DIAGRAMMATIC SETS & TOPOLOGICALLY SOUND REWRITING

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Main reference:

arXiv: 2007.14505

These slides on:

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Central Idea of Higher-Dimensional Rewriting

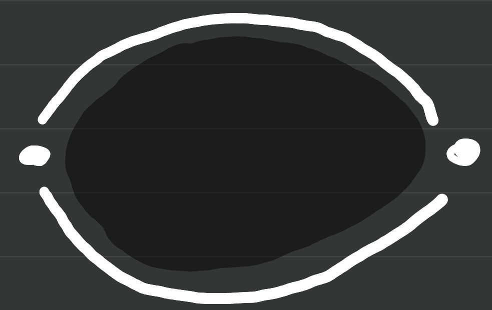
REWRITE SYSTEMS = DIRECTED CELL COMPLEXES

CELL COMPLEX

↳ Space "assembled from n-balls, glued by their (n-1)-sphere boundaries"

Minimal REGULAR cell structure on n-balls:

1. n-dimensional cell,
2. k-dimensional cells for $k<n$

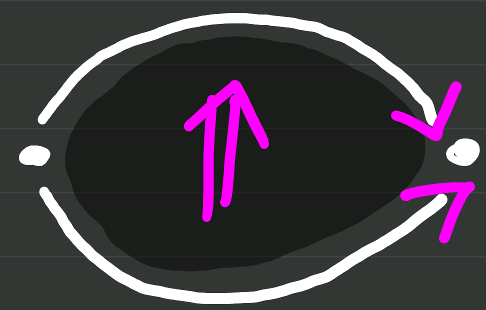


DIRECTED CELL COMPLEX

↳ Space "assembled from DIRECTED n-balls, glued by their (n-1)-sphere boundaries"

Minimal REGULAR cell structure on DIRECTED n-balls:

1. n-dimensional cell,
2. k-dimensional cells for $k<n$



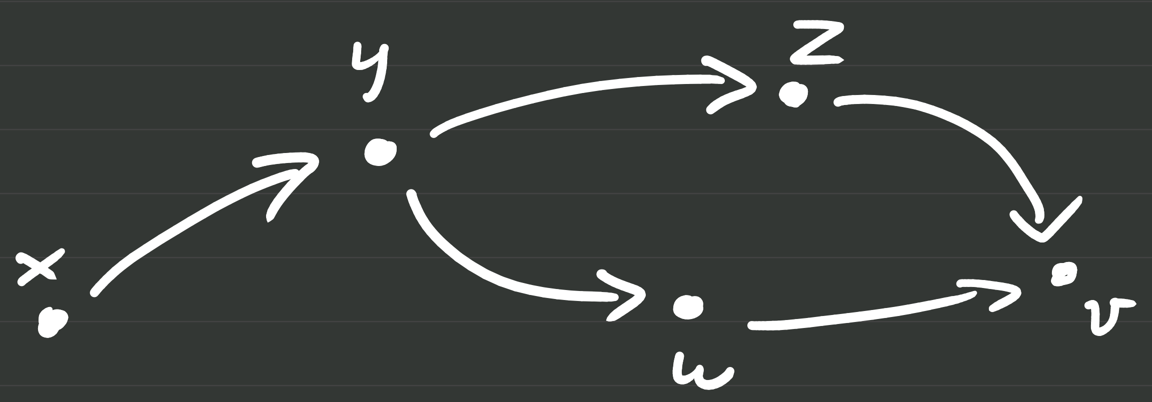
A directed n-ball has its boundary subdivided into an INPUT half and an OUTPUT half, each of which is a directed $(n-1)$-ball.

