Introduction
The distributed computing setting

Task specification:

\((0, 1, 2, 0, 2) \rightarrow (1, 1, 1, 1, 1)\) or \(?\)
The distributed computing setting

Task specification:

\((0, 1, 2, 0, 2) \rightarrow (1, 1, 1, 1, 1) \) or \((0, 1, 2, 0, 2) \rightarrow ? \)
The distributed computing setting

Task specification: $(0, 1, 2, 0, 2) \rightarrow (1, 1, 1, 1, 1)$ or $\text{or}$
The distributed computing setting

Task specification:

\(((0, 1), (2, 0), (2, 2)) \rightarrow ((1, 1), (1, 1), (1, 1), (1, 1), (1, 1))\) or ?
Test-and-set

Task specification: 
\((0, 1, 2, 0, 2) \rightarrow (1, 1, 1, 1, 1)\) or ?
The distributed computing setting

Task specification: 

\[(0, 1, 2, 0, 2) \rightarrow (1, 1, 1, 1, 1)\] or ?
The distributed computing setting

Task specification: \((0, 1, 2, 0, 2) \rightarrow (1, 1, 1, 1, 1)\) \(\checkmark\) or \(\times\)?
Goal: prove **impossibility results** in distributed computing.
**Goal:** prove *impossibility results* in distributed computing.

Various methods:

- Valency arguments (e.g. “FLP impossibility”)
- Epistemic logic (Halpern and Moses 1990)
- Combinatorial topology (Herlihy and Shavit 1999)
Brief overview of this talk

Epistemic logic
(The modal logic of knowledge)

Combinatorial topology
(Simplicial complexes)

Distributed computing
(Fault-tolerant protocols)

Halpern, Moses 1990
Herlihy, Shavit 1999
Saks, Zaharoglou 2000
Brief overview of this talk

Epistemic logic
(The modal logic of knowledge)

Combinatorial topology
(Simplicial complexes)

Distributed computing
(Fault-tolerant protocols)

GLR ’21*
Halpern, Moses 1990

Herlihy, Shavit 1999
Saks, Zaharoglou 2000

*A Simplicial Complex Model for Dynamic Epistemic Logic to study Distributed Task Computability.
Goubault, Ledent, Rajsbaum (2021)
Epistemic Logic
Let $\text{Ag}$ be a finite set of agents and $\text{At}$ a set of atomic propositions.

**Syntax:**

$$\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid K_a \varphi \quad p \in \text{At}, \ a \in \text{Ag}$$

Example formula: $K_a \neg K_b \varphi$ where $a, b \in \text{Ag}$

“$a$ knows that $b$ doesn’t know that the formula $\varphi$ is true.”
Let $Ag$ be a finite set of agents and $At$ a set of atomic propositions.

**Syntax:**

$$\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid K_a \varphi \quad p \in At, \ a \in Ag$$

Example formula: $K_a \neg K_b \varphi$ where $a, b \in Ag$

"a knows that b doesn't know that the formula $\varphi$ is true."

**In distributed computing:**

<table>
<thead>
<tr>
<th>Agents</th>
<th>←→</th>
<th>Processes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Atomic propositions</td>
<td>←→</td>
<td>Facts about the system</td>
</tr>
</tbody>
</table>
Example: the two generals problem

Two divisions of the same army, commanded by general $A$ and general $B$, are surrounding an enemy fortress.
Two divisions of the same army, commanded by general $A$ and general $B$, are surrounding an enemy fortress.

- They must attack simultaneously.
Example: the two generals problem

Two divisions of the same army, commanded by general $A$ and general $B$, are surrounding an enemy fortress.

- They must attack simultaneously.
- They communicate by sending messengers.
Example: the two generals problem

Two divisions of the same army, commanded by general $A$ and general $B$, are surrounding an enemy fortress.

