Tracking Dynamical Features via Continuation and Persistence

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Let $\varphi(x, t) : X \times \mathbb{R} \to X$ be a continuous flow on a compact metric space. Set $S$ is invariant if $S = \text{inv } S := \{x \in S \mid \varphi(x, \mathbb{R}) \subseteq S\}$.

A compact set $N$ is an isolating neighborhood if $\text{inv } N \subseteq \text{int } N$. An invariant set $S$ which admits an isolating neighborhood such that $\text{inv } N = S$ is called an isolated invariant set.
Let \( \varphi_p(x, t) : X \times \mathbb{R} \to X \) be a flow parametrized by \( p \in [a, b] \subset \mathbb{R} \). An isolated invariant set \( S_a \) in \( \varphi_a \) continues to another isolated invariant set \( S_b \) in \( \varphi_b \) if there exist a sequence of compact sets \( N_0, N_1, \ldots, N_k \) and a sequence of intervals \( \{[a_i, b_i] \subset [a, b] \mid i \in \{0, 1, \ldots, k\} \} \) such that

- \( a_0 = a \) and \( b_k = b \),
- \( [a_i, b_i] \cap [a_{i+1}, b_{i+1}] \neq \emptyset \) for all \( i \in \{0, 1, \ldots, k - 1\} \),
- \( N_i \) is an isolating neighbourhood in \( \varphi_p(x, t) \) with \( p \in [a_i, b_i] \),
- \( \text{inv}_{\varphi_a}(N_0) = S_a \) and \( \text{inv}_{\varphi_b}(N_k) = S_b \).

**Theorem 1.7, Conley & Easton, 1971**

Denote \( \Phi(X) \) a space of flows \( \varphi : X \times \mathbb{R} \to X \) on the compact metric space \( X \) endowed with the compact open topology. Let \( N \) be an isolating neighborhood for a flow \( \varphi \in \Phi(X) \). Then there exists an open neighborhood \( U_\varphi \subset \Phi(X) \) such that \( N \) is an isolating neighborhood for every \( \psi \in U_\varphi \).
Multivector fields theory
Multivector as a dynamical black box
Multivector as a dynamical black box
Multivector as a dynamical black box
Multivector as a dynamical black box
Multivector as a dynamical black box
A compact set $N$ is a **Ważewski set** if $N^- := \{ x \in N \mid \forall \epsilon > 0 \varphi(x, [0, \epsilon]) \not\subset N \}$ is closed.

**Ważewski principle**

If $N$ is a Ważewski set and $H_*(N, N^-) \neq 0$ then $\text{inv } N \neq \emptyset$. 

\[ H(p_1, p_2) = F \]
\[ H(p_1, p_2) = O \oplus F \]
\[ H(p_1, p_2) = O \oplus F \oplus F \]
A compact set $N$ is a **Ważewski set** if $N^- := \{ x \in N \mid \forall \epsilon > 0 \varphi(x, [0, \epsilon)) \not\subset N \}$ is closed.

**Ważewski principle**

If $N$ is a Ważewski set and $H_\ast(N, N^-) \neq 0$ then $\text{inv } N \neq \emptyset$. 
Alexandrov Theorem

For a preorder $\leq$ on a finite set $X$, there is a topology $\mathcal{T}_{\leq}$ on $X$ whose open sets are the upper sets with respect to $\leq$. For a topology $\mathcal{T}$ on a finite set $X$, there is a preorder $\leq_{\mathcal{T}}$ where $x \leq_{\mathcal{T}} y$ if and only if $x \in \text{cl}_{\mathcal{T}} y$. The correspondences $\mathcal{T} \leftrightarrow \leq_{\mathcal{T}}$ and $\leq \leftrightarrow \mathcal{T}_{\leq}$ are mutually inverse. Under these correspondences continuous maps are transformed into order-preserving maps and vice versa. Moreover, the topology $\mathcal{T}$ is $T_0$ (Kolmogorov) if and only if the preorder $\leq_{\mathcal{T}}$ is a partial order.