A particular kind of orientation, inspired by higher category theory

$\boxed{1D}$ Abstract Rewrite Systems

$\sim$ Directed Graphs

$\sim$ Directed 1D cell complexes

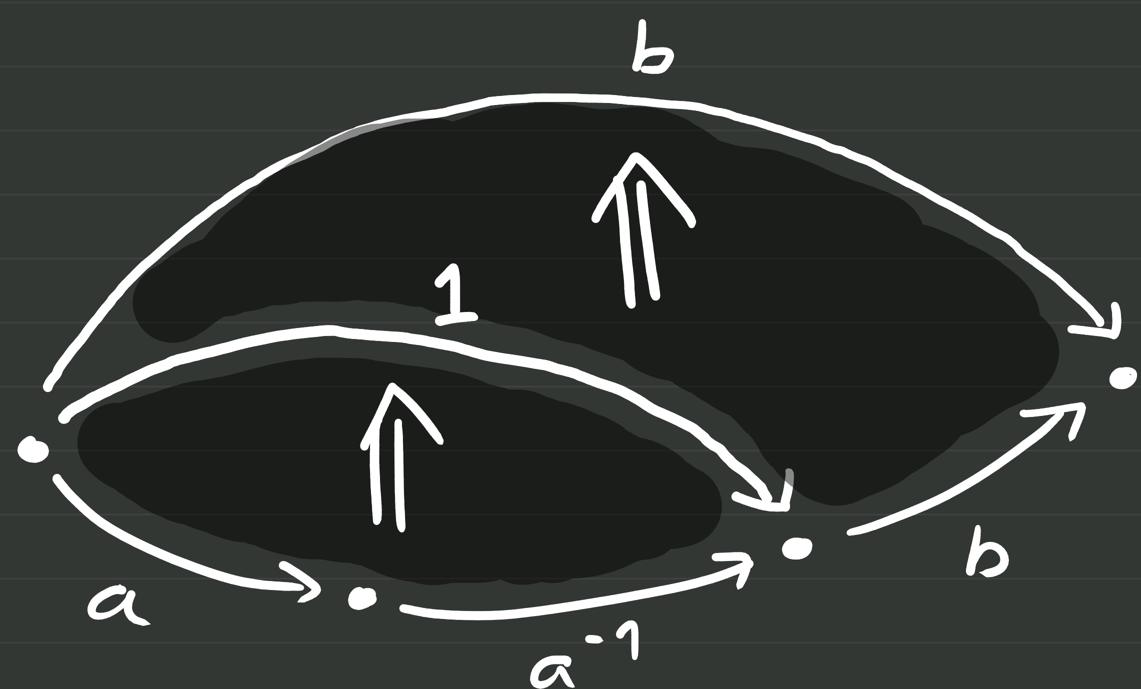


Rewrite sequence $\sim$ Directed homotopy

$\boxed{2D}$ String Rewrite Systems

~ Rewriting paths in graphs

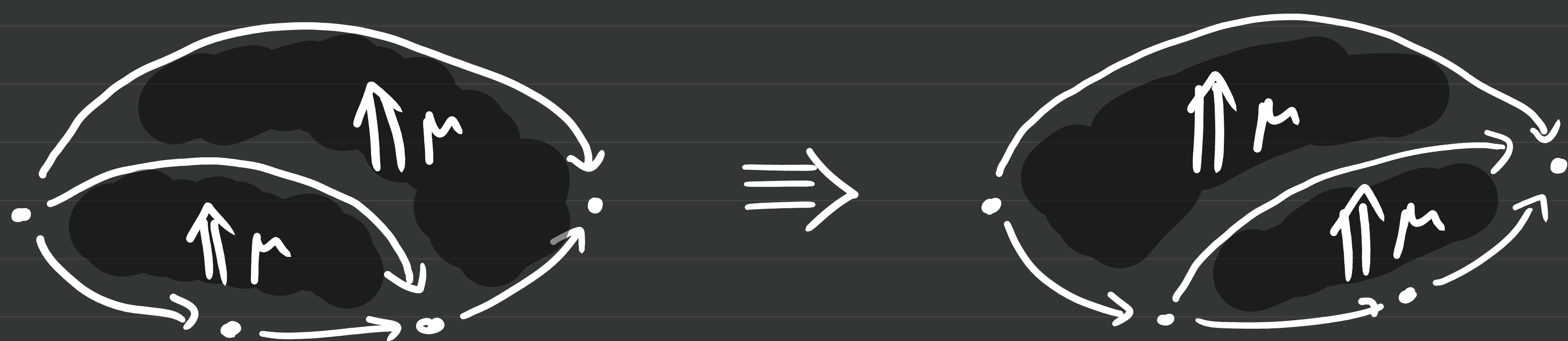
~ Directed 2D cell complexes



$\boxed{\text{3D}}$ Term Rewriting Systems

~ Diagram rewriting in PRO(P)s

~ Directed 3D cell complexes



$$\mu(\mu(x,y),z) \Rrightarrow \mu(x,\mu(y,z))$$

To model a (directed) cell complex,
we need:

1. Models of n-cells & their boundaries

2. Models of "gluing maps", specifying how cells can be put together

Models could be ...

- POINT-SET
- COMBINATORIAL
- ALGEBRAIC
- LOGICAL / SYNTACTIC

Example : POLYGRAPHS

- An n-cell is modelled by the (free n-category on) the n-globe

- It can be glued along any functor of strict ω-categories

3 PROBLEMS FOR A HIGHER-DIMENSIONAL REWRITING THEORY

$\boxed{\#0}$ EXPRESSIVENESS

$\boxed{\#1}$ TOPOLOGICAL SOUNDNESS

$\boxed{\#2}$ HIGHER-CATEGORICAL SEMANTICS

$\boxed{\#0}$ EXPRESSIVENESS