- They must attack simultaneously.
- They communicate by sending messengers.
- Messengers might be captured by the enemy, in which case, the message is never received.
Example: the two generals problem

Two divisions of the same army, commanded by general $A$ and general $B$, are surrounding an enemy fortress.

- They must attack simultaneously.
- They communicate by sending messengers.
- Messengers might be captured by the enemy, in which case, the message is never received.

Fortunately, on this particular night, the enemy guards are asleep. How long will it take to coordinate the attack?
Two divisions of the same army, commanded by general $A$ and general $B$, are surrounding an enemy fortress.

- They must attack simultaneously.
- They communicate by sending messengers.
- Messengers might be captured by the enemy, in which case, the message is never received.

Fortunately, on this particular night, the enemy guards are asleep. How long will it take to coordinate the attack?
Two divisions of the same army, commanded by general $A$ and general $B$, are surrounding an enemy fortress.

- They must attack simultaneously.
- They communicate by sending messengers.
- Messengers might be captured by the enemy, in which case, the message is never received.

Fortunately, on this particular night, the enemy guards are asleep. How long will it take to coordinate the attack?
Example: the two generals problem

Two divisions of the same army, commanded by general $A$ and general $B$, are surrounding an enemy fortress.

- They must attack simultaneously.
- They communicate by sending messengers.
- Messengers might be captured by the enemy, in which case, the message is never received.

Fortunately, on this particular night, the enemy guards are asleep. How long will it take to coordinate the attack?
Simplicial Models for Epistemic Logic
Definition

A chromatic simplicial complex is given by \((V, S, \chi)\) where:

- \((V, S)\) is a simplicial complex,
- \(\chi : V \to Ag\) is a coloring map,

such that every simplex \(X \in S\) has all vertices of distinct colors.

Example: a pure chromatic simplicial complex of dimension 2.
Assume the number of agents is $|\text{Ag}| = n + 1$.

**Definition**

A pure simplicial model is given by $\mathcal{C} = (V, S, \chi, \ell)$ where:

- $(V, S, \chi)$ is a pure chromatic simplicial complex of dimension $n$.
- $\ell : V \rightarrow \mathcal{P}(\text{At})$ is a valuation function.

**Example:** Consider four cards, 1, 2, 3, 4, and three agents. We deal one card to each agent, and keep the remaining card hidden.
Assume the number of agents is $|\text{Ag}| = n + 1$.

**Definition**

A pure simplicial model is given by $\mathcal{C} = (V, S, \chi, \ell)$ where:

- $(V, S, \chi)$ is a pure chromatic simplicial complex of dimension $n$.
- $\ell : V \to \mathcal{P}(\text{At})$ is a valuation function.

**Example:** Consider four cards, 1, 2, 3, 4, and three agents, $a, b, c$. We deal one card to each agent, and keep the remaining card hidden.
Assume the number of agents is $|\text{Ag}| = n + 1$.

**Definition**

A pure simplicial model is given by $\mathcal{C} = (V, S, \chi, \ell)$ where:

- $(V, S, \chi)$ is a pure chromatic simplicial complex of dimension $n$.
- $\ell : V \to \mathcal{P}(\text{At})$ is a valuation function.

**Example:** Consider four cards, 1, 2, 3, 4, and three agents, $a$, $b$, $c$. We deal one card to each agent, and keep the remaining card hidden.
Assume the number of agents is $|\text{Ag}| = n + 1$.

**Definition**

A **pure simplicial model** is given by $\mathcal{C} = (V, S, \chi, \ell)$ where:

- $(V, S, \chi)$ is a pure chromatic simplicial complex of dimension $n$.
- $\ell : V \to \mathcal{P}(\text{At})$ is a valuation function.

**Example:** Consider four cards, 1, 2, 3, 4, and three agents, $a$, $b$, $c$. We deal one card to each agent, and keep the remaining card hidden.
Assume the number of agents is $|Ag| = n + 1$.

