COLIMITS OF LOCAL ORDERS

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What is a local order?
A partially ordered space (or pospace) is $(X, \sqsubseteq)$ such that:

- $X$: topological space
- $\sqsubseteq$: partial order on $X$
- $\{ (a, b) \in X \times X | a \sqsubseteq b \}$ closed in $X \times X$

A pospace morphism is an order-preserving continuous map.

Pospaces and their morphisms form the category $\text{PoSp}$.

The underlying space of a pospace is Hausdorff.
A partially ordered space (or pospace) is \((X, \sqsubseteq)\) such that:
- \(X\) : topological space
- \(\sqsubseteq\) : partial order on \(X\)
- \(\{(a, b) \in X \times X \mid a \sqsubseteq b\}\) closed in \(X \times X\)

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Local order

A collection $\mathcal{U}$ of pospaces such that:
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for all points $p \in |\mathcal{U}|$,
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A collection $\mathcal{U}$ of pospaces such that:

for all points $p \in |\mathcal{U}|$, for all $U_p \in \mathcal{U}$, for all $V_p \in \mathcal{U}$,

there exists $W_p$ open in both $U_p$ and $V_p$ on which $\leq_{U_p}$ and $\leq_{V_p}$ match
Local order morphism from $\mathcal{U}$ to $\mathcal{V}$

A map $f : |\mathcal{U}| \to |\mathcal{V}|$ such that:

...
Local order morphism from $\mathcal{U}$ to $\mathcal{V}$

A map $f : |\mathcal{U}| \rightarrow |\mathcal{V}|$ such that:

for all points $p \in |\mathcal{U}|$,
Local order morphism from $\mathcal{U}$ to $\mathcal{V}$

A map $f : \mathcal{U} \rightarrow \mathcal{V}$ such that:
for all points $p \in \mathcal{U}$, all $V_{f(p)} \subset \mathcal{V}$,
Local order morphism from $\mathcal{U}$ to $\mathcal{V}$

A map $f : |\mathcal{U}| \rightarrow |\mathcal{V}|$ such that:

for all points $p \in |\mathcal{U}|$, all $V_{f(p)} \in \mathcal{V}$, and all $B_{f(p)}$ open subsets of $V$, 
Local order morphism from $\mathcal{U}$ to $\mathcal{V}$

A map $f : |\mathcal{U}| \to |\mathcal{V}|$ such that:
for all points $p \in |\mathcal{U}|$, all $V_{f(p)} \in \mathcal{V}$, and all $B_{f(p)}$ open subsets of $\mathcal{V}$, there exist $U_p \in \mathcal{U}$,
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A map $f : |\mathcal{U}| \to |\mathcal{V}|$ such that:

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there exist $U_p \in \mathcal{U}$, and $A_p$ open subset of $U_p$ such that . . .
Local order morphism from $\mathcal{U}$ to $\mathcal{V}$

A map $f : |\mathcal{U}| \to |\mathcal{V}|$ such that:
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there exist $U_p \in \mathcal{U}$, and $A_p$ open subset of $U_p$ such that
$f(A_p) \subseteq B_{f(p)}$ and the corresponding restriction is order preserving.
Where do local orders come from?
Local order

Bibliography

*Mathematical Cosmology and Extragalactic Astronomy,*
I. E. Segal, 1976, p.23.

*Ordered Manifolds, Invariant Conal Fields, and Semigroups,*

*Algebraic topology and concurrency,*
L. Fajstrup, É. Goubault, and M. Raussen, TCS 2006, p.245. (also MFCS 1998)
What local orders have to do with concurrency?
The box category $\square$

\[ \{ \text{Objects of } \square \} = \mathbb{N} \quad \square(a, b) = \{ w \in \{0, 1, x\}^b \mid \#_x w = a \} \]

\[ \#_x w : \text{number of occurrences of } x \text{ in } w \in \{0, 1, x\}^* \]

Given $\alpha \in \square(a, b)$ and $\beta \in \square(b, c)$, the composite $\beta \circ \alpha$ is obtained by replacing the $i^{th}$ occurrence of $x$ in $\beta$ by the $i^{th}$ occurrence of $\alpha$ (for $i \in \{1, \ldots, b\}$).

\[
\begin{array}{c|c|c}
\beta \circ \alpha & \beta & \alpha \\
0 & 0 & 0 \\
0 & x & x \\
x & x & 1 \\
1 & 1 & \\
1 & x & \\
\end{array}
\]
A precubical set $K$ is a functor from $\square^{op}$ to $\text{Set}$.

