Proving Unsolvability of Set Agreement Task with Epistemic μ-Calculus

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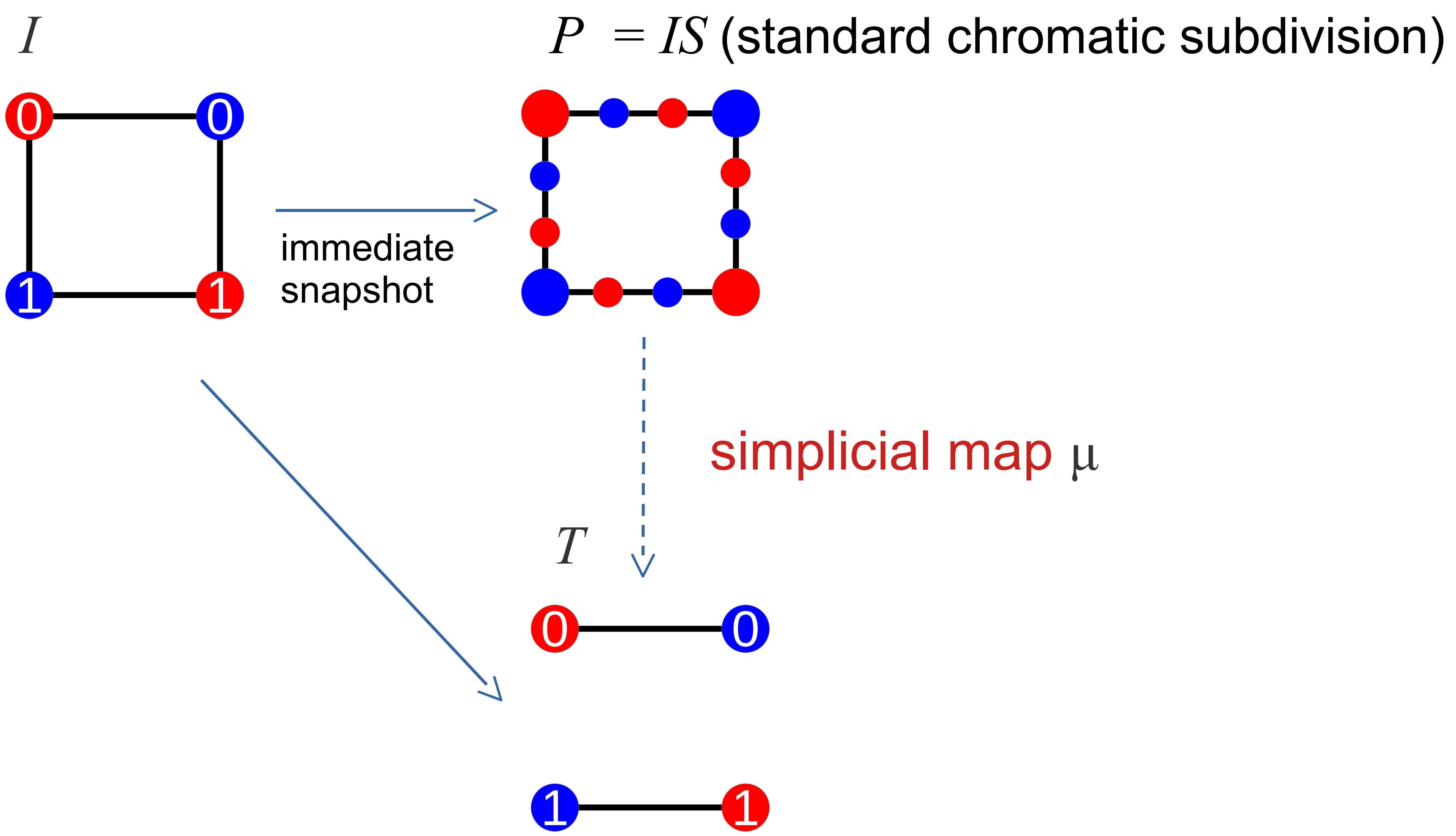
GETCO 2022 — May 30-Jun 3, 2022, Paris

Two methods for task unsolvability

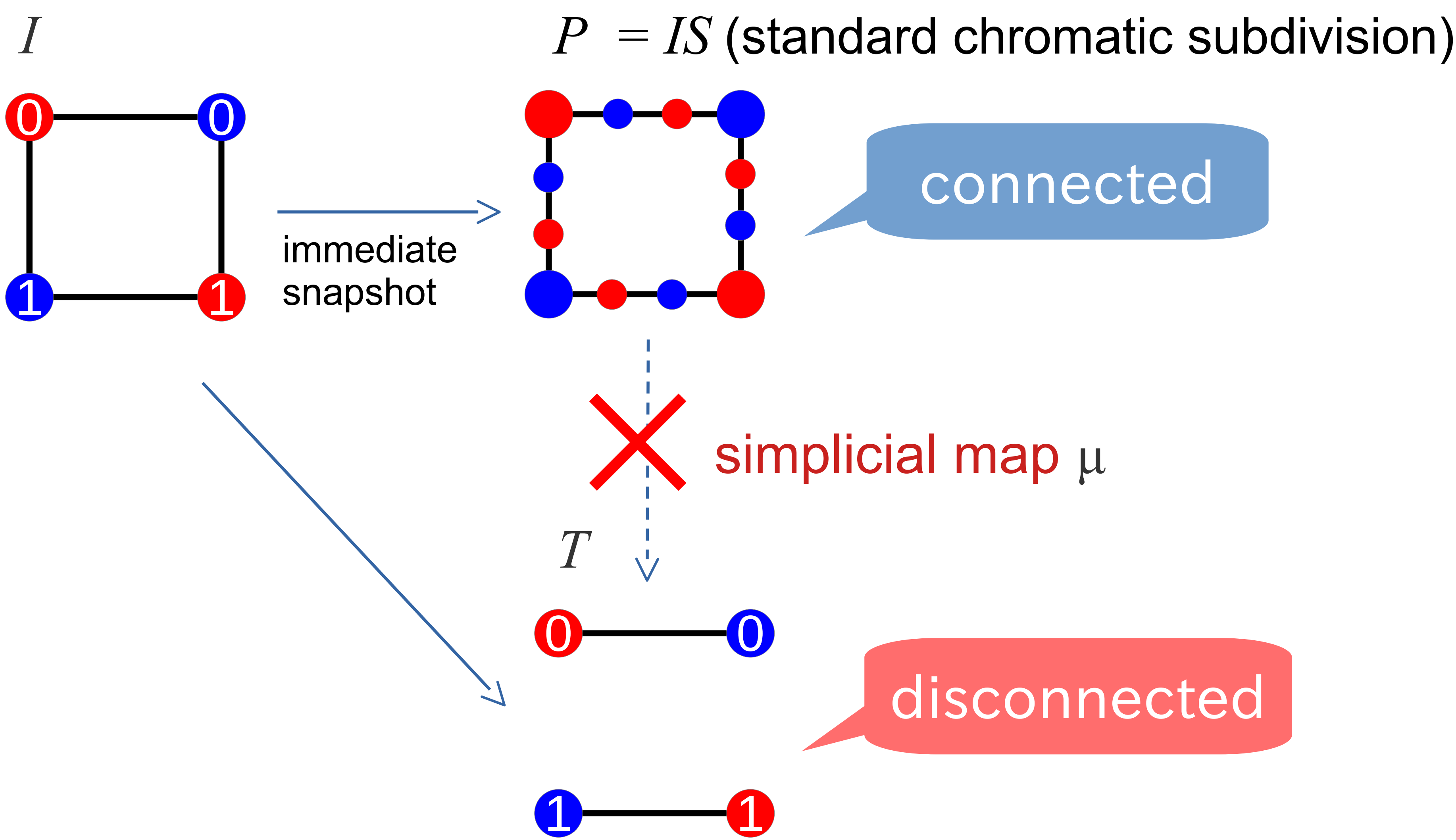
- **Topological method**
  - Model: Simplicial complexes.
  - Strategy: Find a breach in topological invariant.
  - Method: Tools from combinatorial topology.

- **Logical method** [Goubault-Ledent-Rajsbaum2021]
  - Model: (Simplicial) Kripke models.
  - Strategy: Find a logic formula (logical obstruction) that is inconsistent between the models.
  - Method: Epistemic logic reasoning

Unsolvability of 1-set agreement (Topology)



Unsolvability of 1-set agreement (Topology)



- General case argues *higher dimensional connectivity*, resorting to tools from combinatorial topology (Sperner's lemma).

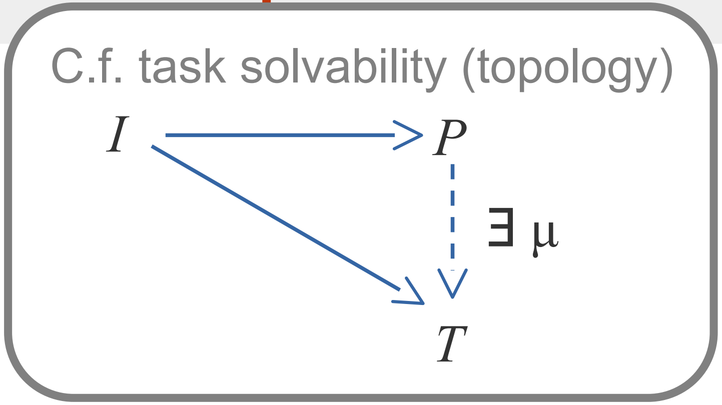
Two methods for task unsolvability

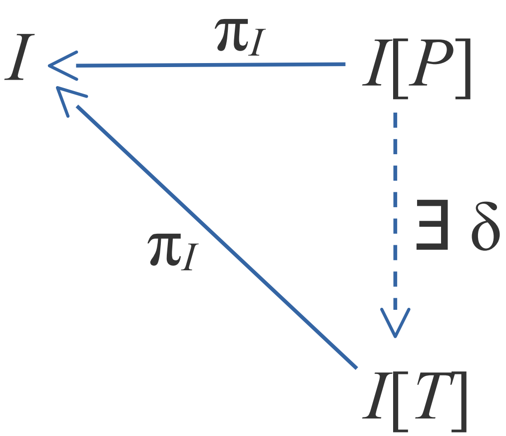
- **Topological method**
  - Model: Simplicial complexes.
  - Strategy: Find a breach in topological invariant.
  - Method: Tools from combinatorial topology.