**Definition**

A **pure simplicial model** is given by $\mathcal{C} = (V, S, \chi, \ell)$ where:

- $(V, S, \chi)$ is a pure chromatic simplicial complex of dimension $n$.
- $\ell : V \to \mathcal{P}(At)$ is a valuation function.

**Example:** Consider four cards, 1, 2, 3, 4, and three agents, a, b, c. We deal one card to each agent, and keep the remaining card hidden.
Assume the number of agents is $|\text{Ag}| = n + 1$.

**Definition**

A *pure simplicial model* is given by $\mathcal{C} = (V, S, \chi, \ell)$ where:

- $(V, S, \chi)$ is a pure chromatic simplicial complex of dimension $n$.
- $\ell : V \to \mathcal{P}(\text{At})$ is a valuation function.

**Example:** Consider four cards, 1, 2, 3, 4, and three agents, a, b, c. We deal one card to each agent, and keep the remaining card hidden.
Assume the number of agents is $|\text{Ag}| = n + 1$.

**Definition**

A pure simplicial model is given by $\mathcal{C} = (V, S, \chi, \ell)$ where:
- $(V, S, \chi)$ is a pure chromatic simplicial complex of dimension $n$.
- $\ell : V \rightarrow \mathcal{P}(\text{At})$ is a valuation function.

**Example:** Consider four cards, 1, 2, 3, 4, and three agents, $a, b, c$. We deal one card to each agent, and keep the remaining card hidden.
Assume the number of agents is $|\text{Ag}| = n + 1$.

**Definition**

A *pure simplicial model* is given by $\mathcal{C} = (V, S, \chi, \ell)$ where:

- $(V, S, \chi)$ is a pure chromatic simplicial complex of dimension $n$.
- $\ell : V \to \mathcal{P}(\text{At})$ is a valuation function.

**Example:** Consider four cards, 1, 2, 3, 4, and three agents, $a, b, c$. We deal one card to each agent, and keep the remaining card hidden.
We define the validity relation $\mathcal{C}, X \models \varphi$, where:

- $\mathcal{C}$ is a simplicial model,
- $X \in \text{Facet}(\mathcal{C})$ is a world of $\mathcal{C}$,
- $\varphi$ is an epistemic logic formula.
We define the validity relation \( \mathcal{C}, X \models \varphi \), where:

- \( \mathcal{C} \) is a simplicial model,
- \( X \in \text{Facet}(\mathcal{C}) \) is a world of \( \mathcal{C} \),
- \( \varphi \) is an epistemic logic formula.

By induction on \( \varphi \):

\[
\begin{align*}
\mathcal{C}, X &\models p \quad \text{iff} \quad p \in \ell(X) \\
\mathcal{C}, X &\models \neg \varphi \quad \text{iff} \quad \mathcal{C}, X \not\models \varphi \\
\mathcal{C}, X &\models \varphi \land \psi \quad \text{iff} \quad \mathcal{C}, X \models \varphi \quad \text{and} \quad \mathcal{C}, X \models \psi \\
\mathcal{C}, X &\models K_a \varphi \quad \text{iff} \quad \mathcal{C}, Y \models \varphi \quad \text{for all} \ Y \in \text{Facet}(\mathcal{C}) \\
&\quad \text{such that} \ a \in \chi(X \cap Y)
\end{align*}
\]
We define the validity relation \( \mathcal{C}, X \models \varphi \), by induction on \( \varphi \):

\[
\begin{align*}
\mathcal{C}, X & \models p \quad \text{iff} \quad p \in \ell(X) \\
\mathcal{C}, X & \models \neg \varphi \quad \text{iff} \quad \mathcal{C}, X \not\models \varphi \\
\mathcal{C}, X & \models \varphi \land \psi \quad \text{iff} \quad \mathcal{C}, X \models \varphi \quad \text{and} \quad \mathcal{C}, X \models \psi \\
\mathcal{C}, X & \models K_a \varphi \quad \text{iff} \quad \mathcal{C}, Y \models \varphi \quad \text{for all} \quad Y \in \text{Facet}(\mathcal{C}) \\
& \quad \text{such that} \quad a \in \chi(X \cap Y)
\end{align*}
\]