A HDRT has to adequately address the existing practice of rewriting theory.

$\boxed{\#1}$ TOPOLOGICAL SOUNDNESS

A directed cell complex also presents a (topological) cell complex.

A well-formed rewrite sequence induces a (cellular) homotopy in the presented space.

$\boxed{\#2}$ HIGHER-CATEGORICAL SEMANTICS

Higher-dimensional rewrite systems should admit a suitably wide class of "semantic universes" in which they can be interpreted

(e.g. when used to present higher algebraic theories)

Higher Rewrite Systems $\longrightarrow$ Higher Categories

SYNTAX $\longrightarrow$ SEMANTICS

HDRSs can be interpreted in higher categories, but they themselves aren't necessarily higher categories

HOW DO POLYGRAPHS DO?

$\boxed{\#0}$ GREAT!

$\boxed{\#1}$ NO SOUND INTERPRETATION of all gluing maps (due to "strict Eckmann-Hilton")

$\boxed{\#2}$ SO-SO...

Semantics only in STRICT higher categories

Notice that #1 fails because
polygraphs are TOO EXPRESSIVE
(too many "cell complexes")

while #2 fails because
strict ω-categories are NOT
EXPRESSIVE ENOUGH
(too few "semantic universes")

"CELL SHAPES" FOR HIGHER CATEGORIES:

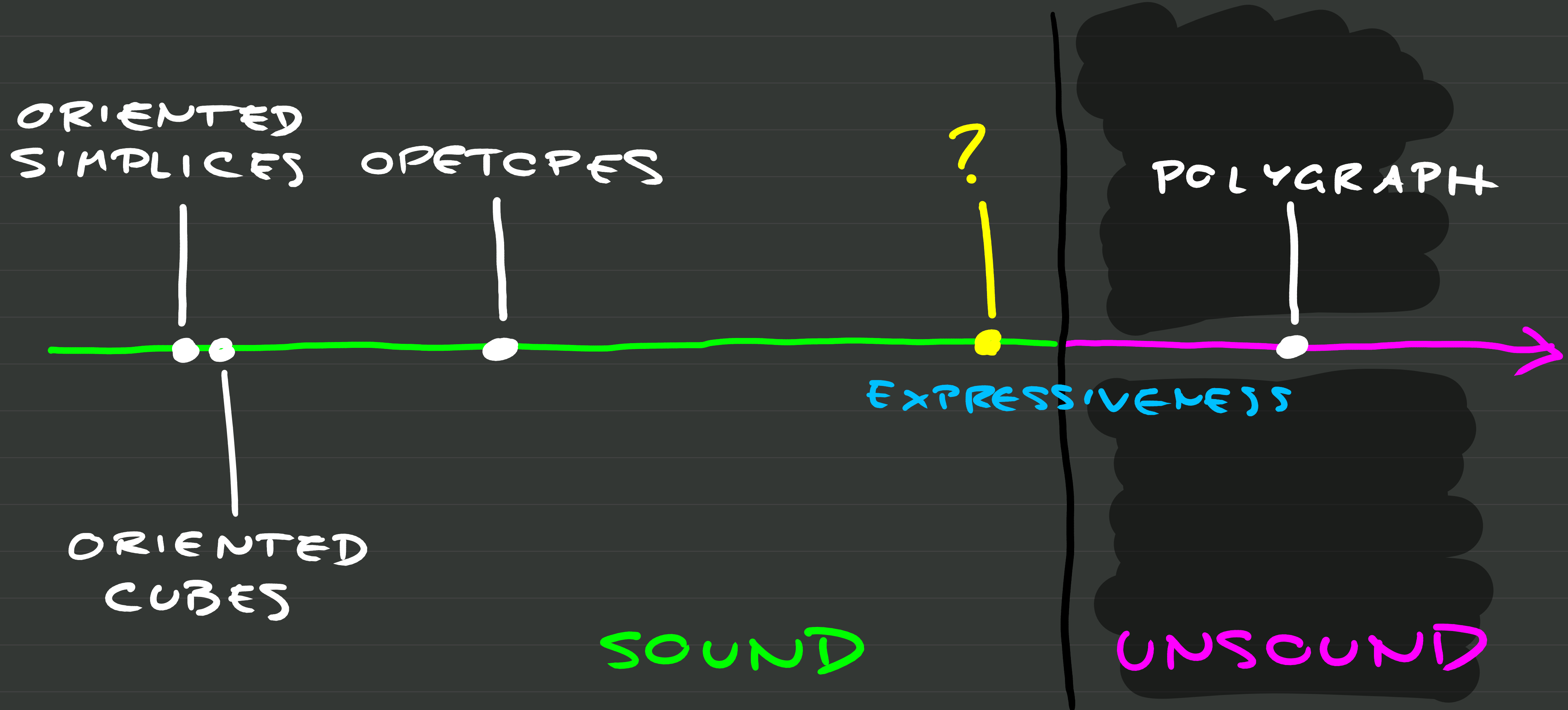
- GLOBES
- ORIENTED SIMPLICES
- ORIENTED CUBES
- OPETOPES

#0

#1 GREAT!

#2 GREAT!

A TRADE-OFF:


ORIENTED SIMPLICES
OPETOPES
?
POLYGRAPH
ORIENTED CUBES
EXPRESSIVENESS
SOUND
UNSOUND

GOAL :

A large, expressive class of directed balls & maps of directed balls, which is still topologically sound

A classical result of combinatorial topology:

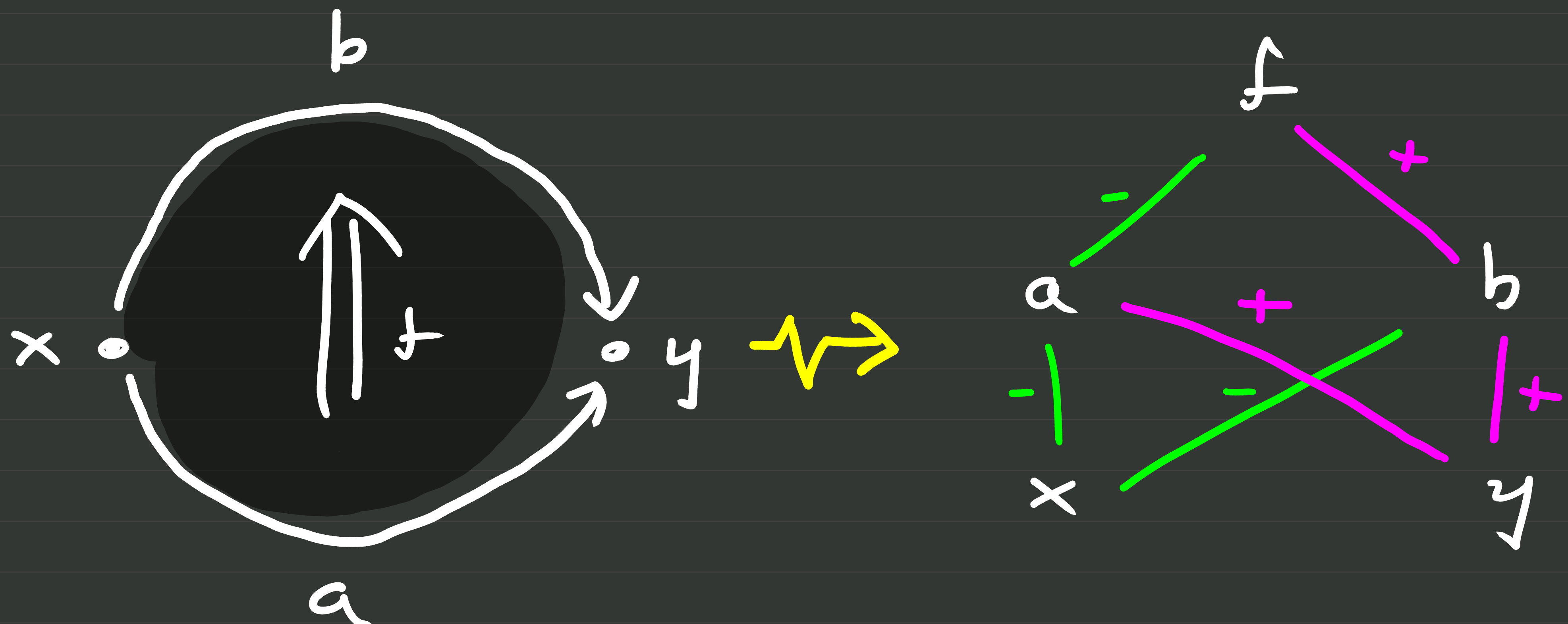
A REGULAR CW COMPLEX is uniquely (up to cellular homeomorphism) described by its FACE POSET

↓

FACE POSETS OF REGULAR CW BALLS ARE COMBINATORIAL MODELS OF BALLS

Encode a DIRECTED BALL as its

ORIENTED FACE POSET



− Source/Input

+ Target/Output

Def A finite poset P is **graded** if $\forall x \in P$, all maximal descending chains under x have the same length.

$$\begin{tikzcd}
 & x \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
y_k \arrow[d, dash] & z_k \arrow[d, dash] & w_k \arrow[d, dash] \\
y_{k-1} & z_{k-1} & w_{k-1} \\
\vdots & \vdots & \vdots \\
y_0 & z_0 & w_0
\end{tikzcd}$$

Rank / Dimension of x : $k+1$

Def An **oriented graded poset** is a graded poset together with an edge-labelling of its Hasse diagram in $\{-, +\}$.

We work mainly with (**downwards**) **closed subsets** of an o.g. poset (which inherit an o.g. poset structure)

Def Boundaries of a closed subset U

$k \in \mathbb{N} \qquad \alpha \in \{-, +\}$

$$\Delta_k^\alpha U := \left\{ x \in U \;\middle|\; \dim(x) = k, \text{ and } \forall y \in U \quad \begin{array}{c} y \\ | \\ x \end{array} \Rightarrow \begin{array}{c} y \\ |\alpha \\ x \end{array} \right\}$$

$$\mathrm{Max}_j U := \left\{ x \in U \;\middle|\; \dim(x) = j, \text{ and } x \text{ is maximal} \right\}$$

$$\partial_k^\alpha U := \mathrm{cl}\left(\Delta_k^\alpha U\right) \cup \bigcup_{j<k} \mathrm{cl}\left(\mathrm{Max}_j U\right)$$

Notation: for $x \in P$, $\partial_k^\alpha x := \partial_k^\alpha \mathrm{cl}\{x\}$

Def A **map** $f : P \to Q$ of o.g. posets is a function satisfying

$$\boxed{f(\partial_k^\alpha x) = \partial_k^\alpha f(x)}$$

for all $x \in P$, $k \in \mathbb{N}$, $\alpha \in \{-,+\}$.