- **Logical method** [Goubault-Ledent-Rajsbaum2021]
  - Model: (Simplicial) Kripke models.
  - Strategy: Find a logic formula (called logical obstruction) that is inconsistent between the models.
  - Method: Epistemic logic reasoning

Task solvability in simplicial Kripke model

[Goubault-Ledent-Rajsbaum2021]





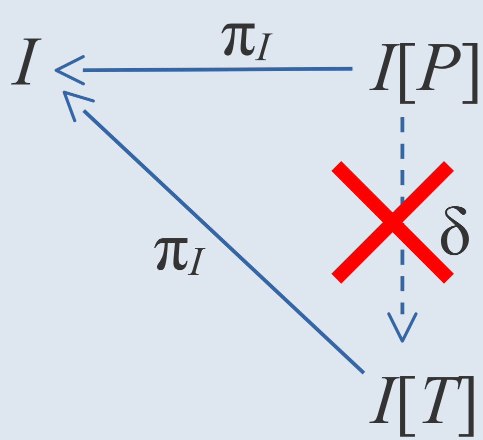
A simplicial map that preserves the set of true atomic propositions.

- Every map $f : I \rightarrow O$ over simplicial complexes induces a product update model $I[O]$, a binary relation encoding of f.
- Every product update model $I\,[O]$ is a simplicial complex, which induces a simplicial Kripke model for epistemic reasoning.

Logical obstruction to task solvability

logical obstruction

- If there exists a *positive* epistemic formula φ and facet $X \in I[P]$ such that, for any δ: $I[P] \to I[T]$,



$I[P], X \nvDash \varphi$

but

$I[T], \delta(X) \vDash \varphi$,

then the task is not solvable (i.e., there is no δ).

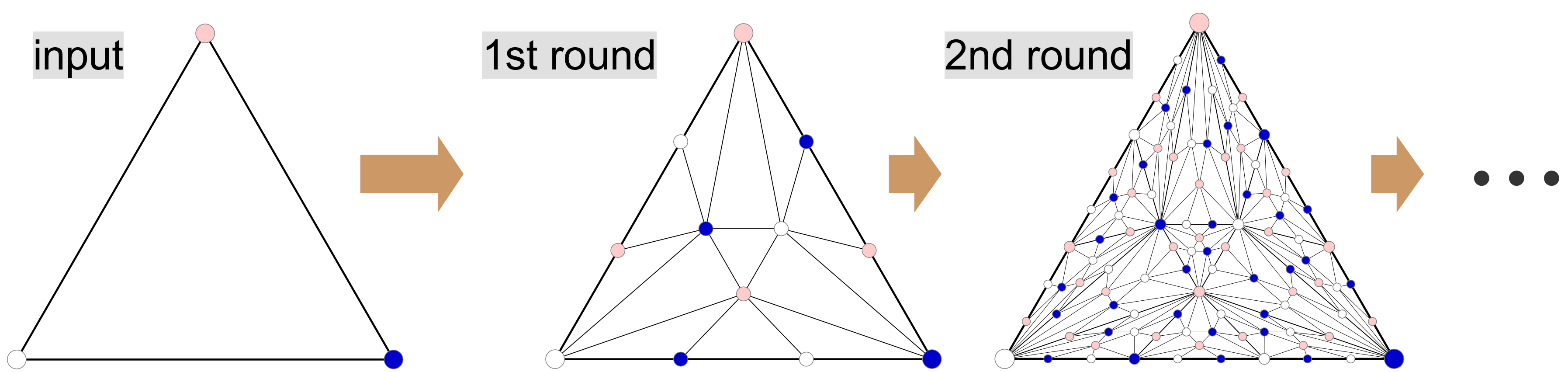
Pros and cons of logical method

- ☺ Just find a logical obstruction φ to show unsolvability.
- ☺ φ accounts for the reason of unsolvability in the formal language of epistemic logic.

- ☹ Limited instances of logical obstructions known to date.
  - 1-set agreement & approximate agreement [Goubault-Ledent-Rajsbaum2021]
  - k-set agreement (k>1) [Nishida2020] (w/ distributed knowledge), later generalized for adversary model [Yagi-Nishimura2020]
    - This works only for *single-round protocol*.
  - General logical obstruction in an extended simplicial model [vanDitramsch-Goubault-Lazic-Ledent-Rajsbaum2021]
    - The general formula involves no epistemic contents and provides no hints for the reason of unsolvability.

Goal of this talk

- Find an epistemic formula Φ such that
  - Φ is a logical obstruction to k-set agreement.
  - Φ contains epistemic contents that account for the reason of unsolvability.
  - Φ works for multi-round protocols (where processes are allowed to communicate arbitrarily many times).



Our strategy

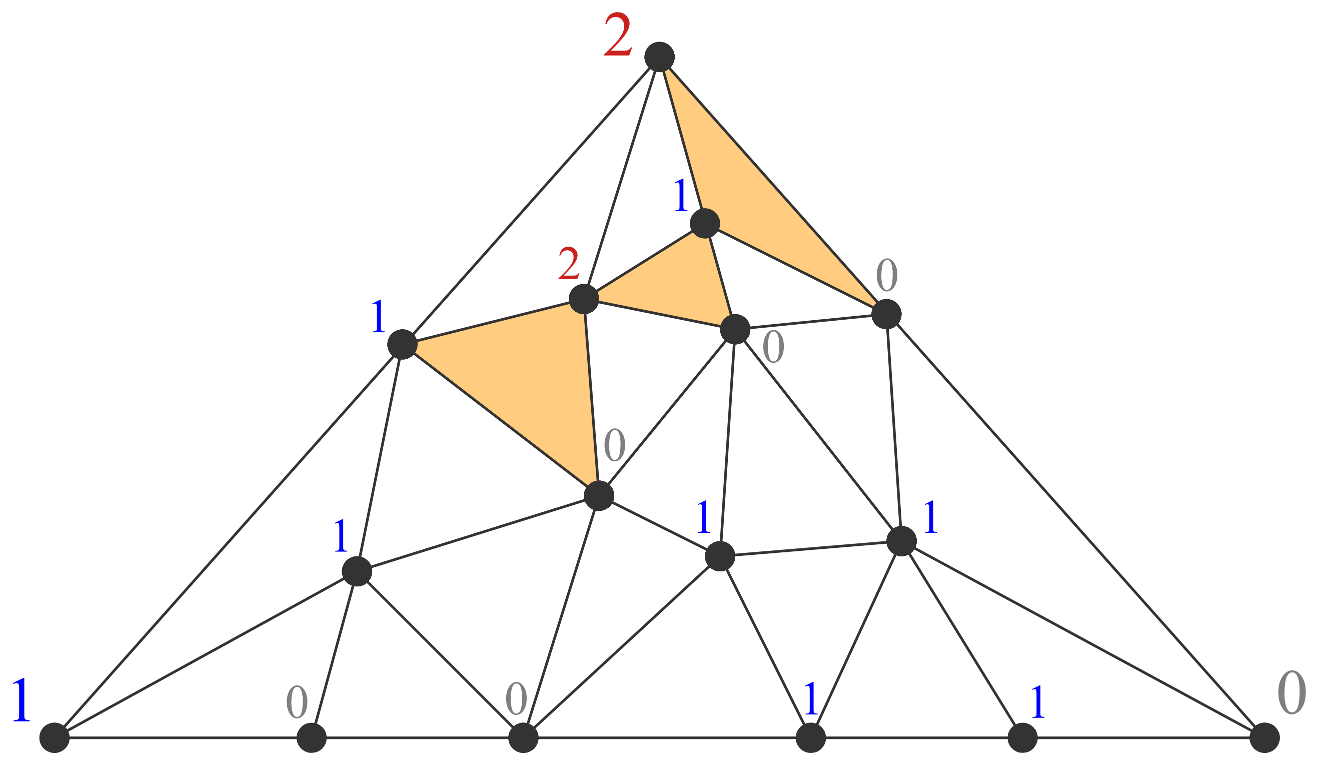
- To find inconsistency between simplicial Kripke models,
  - Rework on “Sperner’s lemma” to rephrase it as a statement on higher dimensional connectivity.

- To express the inconsistency in the language of logic,
  - Use epistemic μ-calculus, which extends epistemic logic with:
    - Distributed knowledge, a modal operator for higher-dimensional connectivity, and
    - Propositional greatest fixpoint for transitive closure.

