Proving Unsolvability of Set Agreement Task with Epistemic $\mu$-Calculus

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GETCO 2022 — May 30-Jun 3, 2022, Paris
Two methods for task unsolvability

- **Topological method**
  - Model: Simplicial complexes.
  - Strategy: Find a breach in topological invariant.

- **Logical method** [Goubault-Ledent-Rajsbaum2021]
  - Model: (Simplicial) Kripke models.
  - Strategy: Find a logic formula (logical obstruction) that is inconsistent between the models.
  - Method: Epistemic logic reasoning
Unsolvability of 1-set agreement (Topology)

\[ I \]

\[ P = IS \text{ (standard chromatic subdivision)} \]

immediate snapshot  

simplicial map \( \mu \)

\[ T \]
Unsolvability of 1-set agreement (Topology)

General case argues higher dimensional connectivity, resorting to tools from combinatorial topology (Sperner’s lemma).

* General case argues *higher dimensional connectivity*, resorting to tools from combinatorial topology (Sperner’s lemma).
Two methods for task unsolvability

- **Topological method**
  - Model: Simplicial complexes.
  - Strategy: Find a breach in topological invariant.

- **Logical method** [Goubault-Ledent-Rajsbaum2021]
  - Model: (Simplicial) Kripke models.
  - Strategy: Find a logic formula (called logical obstruction) that is inconsistent between the models.
  - Method: Epistemic logic reasoning
Every map \( f: I \rightarrow O \) over simplicial complexes induces a **product update model** \( I[O] \), a binary relation encoding of \( f \).

Every product update model \( I[O] \) is a simplicial complex, which induces a simplicial Kripke model for epistemic reasoning.
If there exists a positive epistemic formula $\varphi$ and facet $X \in I[P]$ such that, for any $\delta$: $I[P] \rightarrow I[T]$, $I[P], X \not\vDash \varphi$ but $I[T], \delta(X) \vDash \varphi$, then the task is not solvable (i.e., there is no $\delta$).
Pros and cons of logical method

😊 Just find a logical obstruction $\varphi$ to show unsolvability.

😊 $\varphi$ accounts for the reason of unsolvability in the formal language of epistemic logic.

😞 Limited instances of logical obstructions known to date.

* 1-set agreement & approximate agreement  
  [Goubault-Ledent-Rajsbaum2021]

* k-set agreement ($k>1$) [Nishida2020] (w/ distributed knowledge), later generalized for adversary model [Yagi-Nishimura2020]
  + This works only for single-round protocol.

* General logical obstruction in an extended simplicial model  
  [vanDitramsch-Goubault-Lazic-Ledent-Rajsbaum2021]
  + The general formula involves no epistemic contents and provides no hints for the reason of unsolvability.
Goal of this talk

- Find an epistemic formula $\Phi$ such that
  - $\Phi$ is a logical obstruction to \textit{k-set agreement}.
  - $\Phi$ contains epistemic contents that account for the reason of unsolvability.
  - $\Phi$ works for \textit{multi-round protocols} (where processes are allowed to communicate arbitrarily many times).
Our strategy

◼ To find inconsistency between simplicial Kripke models,
  ◆ Rework on “Sperner’s lemma” to rephrase it as a statement on higher dimensional connectivity.

◼ To express the inconsistency in the language of logic,
  ◆ Use epistemic μ-calculus, which extends epistemic logic with:
    ◆ Distributed knowledge, a modal operator for higher-dimensional connectivity, and
    ◆ Propositional greatest fixpoint for transitive closure.
Sperner’s lemma as connectivity
Sperner’s lemma. Any subdivision of a simplex with Sperner coloring has odd number of fully-colored facets (maximal simplexes).
Proof of Sperner’s lemma (induction on dim.)

dim. = 2
Proof of Sperner’s lemma (induction on dim.)

dim.=2
Proof of Sperner's lemma (induction on dim.)

dim. = 2
Proof of Sperner’s lemma (induction on dim.)

- Each graph node other than special node is of degree 1 or 2.
- A graph node is of odd degree iff it is a fully-colored or a special node (I.H.)

\[
\begin{align*}
\text{(dim.}=2\text{)} \\
\text{(# of fully-colored nodes)} \\
= \text{(# of nodes of odd degree)} \\
- (1 \text{ special node}) = \text{odd}
\end{align*}
\]
Proof of Sperner’s lemma (all dimensions)
Traversing from the initial node of dimension 0, we eventually reach a fully-colored facet.
If there were no fully-colored facet, there would be a cycle-free, ever-lasting path in the graph.
Logical obstruction in epistemic μ-calculus
Epistemic logic for DC

Epistemic logic = Propositional modal logic for knowledge higher dimensional connectivity

* $K_a \phi$  Process $a$ knows $\phi$.
* $D_A \phi$  The collection $A$ of processes know $\phi$.

$M, X \vDash D_A \phi$  iff  $\forall Y \in W. (X \sim_A Y \implies M, Y \vDash \phi)$

where $X \sim_A Y$  iff  $X \sim_a Y$ for every $a \in A$.