The precubical set morphisms are the natural transformations.

Precubical sets and their morphisms form the category $\mathbf{PCS}$.

If $K_n = \emptyset$ for $n \geq 2$ then $K$ is a graph.

A bouquet is a graph with a single vertex.

An automaton is a graph morphism whose target is a bouquet.
A higher dimensional bouquet (HDB) is a precubical set $B$ whose $n$-dimensional elements are words of length $n$.

For any $w \in □(n, m)$, $B_w(a_1 \cdots a_m) = (a_{i_1} \cdots a_{i_n})$
with $\{i_1 < \cdots < i_n\} = \{i \in \{1, \ldots, m\} \mid \text{the } i^{th} \text{ letter of } w \text{ is } x\}$

e.g.: $B_{0xx1x}(abcde) = (bce)$

HDA : precubical set morphism whose target is a bouquet
Precubical realization

From any precubical set $K$ we deduce the diagram $D_K$

$$\{K_w(x)\} \times [0, 1]^n \to \{x\} \times [0, 1]^m$$

$$(K_w(x), t) \mapsto (x, w(t))$$

with $n, m \in \mathbb{N}$, $x \in K_m$, and $w \in \square(n, m)$.

The realization of $K$ is the colimit of $D_K$ in the category in which $[0, 1]$ is interpreted.
Which category to choose?
\textbf{d-spaces}

\textit{Directed homotopy theory, I, M. Grandis, 2003}

\[ X : \text{topological space} \]

\[ dX : \text{collection of paths on } X \text{ containing constant paths and stable under concatenation, reparametrization, and subpaths}. \]

A morphism from \((X, dX)\) to \((Y, dY)\) is a continuous map \(f : X \to Y\) such that \(f \circ dX \subseteq dY\).

d-spaces and morphisms form the category \(dSpc\).
Vortex from the computer science viewpoint
allow infinitely many turns in finite time
Vortex from the computer science viewpoint
allow loop deletion
Vortex from the computer science viewpoint
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allow loop deletion
Local orders and d-spaces

Introducing vortex

\[ U : \text{open subset of } X \]
Local orders and d-spaces

Introducing vortex

\[ U : \text{open subset of } X \]

\[ p \preceq_u q : \text{there exists } \delta \in dX \text{ from } p \text{ to } q \text{ with } \text{img}(\delta) \subseteq U \]
Local orders and d-spaces

Introducing vortex

\[ U : \text{open subset of } X \]

\[ p \triangleleft_U q : \text{there exists } \delta \in dX \text{ from } p \text{ to } q \text{ with } \text{img}(\delta) \subseteq U \]

Is there an open covering \( \mathcal{U} \) of \( X \) such that \( \{(U, \triangleleft_U) \mid U \in \mathcal{U}\} \) is a local order?
q.

p.
Lawson correspondence
Wedges and cones

$E : \text{real vector space}$

$W \subseteq E$ is a **wedge** when $\lambda W + W \subseteq W$ for all $\lambda \in \mathbb{R}_+$

A wedge $C$ is a **cone** when $C \cap (-C) = \{0\}$
A cone field $C$ is a map $p \in M \mapsto (C_p : \text{cone of } T_p M)$

[ Wedge fields are defined the same way ]

$(M, C)$ is a conal manifold

A curve on $M$ is conal when $\dot{\gamma}(t) \in C(\gamma(t))$ for all $t \in \text{dom}(\gamma)$
$(\mathcal{M}, dC)$ is a d-space with $dC = \{\text{conal curves on } (\mathcal{M}, C)\}$

If $C$ is upper semicontinuous then there is an open covering $\mathcal{U}$ of $\mathcal{M}$ such that:

1. $\{(U, \preceq_U) \mid U \in \mathcal{U}\}$ is a local order, and
2. each $(U, \preceq_U)$ is locally convex (i.e. the order convex open subsets of $U$ form a basis of topology)
Cone fields $\leftrightarrow$ d-spaces and local orders