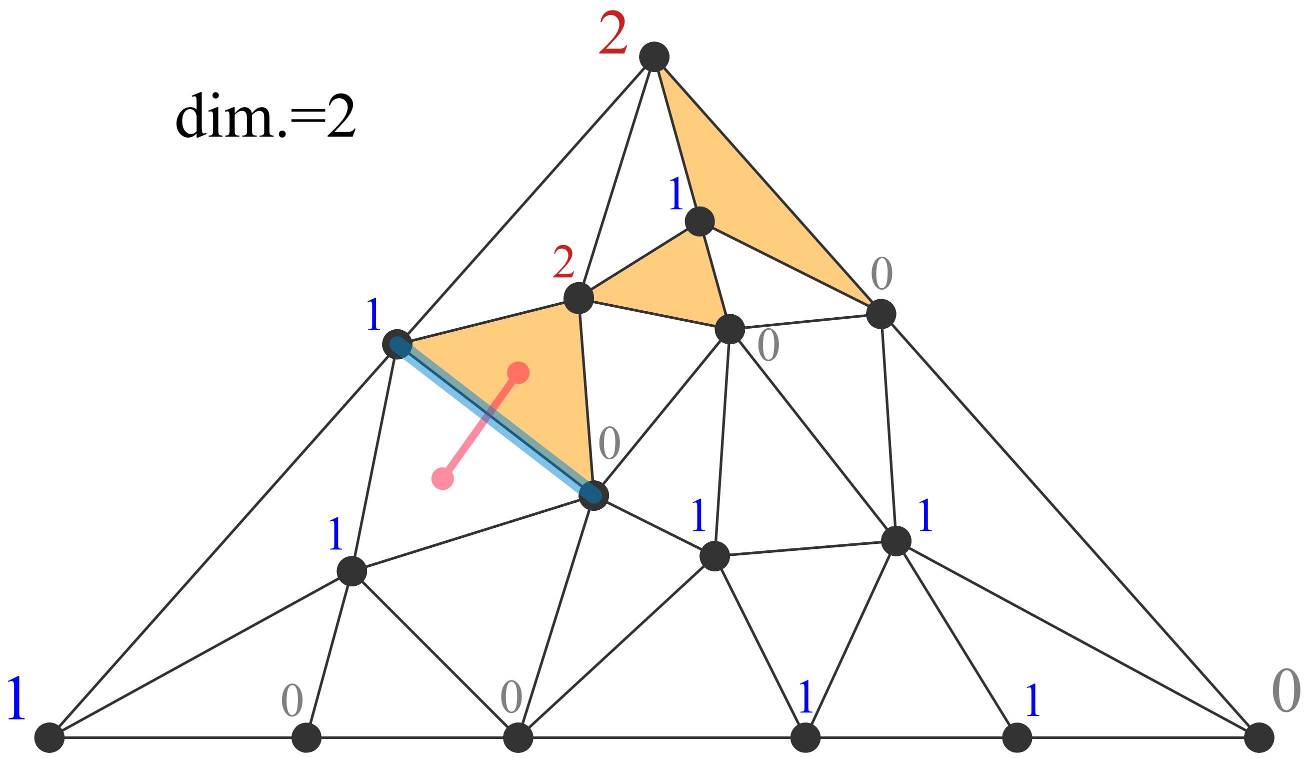
Sperner’s lemma as connectivity

Sperner's lemma

Sperner's lemma. Any subdivision of a simplex with Sperner coloring has odd number of fully-colored facets (maximal simplexes).



Proof of Sperner's lemma (induction on dim.)



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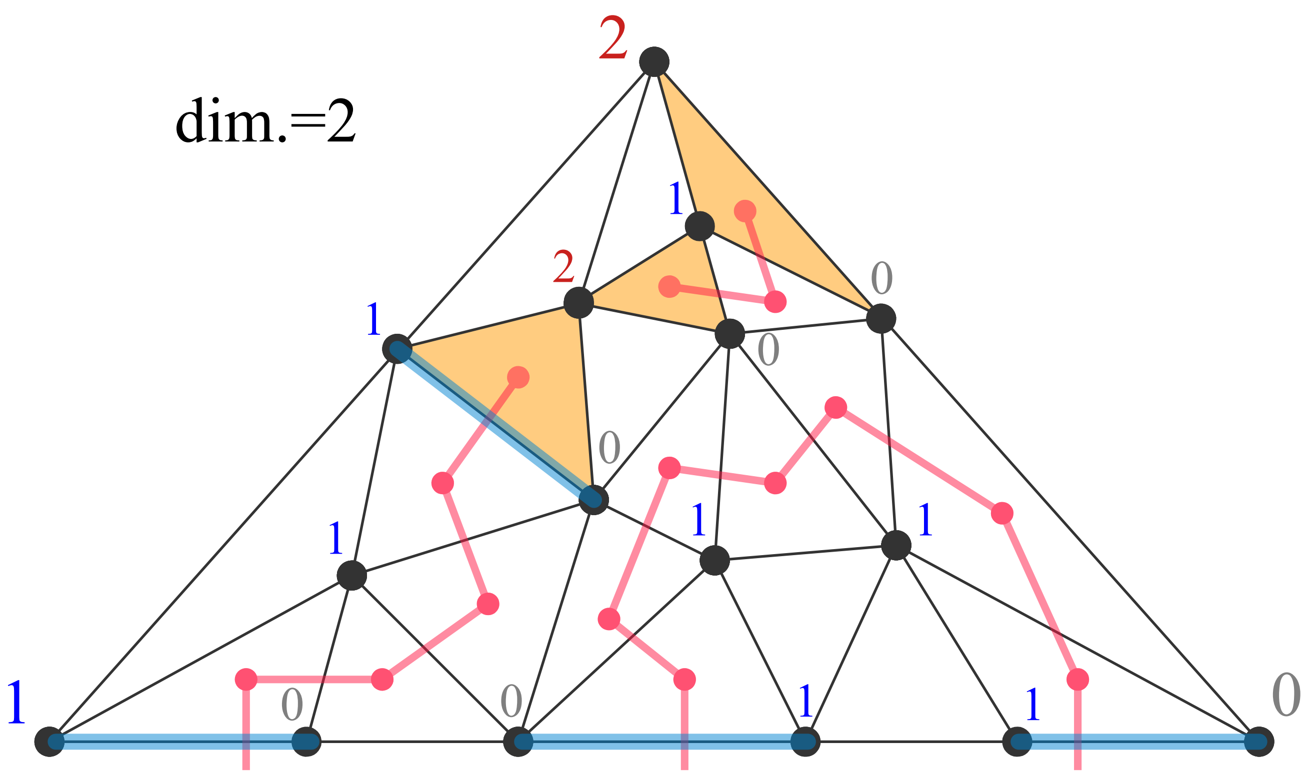

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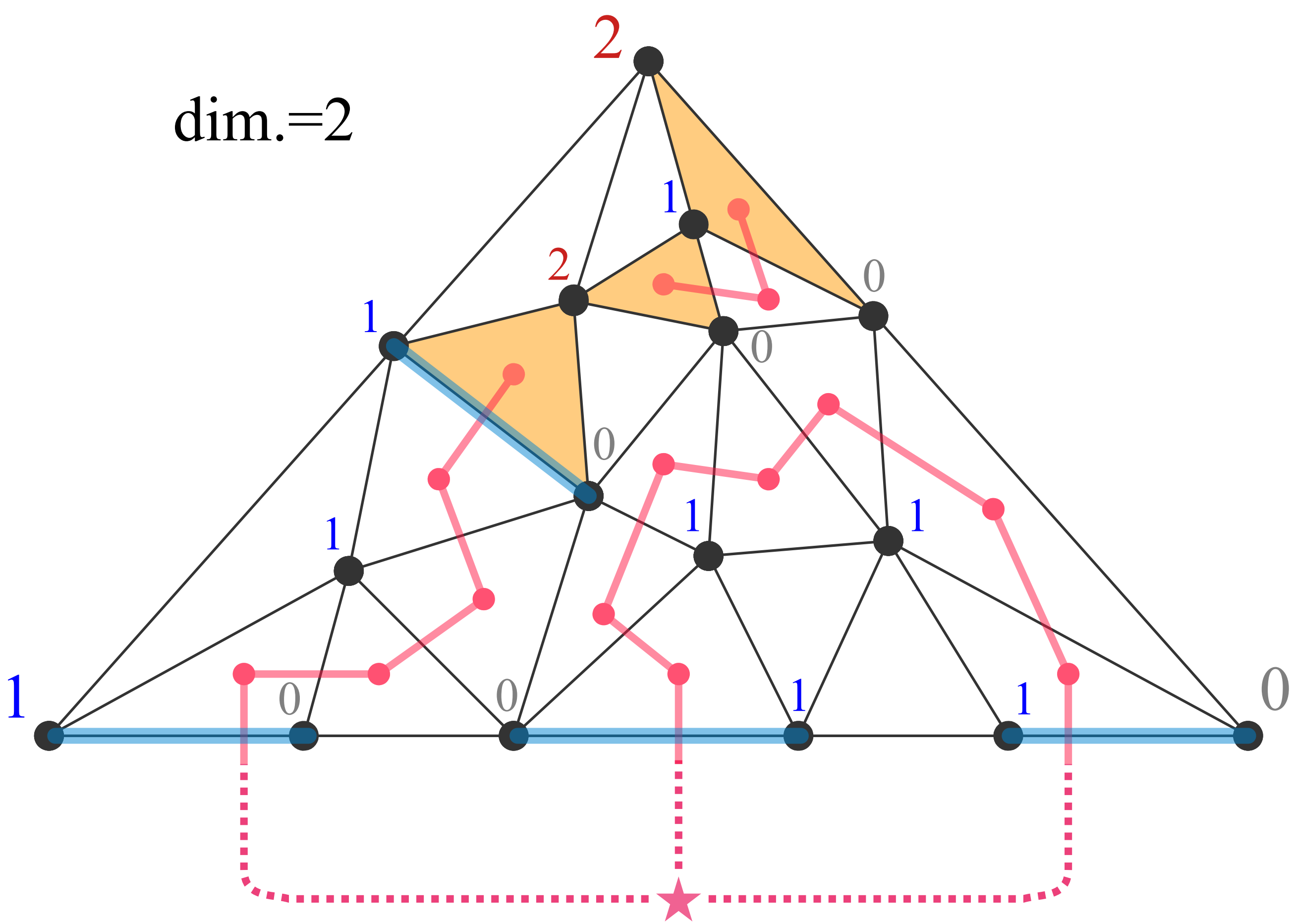