**Example:** \( \mathcal{C}, X \models K_a K_b \text{ value(c)} \neq 1 \)

Agents: \( a, b, c \)
We define the validity relation $C, X \models \varphi$, by induction on $\varphi$:

$C, X \models p$  iff  $p \in \ell(X)$

$C, X \models \neg \varphi$  iff  $C, X \not\models \varphi$

$C, X \models \varphi \land \psi$  iff  $C, X \models \varphi$ and $C, X \models \psi$

$C, X \models K_a \varphi$  iff  $C, Y \models \varphi$ for all $Y \in \text{Facet}(C)$ such that $a \in \chi(X \cap Y)$

Example:  $C, X \models K_a K_b \text{ value}(c) \neq 1$  

Agents:  $a, b, c$
We define the validity relation $\mathcal{C}, X \models \varphi$, by induction on $\varphi$:

- $\mathcal{C}, X \models p$ iff $p \in \ell(X)$
- $\mathcal{C}, X \models \neg \varphi$ iff $\mathcal{C}, X \not\models \varphi$
- $\mathcal{C}, X \models \varphi \land \psi$ iff $\mathcal{C}, X \models \varphi$ and $\mathcal{C}, X \models \psi$
- $\mathcal{C}, X \models K_a \varphi$ iff $\mathcal{C}, Y \models \varphi$ for all $Y \in \text{Facet}(\mathcal{C})$ such that $a \in \chi(X \cap Y)$

Example: $\mathcal{C}, X \models K_a K_b \text{ value}(c) \neq 1$
We define the validity relation $\mathcal{C}, X \models \varphi$, by induction on $\varphi$:

- $\mathcal{C}, X \models p$ iff $p \in \ell(X)$
- $\mathcal{C}, X \models \neg \varphi$ iff $\mathcal{C}, X \not\models \varphi$
- $\mathcal{C}, X \models \varphi \land \psi$ iff $\mathcal{C}, X \models \varphi$ and $\mathcal{C}, X \models \psi$
- $\mathcal{C}, X \models K_a \varphi$ iff $\mathcal{C}, Y \models \varphi$ for all $Y \in \text{Facet}(\mathcal{C})$ such that $a \in \chi(X \cap Y)$

Example: $\mathcal{C}, X \models K_a K_b$ value(c) $\neq 1$  

Agents: a, b, c
Suppose the number of agents is $|\text{Ag}| = n+1$.

**Theorem (Goubault, Ledent, Rajsbaum (2018, 2021))**

The category of pure simplicial models of dimension $n$ is equivalent to the category of proper and local Kripke models.

**Example:** with three agents, $\text{Ag} = \{ a, b, c \}$,

\[ w_1 \cong w_2 \cong w_3 \]

\[ w_1 \xrightarrow{a} w_2 \xrightarrow{b} w_3 \]

\[ w_1 \xrightarrow{c} w_2 \]
Suppose the number of agents is $|\text{Ag}| = n + 1$.

**Theorem (Goubault, Ledent, Rajsbaum (2018, 2021))**

The category of pure simplicial models of dimension $n$ is equivalent to the category of proper and local Kripke models.

**Example:** with three agents, $\text{Ag} = \{a, b, c\}$,
Generalizing Simplicial Models
What about impure simplicial models?

Impure simplicial complexes.

- Common in distributed computing.
- They model systems with detectable crashes.

Contributions:
- Find an equivalent class of Kripke models.
- Axiomatise the logic.

A Simplicial Model for KB4: Epistemic Logic with Agents That May Die

Goubault, Ledent, Rajsbaum (STACS 22)
What about impure simplicial models?