Finite topological spaces $\leftrightarrow$ Partially ordered sets

Continuous maps $\leftrightarrow$ Order preserving maps
Simplicial complex as a finite topological space
Homology of finite topological spaces

McCord Theorem, (McCord, 1966)

There exists a map

$$\mu(X, T) : |K(X, T)| \to (X, T)$$

such that it is continuous and a weak homotopy equivalence. Moreover, if $f : (X, T_X) \to (Y, T_Y)$ is a continuous map of two finite $T_0$ topological spaces, then the following diagrams commute:

\[ |K(X, T_X)| \xrightarrow{|K(f)|} |K(Y, T_Y)| \]
\[ \downarrow \mu(X, T_X) \quad \downarrow \mu(Y, T_Y) \]
\[ (X, T_X) \xrightarrow{f} (Y, T_Y) \]
\[ H(|K(X, T_X)|) \xrightarrow{|K(f)|^*} H(|K(Y, T_Y)|) \]
\[ \downarrow \mu(X, T_X)^* \quad \downarrow \mu(Y, T_Y)^* \]
\[ H(X, T_X) \xrightarrow{f_*} H(Y, T_Y) \]

Let $X$ be a finite topological space and $A \subset X$. Then

$$H(X) \cong H(|K(X)|) \cong H^A(K(X)).$$

$$H(X, A) \cong H(|K(X)|, |K(A)|) \cong H^A(K(X), K(A)).$$
Let \((\mathcal{P}, \leq)\) be a partial order.

\(A \subset \mathcal{P}\) is an **upper set** (**open**) iff \(x \in A\) and \(y \geq x\) implies \(y \in A\).

\(A \subset \mathcal{P}\) is a **down set** (**closed**) iff \(x \in A\) and \(y \leq x\) implies \(y \in A\).

\(A \subset \mathcal{P}\) is **convex** (**locally closed**) iff \(x \leq y \leq z\) with \(x, z \in A\), \(y \in \mathcal{P}\) implies \(y \in A\).
Let \((\mathcal{P}, \leq)\) be a partial order.

\(A \subseteq \mathcal{P}\) is an **upper set (open)** iff \(x \in A\) and \(y \geq x\) implies \(y \in A\).  
\(A \subseteq \mathcal{P}\) is a **down set (closed)** iff \(x \in A\) and \(y \leq x\) implies \(y \in A\).  
\(A \subseteq \mathcal{P}\) is **convex (locally closed)** iff \(x \leq y \leq z\) with \(x, z \in A\), \(y \in \mathcal{P}\) implies \(y \in A\).
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\(A \subseteq \mathcal{P}\) is **convex** (locally closed) iff \(x \leq y \leq z\) with \(x, z \in A\), \(y \in \mathcal{P}\) implies \(y \in A\).
Let $X$ be a finite topological space.

A **multivector** is a locally closed subset of $X$. A **combinatorial multivector field (MVF)** $\mathcal{V}$ on $X$ is a collection of multivectors, such that $\mathcal{V}$ is a partition of $X$. 
Dynamics of MVF for FTop
Dynamics of MVF for FTTop

\[ \mathbf{F}_v(x) := [x]_v \circ dx \]

where \([x]_v\) is the unique MV to which \(x\) belongs.
Dynamics of MVF for FTop
set $A$ is $\mathcal{U}$-compatible if $A$ can be expressed as a union of multivectors from $\mathcal{U}$.

$\langle A \rangle_{\mathcal{U}}$ - the minimal $\mathcal{U}$-compatible and locally closed set containing $A$.

\[
\langle \{ \nu_0, e_2 \} \rangle_{\mathcal{U}} = V_0 \cup V_1 \cup V_2 \cup V_4
\]
A map \( \varphi : \mathbb{Z} \to X \) is a **full solution** for \( \mathcal{V} \) iff \( \forall i \in \mathbb{Z} \): \( \varphi(i + 1) \in F_{\mathcal{V}}(\varphi(i)) \). We denote a set of full solutions in \( X \) by \( \text{Sol}(X) \).

A multivector \( V \in \mathcal{V} \) is **critical** if \( H(\text{cl} \ V, \text{mo} \ V) \neq 0 \), otherwise \( V \) is **regular**.

A full solution \( \varphi : \mathbb{Z} \to X \) is **essential** if for every regular \( x \in \text{im} \varphi \) the set \( \{ t \in \mathbb{Z} \mid \varphi(t) \notin [x]_{\mathcal{V}} \} \) is either left- and right-unbounded. A set of all essential solutions in a set \( A \subseteq X \) with \( \varphi(0) = x \) is denoted \( \text{eSol}(x, A) \).
**Isolated invariant sets**

**Invariant part** of $A \subseteq X$ is

$$\text{Inv}(A) := \{x \in A \mid e\text{Sol}(x, A) \neq \emptyset\}$$

We say that $A$ is **invariant** iff $\text{Inv}(A) = A$.