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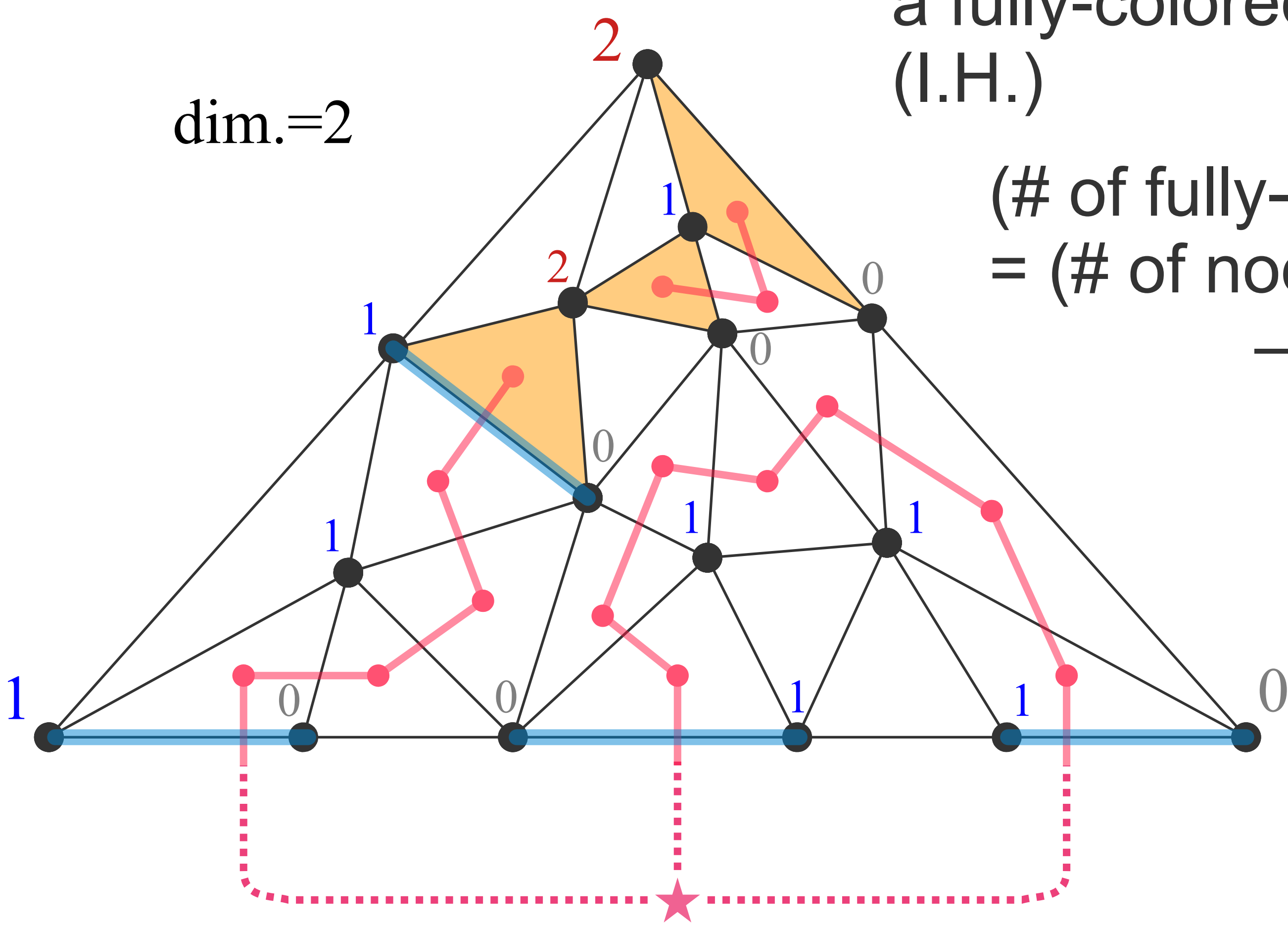
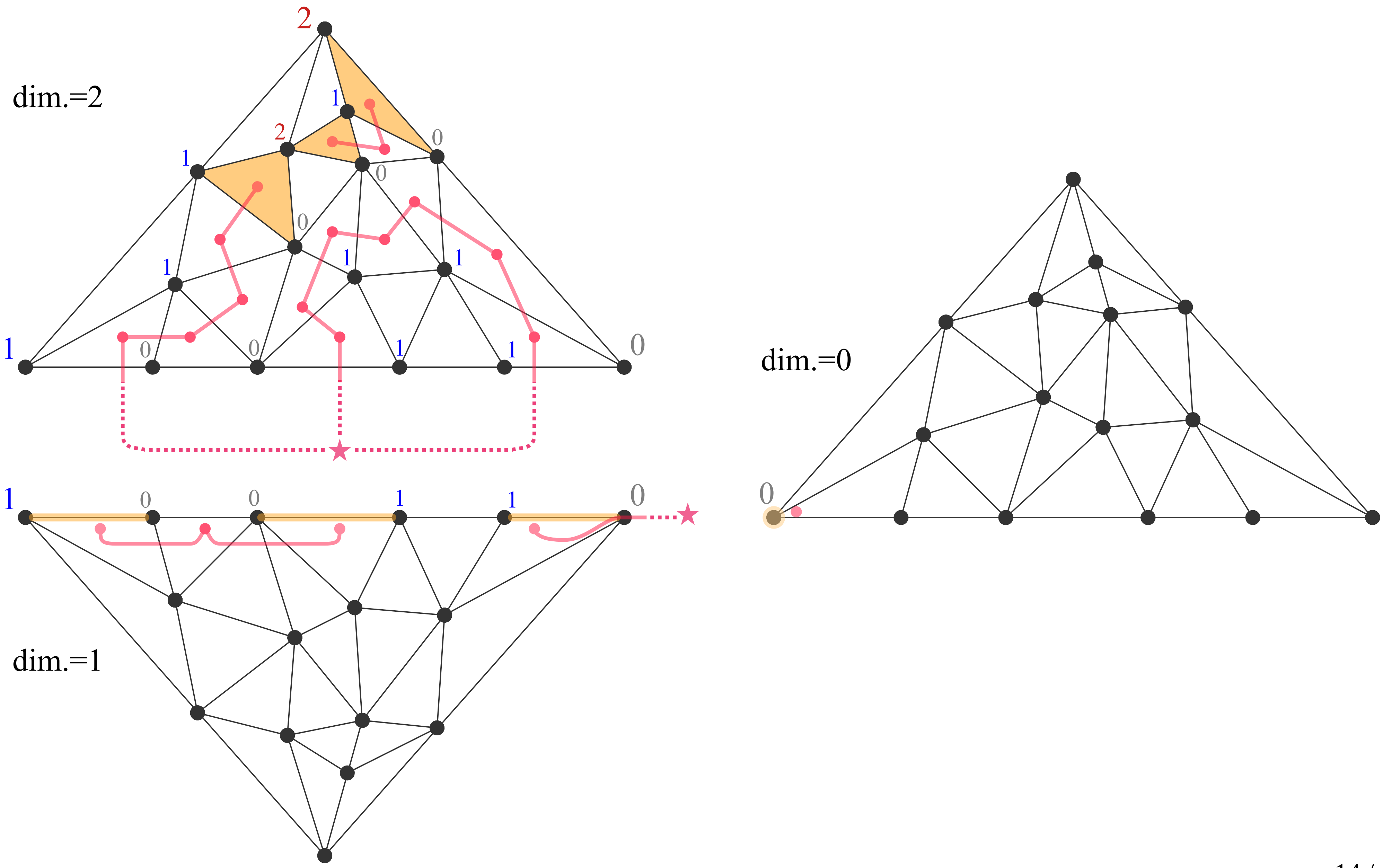


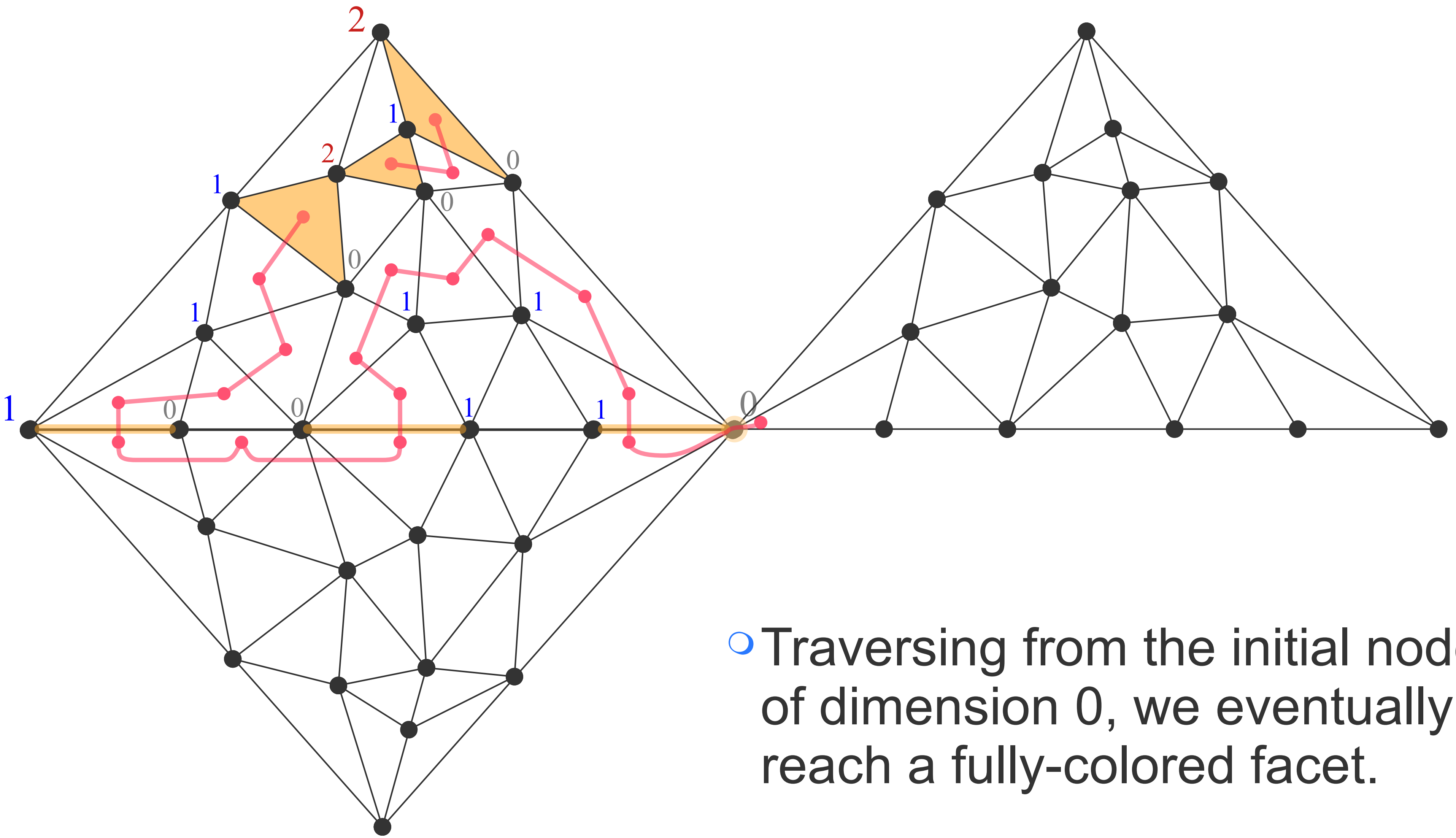
- Each graph node other than special node is of degree 1 or 2.
- A graph node is of odd degree iff it is a fully-colored or a special node (I.H.)

