{Type} \]
\[ \rightarrow R \text{ can be transported along paths} \]

Safety:
\[ x : F + S(x) := \text{from}((x, (f_A, f_B)) : \text{Type} \]
\[ \rightarrow S \text{ should be transported backwards along paths} \]
Example

Reachability:

\[ x : F^+ R(x) := \text{hom}((i_A, i_B), x) : \text{Type} \]
\[ \rightarrow R \text{ can be transported along paths} \]

Safety:

\[ x : F^- S(x) := \text{hom}(x, (f_A, f_B)) : \text{Type} \]
\[ \rightarrow S \text{ should be transported backwards along paths} \]
Example

Then

\[ X : F \to \text{hom}_F(x, x) : \text{Type} \]

can only be transported along invertible diverted paths.

And undirected homotopy should be expressible, as in

\[ x : F, y : F, f : \text{hom}_F(x, y), g : \text{hom}_F(x, y) \]

\[ + \text{id}_{\text{hom}_F(x, y)} (f, g) \]
Example

Then

\[ X : F \rightarrow \text{hom}(x,x) : \text{Type} \]

can only be transported along invertible diverted paths.

And undirected homotopy should be expressible, as in

\[ X : F, y : F, f : \text{hom}_F(x,y), g : \text{hom}_F(x,y) \]

\[ \rightarrow \text{Id}_{\text{hom}_F(x,y)}(f,g) \]
More notions of transport

Directed spaces:

- invertible directed paths ≤ directed paths ≤ undirected paths

Categories:

- isomorphisms ≤ morphisms ≤ localization
Even more notions of transport

Consider \( x: A, y: A \vdash \text{hom}_A(x, y) \).

We should be able to transport \( \text{hom}_A(x, y) \) along paths in the \( x \) or \( y \) variable without disturbing the other.

In \( \text{Cat} \), we have a two-sided fibration.

\[
\begin{array}{c}
\uparrow \\
? \\
\downarrow \\
? \\
\end{array}
\xRightarrow{	ext{?}}
\begin{array}{c}
\downarrow \\
\text{hom}_A(x, y) \\
\downarrow \\
\text{dom}/
\end{array}
\xrightarrow{	ext{cod}}
\begin{array}{c}
A \\
A
\end{array}
\]

For longer contexts, we have more complicated diagrams...
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Results

Syntax
(type theory)
(N)

Comprehension
(Ahrens-N-vdWeide)

Semantics
(generalized weak factorization systems)
(vdBerg - McCloskey - N)
Syntax: first approximation

- Only models - +

- Uses operators $\text{op}$ and $\text{core}$ on types with $\varepsilon: T^{\text{core}} \to T$, $\varepsilon^\circ: T^{\text{core}} \to T^{\text{op}}$

\[
\begin{align*}
A : \text{Type} & \quad a : A^{\text{op}} & \quad b : A \\
\text{hom}_A (a,b) : \text{Type} & \quad (\text{hom-form}) & \quad (\text{hom-intro})
\end{align*}
\]

- There are left and right versions of the elimination and computation rules that allow for
  - forward transport along homomorphisms in $A$
  - backward transport along homomorphisms in $A^{\text{op}}$
  - both along homomorphisms in $A^{\text{core}}$

- Model in Yeat.
Syntax: first approximation

- Only models

\[
\begin{array}{c}
A : \text{Type} \\
a : A^\text{op} \\
b : A
\end{array}
\]
\[
\text{hom}_A(a,b) : \text{Type}
\]

Problems

- Uses operators op and core on types with \(e : T^\text{core} \rightarrow T\), \(\circ : T^\text{core} \rightarrow T^\text{op}\)

\[
\begin{array}{c}
A : \text{Type} \\
a : A^\text{core}
\end{array}
\]
\[
\text{hom}_A(a,a) : \text{Type}
\]

- There are left and right versions of the elimination and computation rules that allow for
  - forward transport along homomorphisms in \(A\)
  - backward transport along homomorphisms in \(A^\text{op}\)
  - both along homomorphisms in \(A^\text{core}\)

- Model in Yeat.
Syntax: second approximation  (to appear)

- Use them rules from above.

- Change the notion of dependency so that

\[ x : A + D(x) : \text{Type} \]
\[ x : A + D(x) : \text{Type} \]
\[ x : A + D(x) : \text{Type} \]
\[ x : A + D(x) : \text{Type} \]

produce the four kinds of transport.

- This walls 1d off from how to prevent them from collapsing into each other.

- Models in any category \( C \) with the following kind of weak factorization system.
Semantics: generalized weak factorization systems

- Recall: Models of the identity type are generated by a symmetric functorial path object:
  \[
  X \xrightarrow{\pi} \Sigma \Pi x,y \Pi x,y \xrightarrow{\Delta_0 \times \varepsilon} X \times X
  \]

- In intended models of the hom-type we also have a functorial path object:
  \[
  X \xrightarrow{\pi} \Sigma \Pi x,y \Pi x,y \xrightarrow{\varepsilon_0} X
  \]

- We generalize the notion of weak factorization system to encompass various shapes.
- Two-sided fibrations in \( \text{Cat} \) are captured by the theory.
Thank you!