A closed set $N$ **isolates** an invariant set $S \subseteq N$ if the following two conditions hold:

a) every path in $N$ with endpoints in $S$ is a path in $S$,

b) $\Pi_Y(S) \subseteq N$.

In this case, we also say that $N$ is an isolating set for $S$. An invariant set $S$ is **isolated** if there exists a closed set $N$ meeting the above conditions.
Let $S$ be isolated invariant set under $\mathcal{V}$, and let $P$ and $E$ denote closed sets where $E \subseteq P$. If the following conditions hold, then $(P, E)$ is an **index pair** for $S$:

1) $F_{\mathcal{V}}(P \setminus E) \subseteq P$,
2) $F_{\mathcal{V}}(E) \cap P \subseteq E$,
3) $S = \text{inv}_{\mathcal{V}}(P \setminus E)$.

The **combinatorial homology Conley index** of an isolated invariant set $S$ is defined as $\text{Con}(S) := H(P, E)$, where $(P, E)$ is an index pair for $S$.

**Theorem 5.16 (LKMW, 2020)**

Let $(P, E)$ and $(P', E')$ be index pairs for an isolated invariant set $S$ then $H(P, E) \cong H(P', E')$. 

Conley index

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Let $(P, E)$ and $(P', E')$ be index pairs for an isolated invariant set $S$ then $H(P, E) \cong H(P', E')$. 
Combinatorial perturbation
Combinatorial perturbation
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Combinatorial perturbation
Combinatorial perturbation
Combinatorial perturbation
Combinatorial perturbation
Combinatorial perturbation
Combinatorial perturbation
Combinatorial perturbation
Multivector fields space

For two families of sets $\mathcal{A}$ and $\mathcal{B}$ we write $\mathcal{A} \sqsubseteq \mathcal{B}$ if for every $A \in \mathcal{A}$ there exists a $B \in \mathcal{B}$ such that $A \subseteq B$.

An atomic rearrangements of multivector fields:

- $\mathcal{V}$ is an **atomic refinement** of $\mathcal{W}$ if $\mathcal{V} \sqsubseteq \mathcal{W}$ and $|\mathcal{V} \setminus \mathcal{W}| = 1$
- $\mathcal{V}$ is an **atomic coarsening** of $\mathcal{W}$ if $\mathcal{V} \sqsupseteq \mathcal{W}$ and $|\mathcal{V} \setminus \mathcal{W}| = 2$
MVF($X$) - a family of all multivector fields on $X$ with a topology induced by $\sqsubseteq$. 
Combinatorial continuation of an isolated invariant set
Let $S_1, S_2, \ldots, S_n$ denote a sequence of isolated invariant sets under the multivector fields $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$, where each $\mathcal{V}_i$ is defined on a fixed simplicial complex $K$. We say that isolated invariant set $S_1$ continues to isolated invariant set $S_n$ whenever there exists a sequence of index pairs $(P_1, E_1), (P_2, E_2), \ldots, (P_{n-1}, E_{n-1})$ where $(P_i, E_i)$ is an index pair for both $S_i$ and $S_{i+1}$. Such a sequence is a sequence of connecting index pairs.
The tracking

$U_0, U_1, U_2, U_3, \ldots, U_n$
The tracking

$\mathcal{U}_0 \quad \mathcal{U}_1 \quad \mathcal{U}_2 \quad \mathcal{U}_3 \quad \cdots \quad \mathcal{U}_n$

$S_0$
The tracking

\[ S_0 \rightarrow ? \]

\[ U_0, U_1, U_2, U_3, \ldots, U_n \]
The tracking

$U_0 \quad U_1 \quad U_2 \quad U_3 \quad \ldots \quad U_n$

$S_0 \xrightarrow{} S_1$

$(P_{01}, E_{01})$
The tracking

\[ S_0 \rightarrow S_1 \rightarrow \ldots \rightarrow S_n \]

\((p_{01}, E_{01})\)
The tracking

\[ \mathcal{U}_0 \quad \mathcal{U}_1 \quad \mathcal{U}_2 \quad \mathcal{U}_3 \quad \ldots \quad \mathcal{U}_n \]

\[ S_0 \longrightarrow S_1 \longrightarrow S_2 \longrightarrow ? \]

\[ (P_{o1}, E_{o1}) \quad (P_{o2}, E_{o2}) \]
The tracking

\( U_0 \quad U_1 \quad U_2 \quad U_3 \quad \ldots \quad U_n \)