(# of fully-colored nodes)
= (# of nodes of odd degree)
– (1 special node) = odd

Proof of Sperner's lemma (all dimensions)

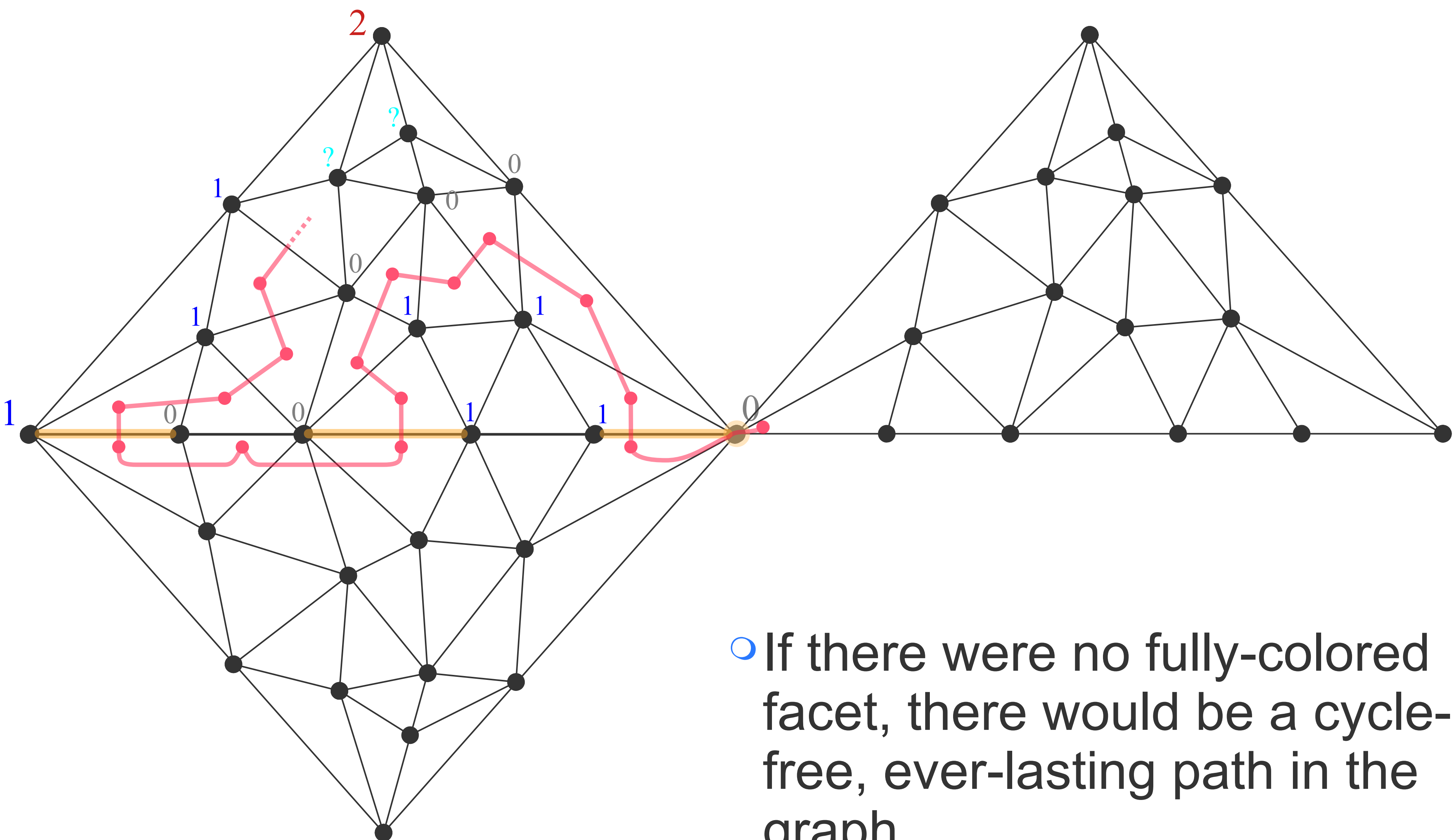


Sperner's lemma in a single unified graph



- Traversing from the initial node of dimension 0, we eventually reach a fully-colored facet.

Sperner's lemma as graph connectivity



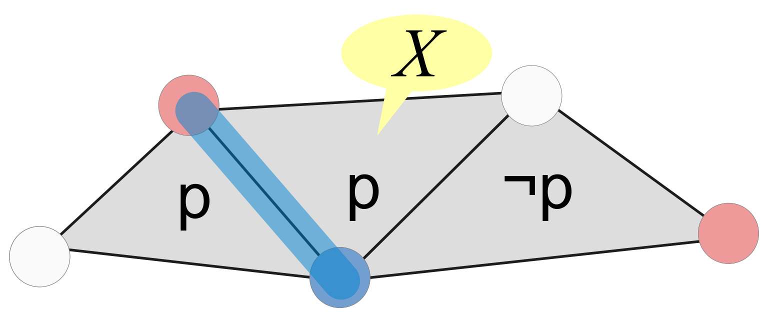
- If there were no fully-colored facet, there would be a cycle-free, ever-lasting path in the graph.

Logical obstruction in epistemic μ-calculus

Epistemic logic for DC

Epistemic logic = Propositional modal logic for ~~knowledge~~ higher dimensional connectivity

- ~~$K_a \varphi$ Process a knows φ.~~
- $D_A \varphi$ The collection A of processes know φ. (distributed knowledge)
  - $M, X \models D_A \varphi$ iff $\forall Y \in W.(X \sim_A Y \Rightarrow M, Y \models \varphi)$ where $X \sim_A Y$ iff $X \sim_a Y$ for every $a \in A$.



$M, X \models D_{\{\bullet, \bullet\}}$ p

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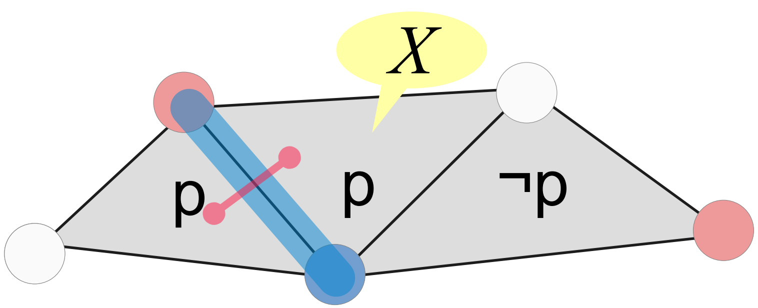
distributed knowledge



$M, X \models D_{\{\bullet, \bullet\}}$ p

Epistemic μ-calculus for DC

$$\varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \mathrm{D}_A \varphi \mid \nu Z.\varphi$$

- Distributed knowledge $\mathrm{D}_{\mathrm{A}}\,\varphi$ for higher dimensional connectivity.
- Greatest fixpoint $\nu Z.\varphi$ for transitive closure of connectivity
  - greatest solution for $Z = \varphi$ (i.e., $\nu Z.\varphi \Leftrightarrow \varphi[\nu Z.\varphi/Z]$)

- Formulas are positive.

Logical obstruction in extended simplicial model

- Extension with atomic propositions on output values. [vanDitramsch-Goubault-Lazic-Ledent-Rajsbaum2021]

Process a has input i

Process a decides output i

$$p ::= \mathsf{input}_a^i \mid \mathsf{decide}_a^i \quad (a \in \Pi, i \in \mathit{Value})$$

logical obstruction

If there exists a *positive* epistemic formula φ and a facet $X \in I[IS^m]$ such that, for any $\delta: I[SA_k] \to I[IS^m]$,

k-set agreement

m rounds

$$\widehat{I[SA_k]}, \delta(X) \models \varphi \quad \text{but} \quad I[IS^m]_\delta, X \not\models \varphi,$$

extended models

then k-set agreement task is not solvable by m-round protocol.

The logical obstruction to k-set agreement

Single output per each process

Validity of agreement

A pair of facets agree on the output of processes that they share.

$$\Phi_k = \nu Z.\Big[\mathrm{OFUN} \wedge \mathrm{VALID} \wedge \bigwedge_{\emptyset \subsetneq A \subseteq \Pi} \big(\mathrm{DEC}_A \Rightarrow \mathrm{D}_A(\mathrm{KNOW} \wedge \mathrm{AGREE}_k \wedge Z)\big)\Big]$$

Collection A of processes decide outputs from the values $\{0, \ldots, |A|\text{-}1\}$.

k-set agreement

$$\mathrm{OFUN} = \bigwedge_{a\in\Pi}\big(\bigwedge_{d,e\in\Pi,d\neq e} \neg(\mathsf{decide}_a^d \wedge \mathsf{decide}_a^e) \wedge \bigvee_{d\in\Pi} \mathsf{decide}_a^d\big)$$

$$\mathrm{VALID} = \bigwedge_{a\in\Pi} \bigwedge_{d\in\Pi}\big(\mathsf{decide}_a^d \Rightarrow \bigvee_{b\in\Pi} \mathsf{input}_b^d\big)$$