Impure simplicial complexes.
- Common in distributed computing.
- They model systems with detectable crashes.

Contributions:
- Find an equivalent class of Kripke models.
- Axiomatise the logic.

A Simplicial Model for KB4: Epistemic Logic with Agents That May Die.

Goubault, Ledent, Rajsbaum (STACS 22)
Satisfaction relation

Recall the definition of the satisfaction relation, \( \mathcal{C}, X \models \varphi \):

\[
\begin{align*}
\mathcal{C}, X \models p & \quad \text{iff} \quad p \in \ell(X) \\
\mathcal{C}, X \models \neg \varphi & \quad \text{iff} \quad \mathcal{C}, X \not\models \varphi \\
\mathcal{C}, X \models \varphi \land \psi & \quad \text{iff} \quad \mathcal{C}, X \models \varphi \quad \text{and} \quad \mathcal{C}, X \models \psi \\
\mathcal{C}, X \models K_{a} \varphi & \quad \text{iff} \quad \mathcal{C}, Y \models \varphi \quad \text{for all} \ Y \in \text{Facet}(\mathcal{C}) \quad \text{such that} \ a \in \chi(X \cap Y)
\end{align*}
\]
Recall the definition of the satisfaction relation, $\mathcal{C}, X \models \varphi$:

- $\mathcal{C}, X \models p$ iff $p \in \ell(X)$
- $\mathcal{C}, X \models \neg \varphi$ iff $\mathcal{C}, X \not\models \varphi$
- $\mathcal{C}, X \models \varphi \land \psi$ iff $\mathcal{C}, X \models \varphi$ and $\mathcal{C}, X \models \psi$
- $\mathcal{C}, X \models K_a \varphi$ iff $\mathcal{C}, Y \models \varphi$ for all $Y \in \text{Facet}(\mathcal{C})$ such that $a \in \chi(X \cap Y)$
Recall the definition of the satisfaction relation, $\mathcal{C}, X \models \varphi$:

$\mathcal{C}, X \models p$ iff $p \in \ell(X)$

$\mathcal{C}, X \models \neg \varphi$ iff $\mathcal{C}, X \not\models \varphi$

$\mathcal{C}, X \models \varphi \land \psi$ iff $\mathcal{C}, X \models \varphi$ and $\mathcal{C}, X \models \psi$

$\mathcal{C}, X \models K_a \varphi$ iff $\mathcal{C}, Y \models \varphi$ for all $Y \in \text{Facet}(\mathcal{C})$

such that $a \in \chi(X \cap Y)$

**Example:** with $\text{Ag} = \{a, b, c\}$ and $\text{At} = \{p\}$.

$p$ is true in $X_1$ only.
Recall the definition of the satisfaction relation, $\mathcal{C}, X \models \varphi$:

- $\mathcal{C}, X \models p$ iff $p \in \ell(X)$
- $\mathcal{C}, X \models \neg \varphi$ iff $\mathcal{C}, X \not\models \varphi$
- $\mathcal{C}, X \models \varphi \land \psi$ iff $\mathcal{C}, X \models \varphi$ and $\mathcal{C}, X \models \psi$
- $\mathcal{C}, X \models K_a \varphi$ iff $\mathcal{C}, Y \models \varphi$ for all $Y \in \text{Facet}(\mathcal{C})$
  such that $a \in \chi(X \cap Y)$

**Example:** with $\text{Ag} = \{a, b, c\}$ and $\text{At} = \{p\}$.

$p$ is true in $X_1$ only.
Recall the definition of the satisfaction relation, $\mathcal{C}, X \models \varphi$:

- $\mathcal{C}, X \models p$ iff $p \in \ell(X)$
- $\mathcal{C}, X \models \neg \varphi$ iff $\mathcal{C}, X \not\models \varphi$
- $\mathcal{C}, X \models \varphi \land \psi$ iff $\mathcal{C}, X \models \varphi$ and $\mathcal{C}, X \models \psi$
- $\mathcal{C}, X \models K_a \varphi$ iff $\mathcal{C}, Y \models \varphi$ for all $Y \in \text{Facet}(\mathcal{C})$ such that $a \in \chi(X \cap Y)$