\( S_0 \quad S_1 \quad S_2 \quad S_3 \quad \ldots \quad ? \)

\((P_{01}, E_{01}) \quad (P_{12}, E_{12}) \quad (P_{23}, E_{23})\)
The tracking

\[ U_0 \rightarrow U_1 \rightarrow U_2 \rightarrow U_3 \rightarrow \ldots \rightarrow U_n \]

\[ S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow \ldots \rightarrow S_n \]

\[ (P_{01}, E_{01}) \rightarrow (P_{12}, E_{12}) \rightarrow (P_{23}, E_{23}) \rightarrow \ldots \rightarrow (P_{n-1,n}, E_{n-1,n}) \]
Attempt to track via continuation:

1. If $\mathcal{V}'$ is an atomic refinement of $\mathcal{V}$, then take $S' := \text{inv}_{\mathcal{V}'}(S)$.

2. If $\mathcal{V}'$ is an atomic coarsening of $\mathcal{V}$, and the unique merged multivector $V$ has the property that $V \subseteq S$, then take $S' := \text{inv}_{\mathcal{V}'}(S)$.

3. If $\mathcal{V}'$ is an atomic coarsening of $\mathcal{V}$, and the unique merged multivector $V$ has the property that $V \cap S = \emptyset$, then take $S' := \text{inv}_{\mathcal{V}'}(S) = S$.

4. If $\mathcal{V}'$ is an atomic coarsening of $\mathcal{V}$ and the unique merged multivector $V$ satisfies the formulae $V \cap S \neq \emptyset$ and $V \not\subseteq S$, then consider $A = \langle S \cup V \rangle_{\mathcal{V}'}$. If $\text{inv}_{\mathcal{V}}(A) = S$, then take $S' := \text{inv}_{\mathcal{V}'}(A)$.

5. Else, it is impossible to track via continuation.
Theorem 11 (Dey, L., Mrozek, Slechta; 2022)

Let $\mathcal{V}$ and $\mathcal{V}'$ denote multivector fields where $\mathcal{V}'$ is an atomic refinement of $\mathcal{V}$. Let $A$ be a $\mathcal{V}$-compatible and convex set. The pair $(\text{cl}(A), \text{mo}(A))$ is an index pair for both $\text{inv}_{\mathcal{V}}(A)$ under $\mathcal{V}$ and $\text{inv}_{\mathcal{V}'}(A)$ under $\mathcal{V}'$.

Theorem 12 (Dey, L., Mrozek, Slechta; 2022)

Let $\mathcal{V}$ and $\mathcal{V}'$ denote multivector fields where $\mathcal{V}'$ is an atomic coarsening of $\mathcal{V}$. Let $A$ be a convex and $\mathcal{V}$-compatible set, and let $V \in \mathcal{V}'$ be the unique merged multivector. If $V \subseteq A$ or $V \cap A = \emptyset$, then $(\text{cl}(A), \text{mo}(A))$ is an index pair for both $\text{inv}_{\mathcal{V}}(A)$ and $\text{inv}_{\mathcal{V}'}(A)$.

Theorem 13 (Dey, L., Mrozek, Slechta; 2022)

Let $S$ denote an isolated invariant set under $\mathcal{V}$ and let $\mathcal{V}'$ denote an atomic coarsening of $\mathcal{V}$ where the unique merged multivector $V \in \mathcal{V}' \setminus \mathcal{V}$ satisfies the formulae $V \cap S \neq \emptyset$ and $V \not\subseteq S$. Furthermore, let $A := \langle S \cup V \rangle_{\mathcal{V}'}$. If $S \neq \text{inv}_{\mathcal{V}}(A)$, then there does not exist an isolated invariant set $S'$ under $\mathcal{V}'$ for which there is an index pair $(P, E)$ satisfying $\text{inv}_{\mathcal{V}}(P \setminus E) = S$ and $\text{inv}_{\mathcal{V}'}(P \setminus E) = S'$.
Continuation of an isolated invariant set
Continuation of an isolated invariant set
Continuation of an isolated invariant set
Continuation of an isolated invariant set
Continuation of an isolated invariant set