$M, X \vDash D_{\{\circ,\bullet\}} p$
Epistemic logic for DC

Epistemic logic = Propositional modal logic for knowledge
higher dimensional connectivity

\[ \text{Process } a \text{ knows } \phi. \]

\[ \text{The collection } A \text{ of processes know } \phi. \]

\[ M, X \models D_A \phi \iff \forall Y \in W. (X \sim_A Y \Rightarrow M, Y \models \phi) \]

where \( X \sim_A Y \iff X \sim_a Y \) for every \( a \in A \).

\[ M, X \models D_{\{\cdot, \cdot\}} p \]
Epistemic $\mu$-calculus for DC

$$\varphi ::= p \mid \neg p \mid \varphi \lor \varphi \mid \varphi \land \varphi \mid D_A \varphi \mid \nu Z.\varphi$$

* **Distributed knowledge** $D_A \varphi$ for higher dimensional connectivity.

* **Greatest fixpoint** $\nu Z.\varphi$ for transitive closure of connectivity
  + greatest solution for $Z = \varphi$ (i.e., $\nu Z.\varphi \Leftrightarrow \varphi[\nu Z.\varphi/Z]$)

* Formulas are positive.
Logical obstruction in extended simplicial model

Extension with atomic propositions on output values.
[vanDitramsch-Goubault-Lazic-Ledent-Rajsbaum2021]

Process \( a \) has input \( i \)

\[ p ::= \text{input}^i_a \mid \text{decide}^i_a \quad (a \in \Pi, i \in Value) \]

If there exists a positive epistemic formula \( \varphi \) and a facet \( X \in I[IS^m] \) such that, for any \( \delta: I[SA_k] \to I[IS^m] \),

\[ I[\widehat{SA_k}], \delta(X) \models \varphi \quad \text{but} \quad I[IS^m]_\delta, X \not\models \varphi, \]

then \( k \)-set agreement task is not solvable by \( m \)-round protocol.
The logical obstruction to $k$-set agreement

Single output per each process

$$\Phi_k = \nu Z. \left[ \text{OFUN} \land \text{VALID} \land \bigwedge_{\emptyset \subsetneq A \subseteq \Pi} \left( \text{DEC}_A \Rightarrow D_A(\text{KNOW} \land \text{AGREE}_k \land Z) \right) \right]$$

Validity of agreement

A pair of facets agree on the output of processes that they share.

Collection $A$ of processes decide outputs from the values $\{0, \ldots, |A|-1\}$.

$k$-set agreement

\[
\begin{align*}
\text{OFUN} &= \bigwedge_{a \in \Pi} \left( \bigwedge_{d,e \in \Pi, d \neq e} \neg (\text{decide}_a^d \land \text{decide}_a^e) \land \bigvee_{d \in \Pi} \text{decide}_a^d \right) \\
\text{VALID} &= \bigwedge_{a \in \Pi} \bigwedge_{d \in \Pi} \left( \text{decide}_a^d \Rightarrow \bigvee_{b \in \Pi} \text{input}_b^d \right) \\
\text{AGREE}_k &= \bigvee_{A \subseteq \Pi, 0 < |A| \leq k} \bigwedge_{a \in \Pi} \bigvee_{d \in A} \text{decide}_a^d \\
\text{KNOW} &= \bigwedge_{A \subseteq \Pi} \bigwedge_{a \in A} \bigwedge_{d \in \Pi} \left( \text{decide}_a^d \Rightarrow D_A \text{decide}_a^d \right) \\
\text{DEC}_A &= \bigwedge_{d=0}^{|A|-1} \bigvee_{a \in A} \text{decide}_a^d
\end{align*}
\]
The logical obstruction to k-set agreement

- $I[SA_k], \delta(X) \models \Phi_k$
  - Obviously holds because OFUN, VALID, etc. are all valid.
- $I[IS^m], X \not\models \Phi_k$
  - $I[IS^m], X \models \Phi_k$ implies a cycle-free ever-lasting path such as:
Combinatorial presentation of facets

- **Facet in** \( I[IS] \) (**1st round**)
  - = ordered set partition  
  - [Kozlov2012]

- **Facet in** \( I[IS^m] \) (**m-th round**)
  - = sequence of \( m \) ordered set partitions
Unsolvability for k-concurrency submodel
k-concurrency

A 2-round immediate snapshot ($IS^2$) where simultaneous execution is restricted up to $k$ processes.

* 2-concurrency in 3-process system

Theorem [Gafni-He-Kuznetsov-Rieutord2016] $\ell$-set agreement task is solvable by $k$-concurrency model iff $\ell \geq k$. 

White facets are excluded because of high congestion.
Take $\Phi_\ell$ as the logical obstruction for $\ell$-set agreement.