**Example:** with $\text{Ag} = \{a, b, c\}$ and $\text{At} = \{p\}$.

- $\mathcal{C}, X_1 \models K_a p$
- $\mathcal{C}, X_1 \models \neg K_b p$

$p$ is true in $X_1$ only.
Recall the definition of the satisfaction relation, $\mathcal{C}, X \models \varphi$:

- $\mathcal{C}, X \models p$ iff $p \in \ell(X)$
- $\mathcal{C}, X \models \neg \varphi$ iff $\mathcal{C}, X \not\models \varphi$
- $\mathcal{C}, X \models \varphi \land \psi$ iff $\mathcal{C}, X \models \varphi$ and $\mathcal{C}, X \models \psi$
- $\mathcal{C}, X \models K_a \varphi$ iff $\mathcal{C}, Y \models \varphi$ for all $Y \in \text{Facet}(\mathcal{C})$ such that $a \in \chi(X \cap Y)$

**Example:** with $\text{Ag} = \{a, b, c\}$ and $\text{At} = \{p\}$.

- $\mathcal{C}, X_1 \models K_a p$
- $\mathcal{C}, X_1 \models \neg K_b p$
- $\mathcal{C}, X_4 \models (K_b \neg p) \land (K_c \neg p)$

$p$ is true in $X_1$ only.
Recall the definition of the satisfaction relation, $\mathcal{C}, X \models \varphi$:

- $\mathcal{C}, X \models p$ iff $p \in \ell(X)$
- $\mathcal{C}, X \models \neg \varphi$ iff $\mathcal{C}, X \not\models \varphi$
- $\mathcal{C}, X \models \varphi \land \psi$ iff $\mathcal{C}, X \models \varphi$ and $\mathcal{C}, X \models \psi$
- $\mathcal{C}, X \models K_a \varphi$ iff $\mathcal{C}, Y \models \varphi$ for all $Y \in \text{Facet}(\mathcal{C})$ such that $a \in \chi(X \cap Y)$

**Example:** with $\text{Ag} = \{a, b, c\}$ and $\text{At} = \{p\}$.

- $\mathcal{C}, X_1 \models K_a p$
- $\mathcal{C}, X_1 \models \neg K_b p$
- $\mathcal{C}, X_4 \models (K_b \neg p) \land (K_c \neg p)$
- $\mathcal{C}, X_2 \models K_a p$

$p$ is true in $X_1$ only.
Recall the definition of the satisfaction relation, $\mathcal{C}, X \models \varphi$:

- $\mathcal{C}, X \models p$ iff $p \in \ell(X)$
- $\mathcal{C}, X \models \neg \varphi$ iff $\mathcal{C}, X \not\models \varphi$
- $\mathcal{C}, X \models \varphi \land \psi$ iff $\mathcal{C}, X \models \varphi$ and $\mathcal{C}, X \models \psi$
- $\mathcal{C}, X \models K_a \varphi$ iff $\mathcal{C}, Y \models \varphi$ for all $Y \in \text{Facet}(\mathcal{C})$ such that $a \in \chi(X \cap Y)$

**Example:** with $\text{Ag} = \{a, b, c\}$ and $\text{At} = \{p\}$.

- $\mathcal{C}, X_1 \models K_a p$
- $\mathcal{C}, X_1 \models \neg K_b p$
- $\mathcal{C}, X_4 \models (K_b \neg p) \land (K_c \neg p)$
- $\mathcal{C}, X_2 \models K_a p$
- $\mathcal{C}, X_1 \models K_b K_a p$

$p$ is true in $X_1$ only.
Define the following formulas, for an agent $a \in Ag$:

$$\text{dead}(a) := K_a \text{false} \quad \text{alive}(a) := \neg \text{dead}(a)$$