Case 4

S
Continuation of an isolated invariant set
Continuation of an isolated invariant set

Case 1
Continuation of an isolated invariant set
Continuation of an isolated invariant set
Continuation of an isolated invariant set

Case 1
Continuation of an isolated invariant set
Continuation of an isolated invariant set
Continuation of an isolated invariant set
Continuation of an isolated invariant set
Continuation of an isolated invariant set

Case 3
Continuation of an isolated invariant set
The canonicity of the choice

Theorem 25; Dey, L., Mrozek, Slechta (2022)

Let $S$ be an isolated invariant set under $\mathcal{V}$, and let $S'$ denote an isolated invariant set under $\mathcal{V}'$ that is obtained by applying the Tracking Protocol. If $S'$ is obtained via Steps 1, 2, or 3, then $S' \subseteq S$.

Theorem 26; Dey, L., Mrozek, Slechta (2022)

Let $S$ be an isolated invariant set under $\mathcal{V}$, and let $S'$ denote an isolated invariant set under $\mathcal{V}'$ that is obtained by applying the Tracking Protocol. If $S'$ is obtained via Step 4 then $S \subseteq S'$ or $S' \subseteq S$. 
Continuation in terms of persistence

\[ U_0 \rightarrow U_1 \rightarrow U_2 \rightarrow U_3 \rightarrow \cdots \rightarrow U_n \]

\[ S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow \cdots \rightarrow S_n \]

\[ (P_{e_{01}}, E_{01}) \rightarrow (P_{e_{12}}, E_{12}) \rightarrow (P_{e_{23}}, E_{23}) \rightarrow \cdots \rightarrow (P_{e_{n,n}}, E_{n,n}) \]
Continuation in terms of persistence
Continuation in terms of persistence
\[(P, E) \supseteq (\text{cl } S, \text{mo } S) \subseteq (P', E')\]
\( (P, E) \supseteq (\text{cl } S, \text{mo } S) \subseteq (P', E') \)

\[
(\text{cl}(S), \text{mo}(S)) \subseteq \\
(\text{pf}_{\nabla'}(\text{cl}(S), P'), \text{pf}_{\nabla'}(\text{mo}(S), P')) \supseteq \\
(P' \cap \text{pf}_{\nabla'}(\text{cl}(S), P'), E' \cap \text{pf}_{\nabla'}(\text{mo}(S), P')) \subseteq (P', E')
\]
\[(P, E) \supseteq (\text{cl } S, \text{mo } S) \subseteq (P', E')\]

\[(P, E) \supseteq (P \cap \text{pf}_\forall (\text{cl}(S), P), E \cap \text{pf}_\forall (\text{mo}(S), P)) \]
\[\subseteq (\text{pf}_\forall (\text{cl}(S), P), \text{pf}_\forall (\text{mo}(S), P)) \]
\[\supseteq (\text{cl}(S), \text{mo}(S)) \subseteq \]
\[(\text{pf}_\forall (\text{cl}(S), P'), \text{pf}_\forall (\text{mo}(S), P')) \supseteq \]
\[(P' \cap \text{pf}_\forall (\text{cl}(S), P'), E' \cap \text{pf}_\forall (\text{mo}(S), P')) \subseteq (P', E')\]
Connecting index pairs

Let \( S \) be isolated invariant set under \( \mathcal{V} \) isolated by \( N \). Let \( P \) and \( E \) be closed sets such that \( E \subseteq P \). If the following conditions hold, then \((P, E)\) is an **index pair in** \( N \) for \( S \):

1. \( \Pi_{\mathcal{V}}(P \setminus E) \subseteq N \),
2. \( \Pi_{\mathcal{V}}(E) \cap N \subseteq E \),
3. \( \Pi_{\mathcal{V}}(P) \cap N \subseteq P \),
4. \( S = \text{inv}_{\mathcal{V}}(P \setminus E) \).

**Theorem 21; Dey, L., Mrozek, Slechta (2022)**

Let \((P, E)\) and \((P', E')\) denote index pairs for \( S \) in \( N \) under \( \mathcal{V} \). The pair \((P \cap P', E \cap E')\) is an index pair for \( S \) in \( N \) under \( \mathcal{V} \).
Theorem 28; Dey, L., Mrozek, Slechta (2022)

Let \((P, E)\) and \((P', E')\) denote index pairs for \(S\) under \(\mathcal{V}\) such that \(P \subseteq P'\) and \(E \subseteq E'\). Then the inclusion \(i : (P, E) \hookrightarrow (P', E')\) induces an isomorphism in homology.