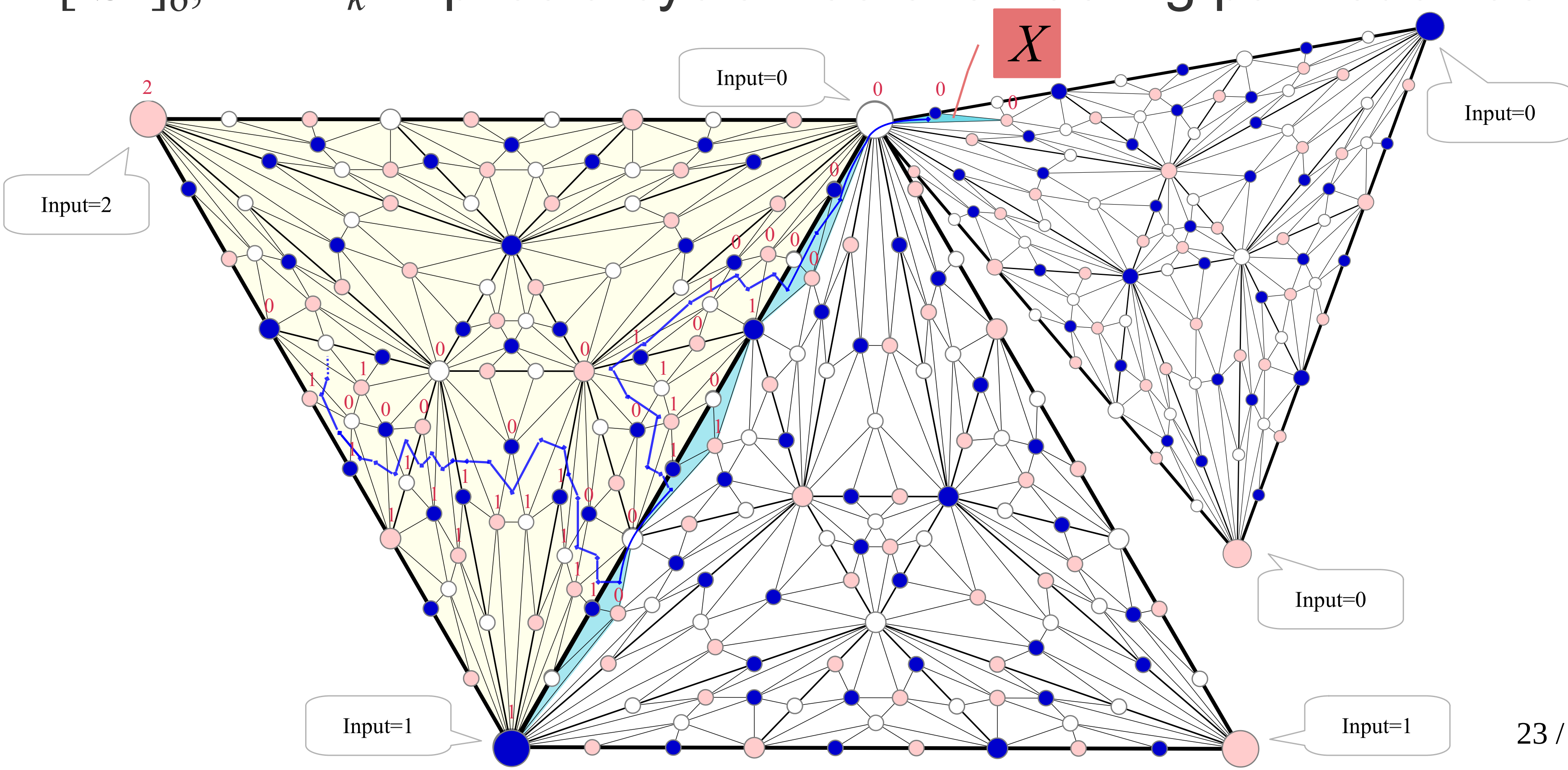
$$\mathrm{AGREE}_k = \bigvee_{A\subseteq\Pi,0<|A|\leq k} \bigwedge_{a\in\Pi} \bigvee_{d\in A} \mathsf{decide}_a^d$$

$$\mathrm{KNOW} = \bigwedge_{A\subseteq\Pi} \bigwedge_{a\in A} \bigwedge_{d\in\Pi}\big(\mathsf{decide}_a^d \Rightarrow \mathrm{D}_A\, \mathsf{decide}_a^d\big)$$

$$\mathrm{DEC}_A = \bigwedge_{d=0}^{|A|-1} \bigvee_{a\in A} \mathsf{decide}_a^d$$

The logical obstruction to k-set agreement

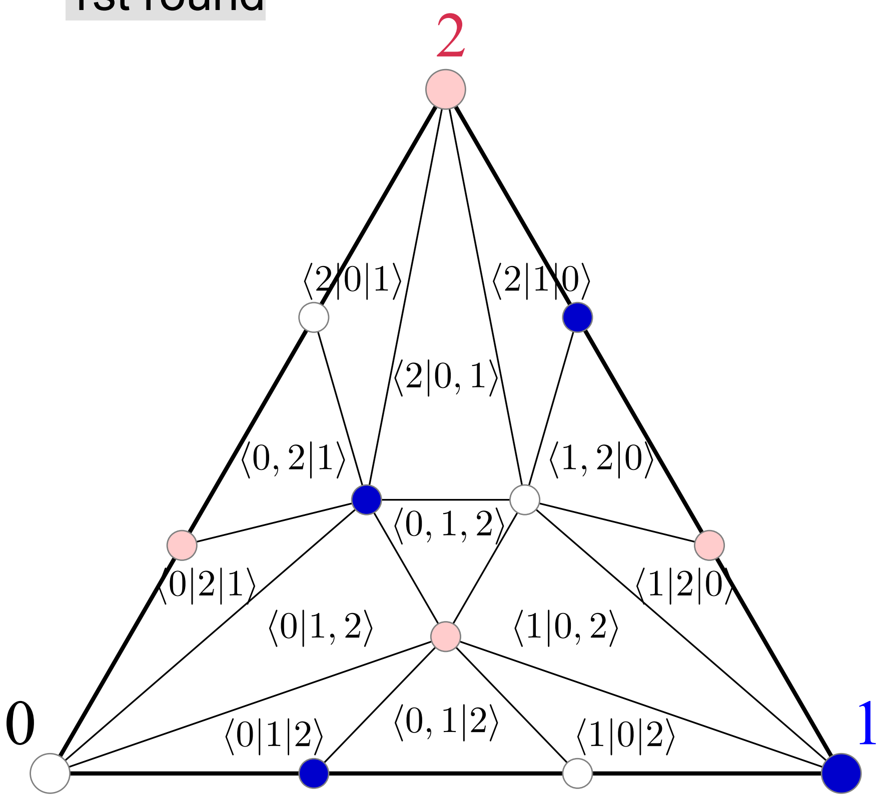
- $I[\widehat{SA_k}], \delta(X) \models \Phi_k$
  - Obviously holds because OFUN, VALID, etc. are all valid.
- $I[IS^m]_\delta, X \not\models \Phi_k$
  - $I[IS^m]_\delta, X \models \Phi_k$ implies a cycle-free ever-lasting path such as:



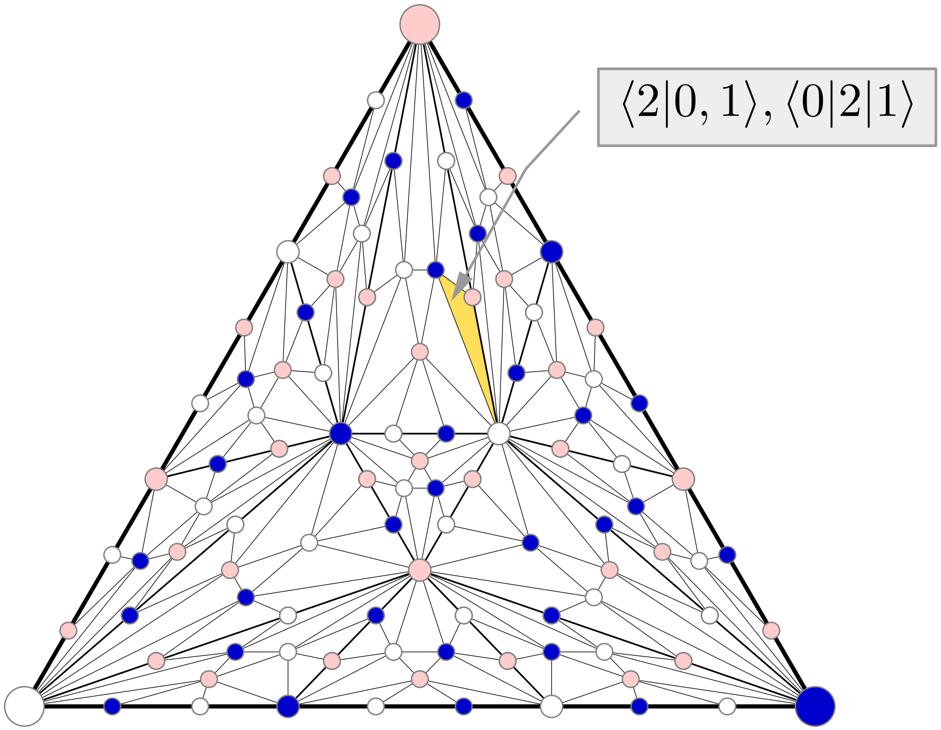
Combinatorial presentation of facets

- Facet in $I[IS]$ (1st round) = ordered set partition [Kozlov2012]
- Facet in $I[IS^m]$ (m-th round) = sequence of m ordered set partitions

1st round



2nd round

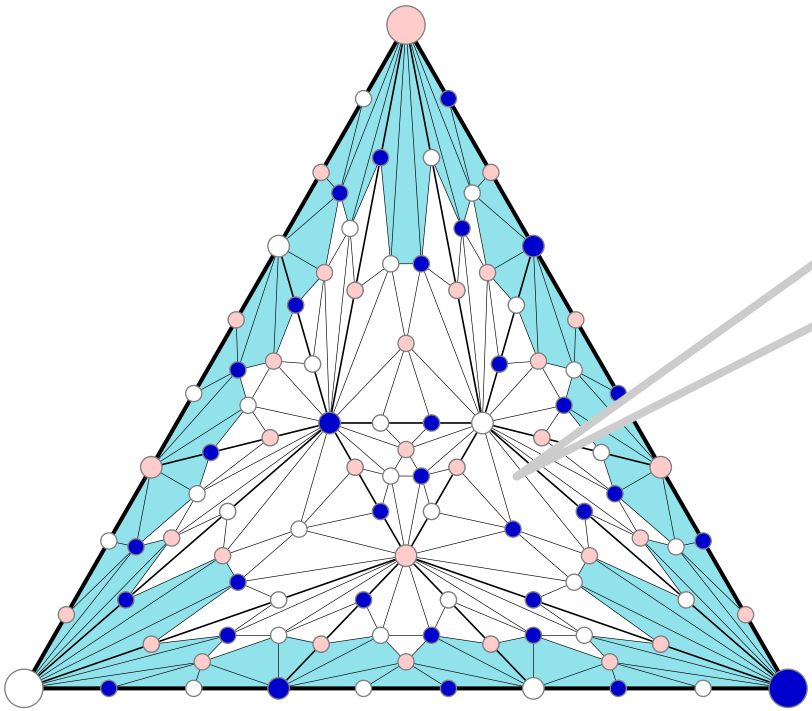


Unsolvability for k-concurrency submodel

k-concurrency

A 2-round immediate snapshot (IS^2) where simultaneous execution is restricted up to k processes.