E.g., in 2-concurrency model, $\Phi_1$ is a logical obstruction to 1-set agreement, because the model includes all the facets relevant to the proof.
Summary and Future Topics
Summary

- Unsolvability of k-set agreement task in logical method:
  - Formula of epistemic μ-caluculs as an account for the reason of unsolvability.
  - Sperner’s lemma as a statement for higher-dimensional connectivity.
  - Greatest fixpoint for expressing long-range, higher-dimensional connectivity.
Future topics

◉ More instances!

◉ From topology to logic
  ✷ Sperner’s lemma
    → higher-dimensional connectivity as a greatest fixpoint in epistemic μ-calculus
  ✷ Others?? (Index lemma, Nerve lemma, ...)

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Thank you for listening.

Manuscript on arXiv:
http://arxiv.org/abs/2205.06452


E. Gafni, Y. He, P. Kuznetsov and T. Rieutord, “Read-write memory and k-set consensus as an affine task”, OPODIS 2016.

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Epistemic logic

Epistemic logic = Propositional logic with modality $K_a \varphi$

* $K_a \varphi$  Process $a$ knows $\varphi$.

*Kripke model semantics  $M = (W, \sim, L)$

* $W$ is the set of epistemic states (possible worlds).
* $L(X)$ gives the set of true propositions in $X \in W$.
* $\sim_a$ (for each $a \in \Pi$) is an equivalence relation over $W$.
  + $M, X \models K_a \varphi$  iff  $\forall Y \in W.(X \sim_a Y \Rightarrow M, Y \models \varphi)$

*Every complex $C$ gives rise to a simplicial Kripke model:
  + $W$ is the set of facets in $C$.
  + $X \sim_a Y$ iff $X \sim_a Y$ share a common vertex of color $a$. 
Simplicial Kripke model semantics

- **Simplicial Kripke model** $M = (W, \sim, L)$
  - $W$ is the set of *facets* (maximal simplexes) in a chromatic simplicial complex.
  - $L(X)$ gives the set of true props. in $X \in W$.
  - $\sim_a (\forall a \in \Pi)$ is an equivalence relation over $W$ defined by:
    \[ X \sim_a Y \iff X \text{ and } Y \text{ are simplexes sharing a common vertex of color } a. \]

- **Semantics of knowledge modality $K_a \varphi$**
  - $M, X \models K_a \varphi \iff \forall Y \in W. (X \sim_a Y \Rightarrow M, Y \models \varphi)$
Unsolvability proof with epistemic logic

Knowledge gain theorem. Suppose $C \xrightarrow{\delta} D$, $X \in C$, and $\phi$ is a positive epistemic formula. Then, $D, \delta(X) \models \phi$ implies $C, X \models \phi$.

There exists no $\delta$ that makes the following diagram commute (hence the task is not solvable),

If there exists a positive epistemic formula $\phi$ and facet $X \in C$ such that $I[T], \delta(X) \models \phi$ but $I[P], X \not\models \phi$. 

logical obstruction
If there exists a positive epistemic formula $\varphi$ and facet $X \in I[IS^m]$ such that $I[SA_k], \delta(X) \models \varphi$ but $I[IS^m], X \not\models \varphi$, then $k$-set agreement task is not solvable by $m$-round protocol.
k-set agreement task

**Input**  Each of (n+1) processes has its private input value.

**Output**  Each process decides an output value satisfying:

* **Validity.** Each process decides a value out of (n+1) inputs.
* **Agreement.** Processes decide at most k different values.

**A 2-set agreement:**

![Diagram showing a 2-set agreement with processes deciding different values]
**Input**  Each of $(n+1)$ processes has its private input value.

**Output**  Each process decides an output value satisfying:
- **Validity.** Each process decides a value out of $(n+1)$ inputs.
- **Agreement.** Processes decide at most $k$ different values.

**A 2-set agreement:**

![Diagram showing 2-set agreement with input values 0, 1, and 2 leading to output values 0 and 2.]

**Fact.** $k$-set agreement task is *not solvable* by a (wait-free, asynchronous) system of $n+1$ processes, unless $k \geq n+1$. 
Topological model for DC

- **Chromatic simplex** of dimension $n$ = system state of $(n+1)$ processes
- **Chromatic simplicial complex** = nondeterministic set of states
- **Task solvability (topological)**

\[ \exists \mu \rightarrow T \]

**Simplicial map**
- Color-preserving vertex-to-vertex mapping s.t. $\mu(X) \in T$ for every simplex $X \in P$. 

**Vertex** = color×value

\[ \{(○,0),(●,1),(●,2)\} \]
P is a common knowledge among the set A of processes.

\[ C_A P \]

\[ \Leftrightarrow \nu Z. \left( P \land \bigwedge_{a \in A} K_a Z \right) \]

\[ \Leftrightarrow P \land \bigwedge_{a \in A} K_a \left( \nu X. \left( P \land \bigwedge_{a \in A} K_a X \right) \right) \]

\[ \Leftrightarrow P \land \bigwedge_{a \in A} K_a \left( P \land \bigwedge_{a \in A} K_a \left( P \land \bigwedge_{a \in A} K_a \cdots \right) \right) \]