The **push-forward of a set** \(A\) in \(N\) is defined as
\[
\text{pf}_\mathcal{V}(A, N) := \{x \in N \mid \exists \rho \in \text{Sol}(x, N), k \in \mathbb{N}, \rho(0) \in A, \rho(k) = x\}.
\]

\[
(\text{cl}(S), \text{mo}(S)) \subseteq (\text{pf}_\mathcal{V}(\text{cl}(S), P'), \text{pf}_\mathcal{V}(\text{mo}(S), P'))
\]
\[
\supseteq (P' \cap \text{pf}_\mathcal{V}(\text{cl}(S), P'), E' \cap \text{pf}_\mathcal{V}(\text{mo}(S), P'))
\]
\[
\subseteq (P', E')
\]
(P, E) in N
\((s, m_0 S)\)
\((dS, mos) \leq (pf(dS, N), pf(mos, N))\)
\((p^f)_{(dS,N)} \circ P, p^f_{(m0S,N)} \circ E)\)
\[(P, E) \supseteq (P \cap pf_{\mathcal{V}}(\text{cl}(S), P), E \cap pf_{\mathcal{V}}(\text{mo}(S), P))\]
\[\subseteq (pf_{\mathcal{V}}(\text{cl}(S), P), pf_{\mathcal{V}}(\text{mo}(S), P))\]
\[\supseteq (\text{cl}(S), \text{mo}(S)) \subseteq\]
\[(pf_{\mathcal{V}'}(\text{cl}(S), P'), pf_{\mathcal{V}'}(\text{mo}(S), P')) \supseteq\]
\[(P' \cap pf_{\mathcal{V}'}(\text{cl}(S), P'), E' \cap pf_{\mathcal{V}'}(\text{mo}(S), P')) \subseteq (P', E')\]

**Theorem 22; Dey, L., Mrozek, Slechta (2022)**

For every \(k \geq 0\), the \(k\)-dimensional barcode of a connecting sequence of index pairs \(\{(P_i, E_i)\}_{i=1}^n\) has \(m\) bars \([1, n]\) if
\[\dim H_k(P_1, E_1) = m.\]
Beyond continuation
If it is impossible to track via continuation, then attempt to track via persistence:

6 If $A := \langle S \cup V \rangle_{\mathcal{V}}$, then take $S' := \text{inv}_{\mathcal{V}'}(A)$. If $S$ and $S'$ have a common isolating set, then use the technique from the next slide to find a zigzag filtration connecting them.

7 Otherwise, there is no natural choice of $S'$. 

Persistence of an isolated invariant set
Persistence of an isolated invariant set
Persistence of an isolated invariant set
The canonicity of the choice

Theorem 23; Dey, L., Mrozek, Slechta (2022)

Let $S'$ denote an isolated invariant set under $\mathcal{V}'$ that is obtained from applying Step 6 of the Tracking Protocol to the isolated invariant set $S$ under $\mathcal{V}$. If $S''$ is an isolated invariant set under $\mathcal{V}'$ where $S \subseteq S''$, then $S' \subseteq S''$. 
\[(\text{cl}(S), \text{mo}(S)) \subseteq (\text{pf}_\mathcal{V}(\text{cl}(S), B), \text{pf}_\mathcal{V}(\text{mo}(S), B)) \supseteq (\text{pf}_\mathcal{V}(\text{cl}(S), B) \cap \text{pf}_\mathcal{V}(\text{cl}(S'), B), \text{pf}_\mathcal{V}(\text{mo}(S), B) \cap \text{pf}_\mathcal{V}(\text{mo}(S'), B)) \subseteq (\text{pf}_\mathcal{V}(\text{cl}(S'), B), \text{pf}_\mathcal{V}(\text{mo}(S'), B)) \supseteq (\text{cl}(S'), \text{mo}(S'))\]
\((P, E) \supseteq (P \cap pf_{\mathcal{V}}(cl(S), P), E \cap pf_{\mathcal{V}}(mo(S), P)) \subseteq (pf_{\mathcal{V}}(cl(S), P), pf_{\mathcal{V}}(mo(S), P)) \supseteq (cl(S), mo(S))\)
Thank you!