- 2-concurrency in 3-process system

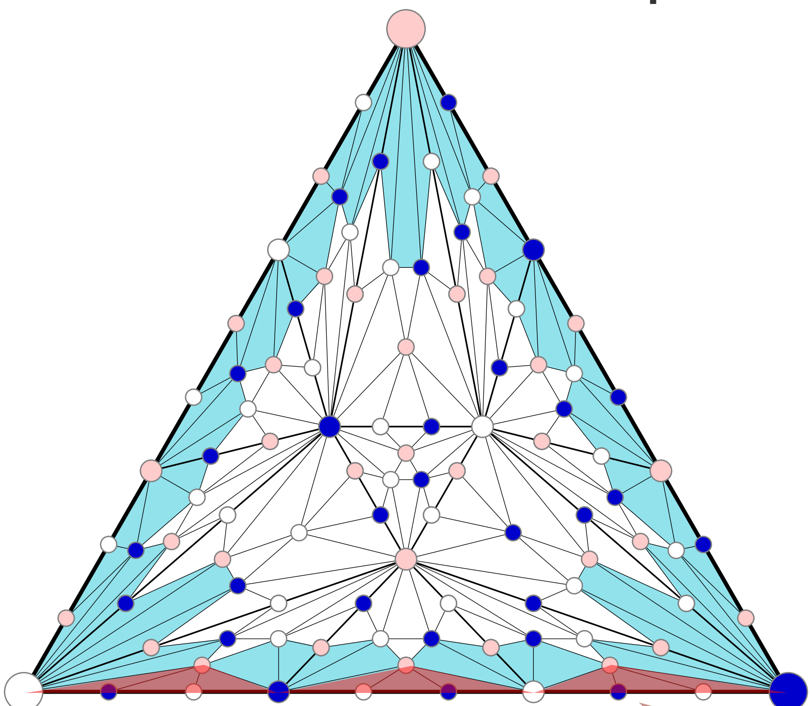


White facets are excluded because of high congestion.

Theorem[Gafni-He-Kuznetsov-Rieutord2016] ℓ-set agreement task is solvable by k-concurrency model iff $\ell \geq k$.

Unsolvability for k-concurrency submodel

- Take Φ_ℓ as the logical obstruction for ℓ-set agreement.
  - E.g., in 2-concurrency model, Φ_1 is a logical obstruction to 1-set agreement, because the model includes all the facets relevant to the proof.



Relevant to 1-set agreement unsolvability.

Summary and Future Topics

Summary

- Unsolvability of k-set agreement task in logical method:
  - Formula of epistemic μ-caluculs as an account for the reason of unsolvability.
  - Sperner's lemma as a statement for higher-dimensional connectivity.
  - Greatest fixpoint for expressing long-range, higher-dimensional connectivity.

Future topics

- More instances!
- From topology to logic
  - Sperner's lemma
    $\longrightarrow$ higher-dimensional connectivity as a greatest fixpoint in epistemic μ-calculus
  - Others?? (Index lemma, Nerve lemma, ...)

Thank you for listening.

Manuscript on arXiv:
http://arxiv.org/abs/2205.06452

References

- É. Goubault, J. Ledent and S. Rajsbaum, *"A simplicial complex model for dynamic epistemic logic to study distributed task computability"*, Inf. Cmpt. 278, 2021. (An earlier version appeared in GandALF 2018).
- Y. Nishida, *"Impossibility of k-set agreement via dynamic epistemic logic"*, RIMS Kôkyûroku 2188, 2020, pp. 96–105.
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- E. Gafni, Y. He, P. Kuznetsov and T. Rieutord, *"Read-write memory and k-set consensus as an affine task"*, OPODIS 2016.
- D. Kozlov, *"Chromatic subdivision of a simplicial complex"*, Homology, Homotopy and Applications 14 (2012).

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Epistemic logic

Epistemic logic = Propositional logic with modality $\mathrm{K}_a\,\varphi$

- $\mathrm{K}_a\,\varphi$ Process a knows φ.

- Kripke model semantics $M = (W, \sim, L)$
  - W is the set of epistemic states (possible worlds).
  - $L(X)$ gives the set of true propsitions in $X \in W$.
  - $\sim_a$ (for each $a \in \Pi$) is an *equivalence relation* over W.
    - $M, X \models \mathrm{K}_a\,\varphi$ iff $\forall Y \in W.(X \sim_a Y \Rightarrow M, Y \models \varphi)$
- Every complex C gives rise to a simplicial Kripke model:
    - W is the set of facets in C.
    - $X \sim_a Y$ iff $X \sim_a Y$ share a common vertex of color a.

Simplicial Kripke model semantics

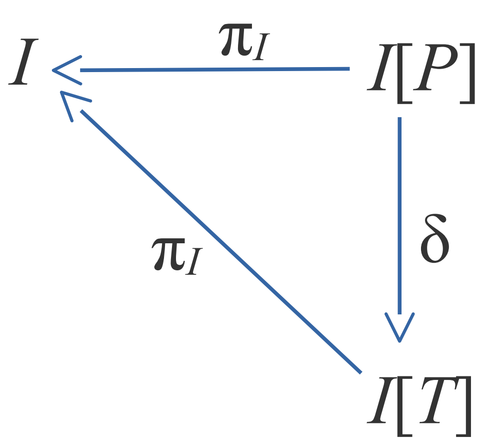
- Simplicial Kripke model $M = (W, \sim, L)$
  - W is the set of *facets* (maximal simplexes) in a chromatic simplicial complex.
  - $L(X)$ gives the set of true props. in $X \in W$.
  - $\sim_a$ ($a \in \Pi$) is an equivalence relation over W defined by:

    $X \sim_a Y \Leftrightarrow X$ and Y are simplexes sharing a common vertex of color a.
- Semantics of knowledge modality $\mathrm{K}_a\, \varphi$
  - $M, X \models \mathrm{K}_a\, \varphi$ iff $\forall Y \in W. (X \sim_a Y \Rightarrow M, Y \models \varphi)$

Unsolvability proof with epistemic logic

Knowledge gain theorem. Suppose $C \xrightarrow{\delta} D$, $X \in C$, and φ is a *positive* epistemic formula. Then, $D, \delta(X) \models \varphi$ implies $C, X \models \varphi$.

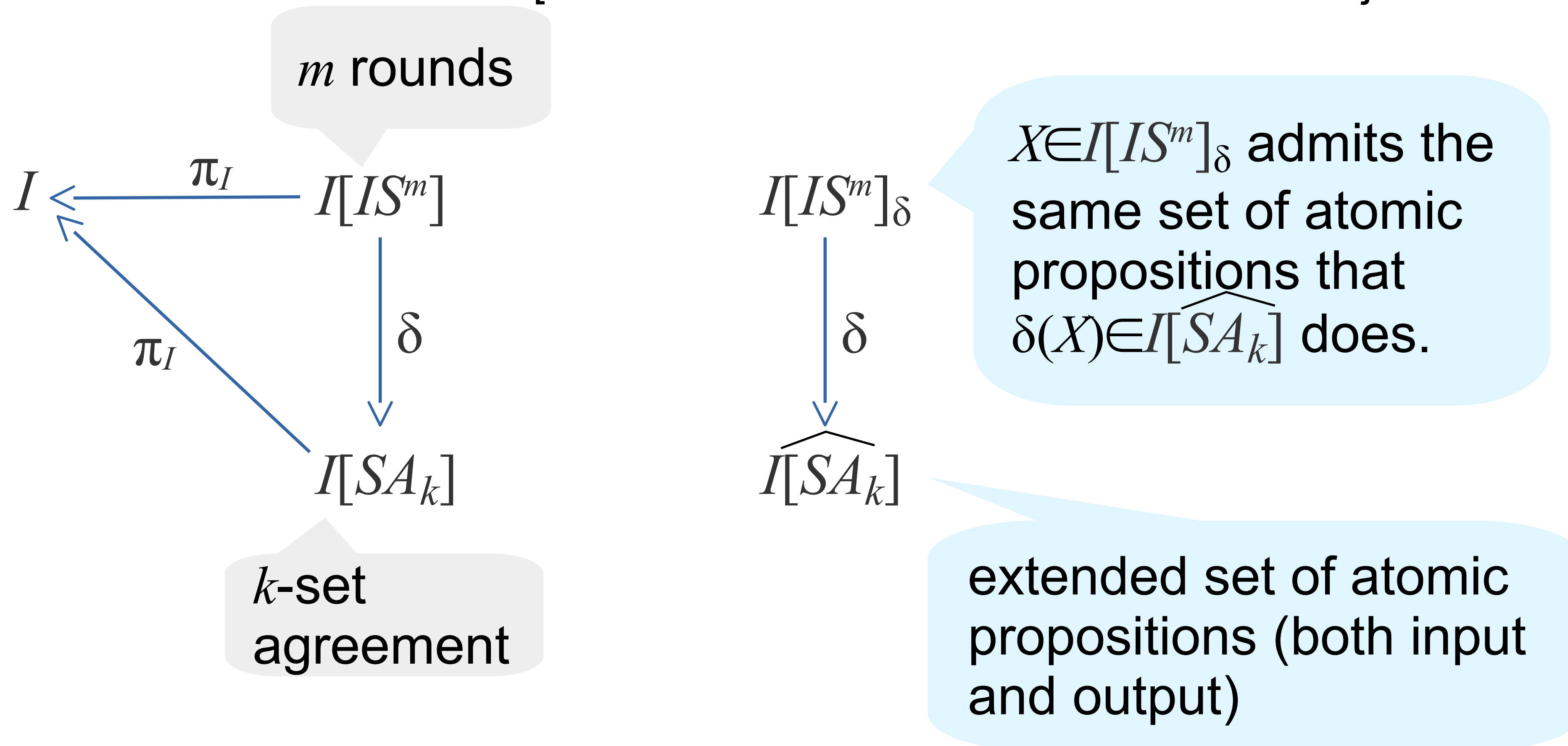
- There exists no δ that makes the following diagram commute (hence the task is not solvable),



if there exists a *positive* epistemic formula φ and facet $X \in C$ such that $I[T], \delta(X) \models \varphi$ but $I[P], X \not\models \varphi$.z

(Un)solvability in extended simplicial model

[vanDitramsch-Goubault-Lazic-Ledent-Rajsbaum2021]



- If there exists a positive epistemic formula φ and facet $X \in I[IS^m]$ such that $\widehat{I[SA_k]}, \delta(X) \models \varphi$ but $I[IS^m]_\delta, X \not\models \varphi$, then k-set agreement task is not solvable by m-round protocol.