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E.g. The local order on the unit circle comes from the vector field $z \mapsto iz$
Local orders $\mapsto$ cone fields

$\mathcal{U}$ : local order on $\mathcal{M}$ whose elements are locally convex
Local orders $\mapsto$ cone fields

$\mathcal{U}$ : local order on $\mathcal{M}$ whose elements are locally convex

For all $p \in \mathcal{M}$, the least wedge of $T_p \mathcal{M}$ containing $\dot{\gamma}(0)$ for all curves $\gamma$ with $\gamma(0) = p$ is a cone $C(p)$, and ...
Local orders $\mapsto$ cone fields

$\mathcal{U}$ : local order on $\mathcal{M}$ whose elements are locally convex

For all $p \in \mathcal{M}$, the least wedge of $T_p \mathcal{M}$ containing $\dot{\gamma}(0)$ for all curves $\gamma$ with $\gamma(0) = p$ is a cone $C(p)$, and

the mapping $p \mapsto C_p$ is upper semicontinuous.
Lawson correspondence

upper semicontinuous cone fields on $\mathcal{M}$

$\uparrow$

locally convex local orders on $\mathcal{M}$
How local order colimits behave?
Pinching a loop

Setting the problem

\( S \) : the directed unit circle

\( \mathcal{U} \) : local order with a distinguished point \( \star \)

\( i_* : z \in S \mapsto (z, \star) \in S \times \mathcal{U} \)

\( c_* : z \in S \mapsto (1, \star) \in S \times \mathcal{U} \)

The coequalizer of \( i_* \) and \( c_* \) exists iff the family of clopens containing \( \star \) has a least element \( O \)

[NB: connected component of \( \star \) \( \subseteq \cap \{ \text{clopens containing } \star \} \), with equality if the topology of \( \mathcal{U} \) is locally connected.]
Pinching a loop

Description of the colimit

For $A \in \mathcal{U}$ and $\alpha$ proper arc of $S$ define

$$U_{\alpha, A} = (O \cap A) \sqcup \alpha \times (O^c \cap A)$$

$$p(u) = \begin{cases} u & \text{if } u \in O \cap A \\ x & \text{if } u = (z, x) \in \alpha \times (O^c \cap A) \end{cases}$$

$u \sqsubseteq u'$ when $p(u) \leq_A p(u')$ and

$$\begin{cases} \{u, u'\} \cap O \neq \emptyset \\ \text{or} \\ u = (z, x), u' = (z', x'), \text{ and } z \leq_{\alpha} z' \end{cases}$$
Pinching a loop

Description of the colimit

The transitive closure $\sqsubseteq^*$ is antisymmetric, and we have $u \sqsubseteq^* u'$ iff there exists $u'' \in O \cap A$ such that $u \sqsubseteq u'' \sqsubseteq u'$.

The collection $\left\{ U_{\alpha, A} \mid \alpha : \text{proper arc} ; A \in \mathcal{U} \right\}$ is a local order we denote by $\mathcal{U}_O$.

The quotient map $q_O : S \times \mathcal{U} \to \mathcal{U}_O$ is the coequalizer of $i_*$ and $c_*$. 

If $\mathcal{U}$ is $\mathbb{R}$ or $S$ (with the obvious local order), then the coequalizer is $\mathcal{U}$.

If $\mathcal{U}$ is $\mathbb{Q}$ (sub local order of $\mathbb{R}$) or $\mathbb{R}_*$ (with the least topology containing all the usual open subsets and all the single elements set but $\{ \star \}$) the coequalizer does not exist.
The zebra cylinder (no more topological trick)
The underlying topology is the standard cylinder $S \times [0, 1]$
Locally ordered realization
of a non-geometric precubical set
A non-geometric precubical set

\[ K_n = \begin{cases} \{n\} & \text{if } n \leq 2 \\ \emptyset & \text{if } n > 2 \end{cases} \]
A non-geometric precubical set

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Neighborhood of a point of the edge
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Neighborhood of the vertex
Neighborhood of the vertex
Neighborhood of the vertex

A \times B

B \times A

\{ \ast \}

B \times B

A \times A

A \times B
Neighborhood of the vertex
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Coherence

\[ p \in W_\star \cap W'_\star \]

\[ p \in W_u \cap W_\star \]

\[ p \in W_u \cap W'_u \]