One can check that:

$$\mathcal{C}, w \models \text{alive}(a) \quad \text{iff} \quad a \in \chi(w)$$
Define the following formulas, for an agent \( a \in \text{Ag} \):

\[
\text{dead}(a) := K_a \text{false} \quad \text{alive}(a) := \neg \text{dead}(a)
\]

One can check that:

\[
\mathcal{C}, w \models \text{alive}(a) \quad \text{iff} \quad a \in \chi(w)
\]

**Example:** Some valid formulas in KB4:

- Dead agents know everything: \( \text{KB4} \vdash \text{dead}(a) \Rightarrow K_a \varphi \).
- Alive agents know they are alive: \( \text{KB4} \vdash \text{alive}(a) \Rightarrow K_a \text{alive}(a) \).
- Alive agents satisfy Axiom T: \( \text{KB4} \vdash \text{alive}(a) \Rightarrow (K_a \varphi \Rightarrow \varphi) \).
Simplicial set models

Definition

A **pre-simplicial set** is given by a sequence of sets \((S_n)_{n \in \mathbb{N}}\), together with maps \(d_i^n : S_n \to S_{n-1}\) for every \(n \in \mathbb{N}\) and \(0 \leq i \leq n\), satisfying the **simplicial identities**.

\[
\begin{align*}
S_0 & \xleftarrow{d_0} S_1 & S_1 & \xleftarrow{d_0} S_2 & S_2 & \xleftarrow{d_0} S_3 & \ldots \\
& \xleftarrow{d_1} & & \xleftarrow{d_1} & & \xleftarrow{d_1} \\
& & \xleftarrow{d_2} & & \xleftarrow{d_2} & \\
& & & \xleftarrow{d_3} & \\
\end{align*}
\]
Simplicial set models

**Definition**

A **pre-simplicial set** is given by a sequence of sets \((S_n)_{n \in \mathbb{N}}\), together with maps \(d^n_i : S_n \to S_{n-1}\) for every \(n \in \mathbb{N}\) and \(0 \leq i \leq n\), satisfying the **simplicial identities**.

\[
\begin{align*}
S_0 & \xleftarrow{d_0} S_1 \xleftarrow{d_1} S_2 \xleftarrow{d_2} S_3 \ldots \\
S_1 & \xleftarrow{d_0} S_2 \xleftarrow{d_1} S_3 \ldots \\
S_2 & \xleftarrow{d_0} S_3 \ldots \\
S_3 & \ldots
\end{align*}
\]

**Idea:**

- Define simplicial models based on (pre-)simplicial sets.
- What is the associated logic?
- What are some use cases?
Applications to Distributed Computing
Topological characterization of task solvability (Herlihy et al.)

Input complex
Topological characterization of task solvability (Herlihy et al.)
Topological characterization of task solvability (Herlihy et al.)

Protocol complex

Execution

Input complex

Output complex

Task specification
Topological characterization of task solvability (Herlihy et al.)

∃ Decision?

Protocol complex

Output complex

Execution

Task specification

Input complex
Idea: find a logical obstruction to the existence of the simplicial map $\delta$. 
**Epistemic proofs of impossibility**

**Idea:** find a **logical obstruction** to the existence of the simplicial map $\delta$.

**Lemma (Knowledge Gain)**

Let $\delta : \mathcal{C} \to \mathcal{C}'$ be a morphism of simplicial models, and let $\varphi$ be a **positive formula**. Then:

$$\mathcal{C}', \delta(X) \models \varphi \quad \text{implies} \quad \mathcal{C}, X \models \varphi$$
Epistemic proofs of impossibility

Idea: find a logical obstruction to the existence of the simplicial map $\delta$.