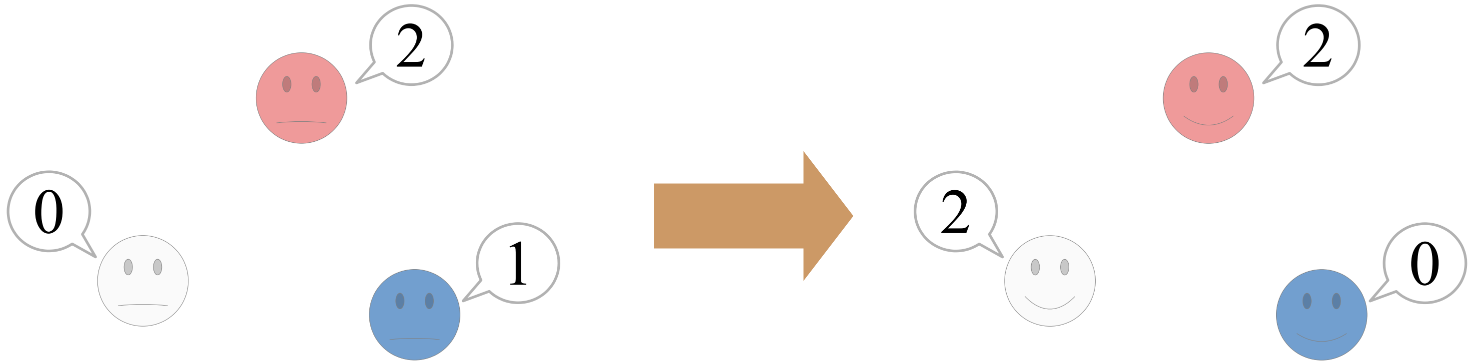
k-set agreement task

Input Each of (n+1) processes has its private input value.

Output Each process decides an output value satisfying:

- Validity. Each process decides a value out of (n+1) inputs.
- Agreement. Processes decide at most k different values.

A 2-set agreement:



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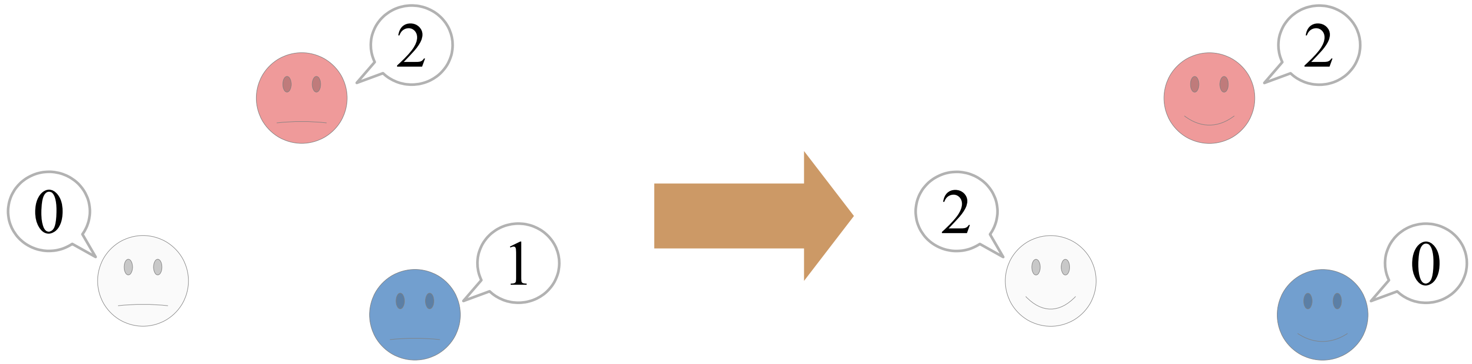
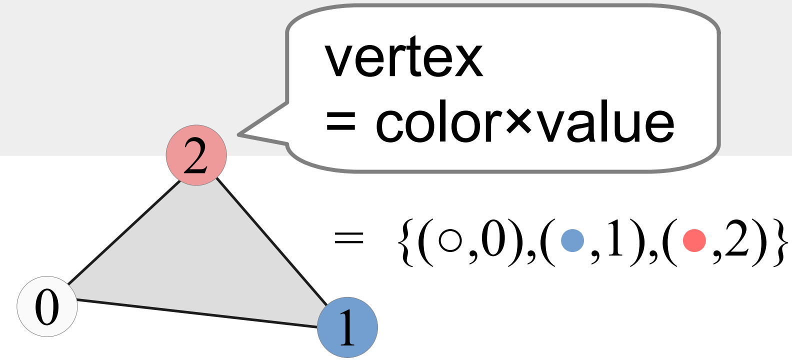
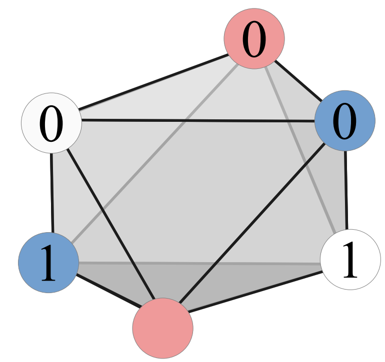


Fact. k-set agreement task is *not solvable* by a (wait-free, asynchronous) system of n+1 processes, unless k≥n+1.

Topological model for DC

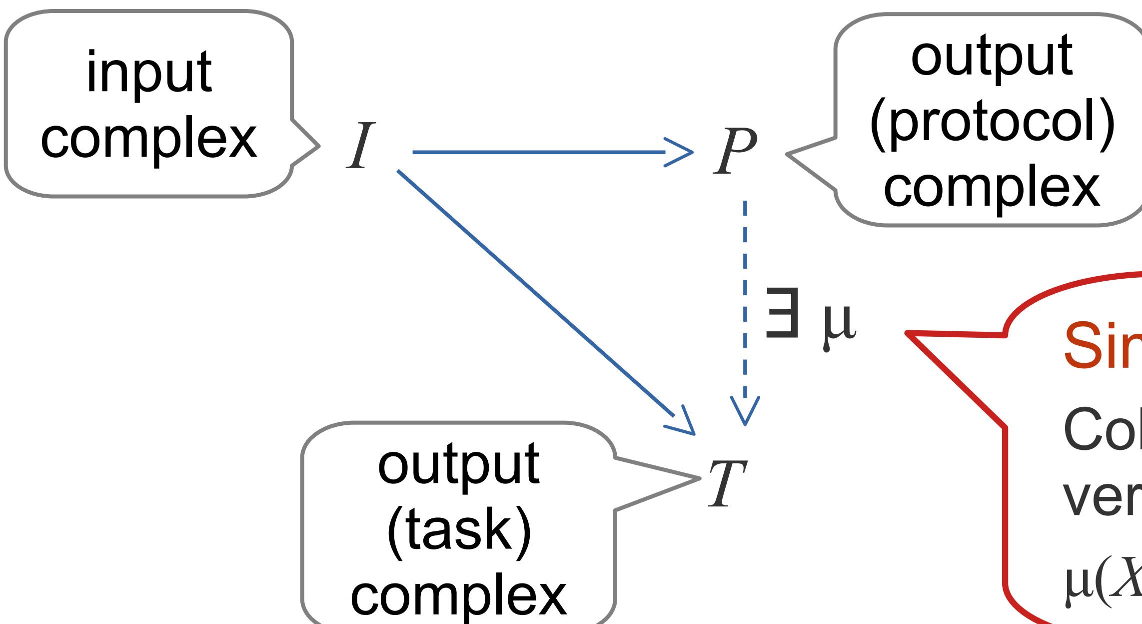


- chromatic **simplex** of dimension n
  = system state of $(n+1)$ processes



- chromatic **simplicial complex**
  = nondeterministic set of states

- Task solvability (topological)



Simplicial map

Color-preserving vertex-to-vertex mapping s.t.
$\mu(X) \in T$ for every simplex $X \in P$.

Common knowledge as fixpoint

- P is a common knowledge among the set A of processes.

$$
\begin{aligned}
& \mathrm{C}_A P \\
\Leftrightarrow\ & \nu Z.\Big(P \wedge \bigwedge_{a \in A} \mathrm{K}_a\, Z\Big) \\
\Leftrightarrow\ & P \wedge \bigwedge_{a \in A} \mathrm{K}_a \Big(\nu X.\Big(P \wedge \bigwedge_{a \in A} \mathrm{K}_a\, X\Big)\Big) \\
\Leftrightarrow\ & P \wedge \bigwedge_{a \in A} \mathrm{K}_a \Big(P \wedge \bigwedge_{a \in A} \mathrm{K}_a \Big(P \wedge \bigwedge_{a \in A} \mathrm{K}_a \cdots\Big)\Big)
\end{aligned}
$$