Lemma (Knowledge Gain)

Let $\delta : \mathcal{C} \rightarrow \mathcal{C}'$ be a morphism of simplicial models, and let $\varphi$ be a positive formula. Then:

$$\mathcal{C}', \delta(X) \models \varphi \quad \text{implies} \quad \mathcal{C}, X \models \varphi$$

Recipe for impossibility proofs:

- Assume by contradiction that $\delta : \mathcal{P} \rightarrow \mathcal{O}$ exists.
- Choose a suitable formula $\varphi$ such that:
  - $\varphi$ is true everywhere in the output model
  - $\varphi$ is false somewhere in the protocol model

- ✔️ **Consensus**: impossibility proof using common knowledge.
- ✔️ **Approximate agreement**: impossibility proof using iterated knowledge.
- ✗ **Set agreement**: an impossibility proof is given, but the formula is unsatisfactory.
Results


- ✔ **Consensus**: impossibility proof using common knowledge.
- ✔ **Approximate agreement**: impossibility proof using iterated knowledge.
- ✗ **Set agreement**: an impossibility proof is given, but the formula is unsatisfactory.

Nishimura, Yagi, *Logical Obstruction to Set Agreement Tasks for Superset-Closed Adversaries* (2020)

- ✔ **Set agreement**: a more informative impossibility formula is used, but only for one round.

- ✔️ **Consensus**: impossibility proof using common knowledge.
- ✔️ **Approximate agreement**: impossibility proof using iterated knowledge.
- ✗ **Set agreement**: an impossibility proof is given, but the formula is unsatisfactory.

Nishimura, Yagi, *Logical Obstruction to Set Agreement Tasks for Superset-Closed Adversaries* (2020)

- ✔️ **Set agreement**: a more informative impossibility formula is used, but only for one round.


- ✗ **Equality negation**: no formula can prove impossibility.
Research directions
Distributed knowledge. $D_B \varphi$, where $B \subseteq \text{Ag}$.

- A group of agents put their knowledge in common.
- In simplicial models: simplexes sharing a $B$-coloured face.
Distributed knowledge. $D_B \varphi$, where $B \subseteq Ag$.

- A group of agents put their knowledge in common.
- In simplicial models: simplexes sharing a $B$-coloured face.

Common Distributed Knowledge. $CD_\beta \varphi$, where $\beta \subseteq \mathcal{P}(Ag)$

- Infinitary iteration of distributed knowledge.
- Subsumes both common knowledge and distributed knowledge.
Distributed knowledge. $D_B \varphi$, where $B \subseteq \text{Ag}$.

- A group of agents put their knowledge in common.
- In simplicial models: simplexes sharing a $B$-coloured face.

Common Distributed Knowledge. $CD_\beta \varphi$, where $\beta \subseteq \mathcal{P}(\text{Ag})$

- Infinitary iteration of distributed knowledge.
- Subsumes both common knowledge and distributed knowledge.

Other topological operators?

<table>
<thead>
<tr>
<th>Distributed computing</th>
<th>Topology</th>
<th>Logic</th>
</tr>
</thead>
<tbody>
<tr>
<td>consensus</td>
<td>connectedness</td>
<td>common knowledge</td>
</tr>
<tr>
<td>$k$-set agreement</td>
<td>$k$-connectedness</td>
<td>???</td>
</tr>
</tbody>
</table>
Topology vs logic: can we characterize topological properties via logical formulas?
Topology vs logic: can we characterize topological properties via logical formulas?

Examples:

- Is there a $\varphi$ such that $C \models \varphi$ iff $C$ is a (pseudo-)manifold?
- Is there a sound and complete axiomatization for the class of collapsible simplicial models?
- Which logical formulas are preserved under subdivision?
Theorem

There is a bijection between facets of the $n$-dimensional chromatic subdivision and cube chains in the $(n+1)$-dimensional cube.
There is an order isomorphism between the face poset of the $n$-dimensional chromatic subdivision and the poset of partial cube chains in the $(n+1)$-dimensional cube.
Thanks!