Prop A map of o.g. posets is

- order-preserving, and
- dimension-non-increasing.

The class $\mathcal{R}$ of **REGULAR MOLECULES**:

1. **(POINT)** The terminal o.g. poset $\bullet \in \mathcal{R}$.

2. **(ATOM)** If $U, V \in \mathcal{R}$,
   a) $\dim(U) = \dim(V) = n$,
   b) $\partial^{\alpha}_{n-1} U \simeq \partial^{\alpha}_{n-1} V$ for all $\alpha \in \{-,+\}$,
   c) U, V are **ROUND**,

   then $U \Rightarrow V \in \mathcal{R}$.

3. **(PASTE)** If $U, V \in \mathcal{R}$, $\partial^{+}_{k} U \simeq \partial^{-}_{k} V$, then $U \#_k V \in \mathcal{R}$.

2. (ATOM):

- Glue U, V along the (unique!) isomorphism of their boundaries

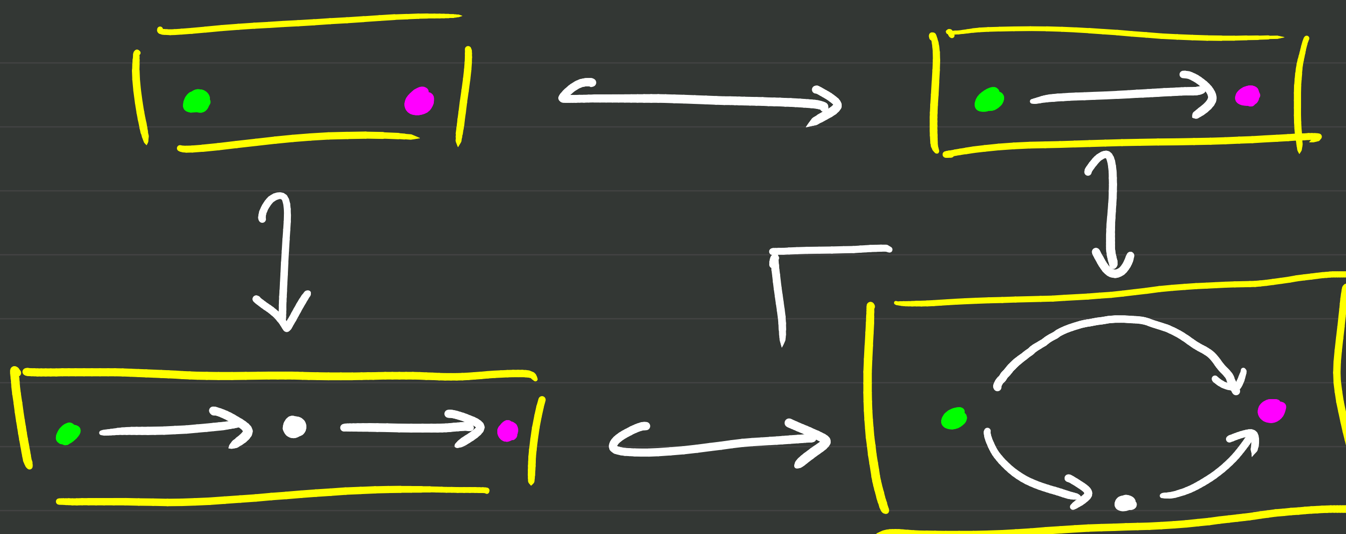
$$\begin{tikzcd} \partial_{n-1} U \arrow[r, "\sim"] \arrow[d, hook] & \partial_{n-1} V \arrow[r, hook] & V \arrow[d] \\ U \arrow[rr] & & \partial(U \Rightarrow V) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

- Add a greatest element $\top$ with $\partial_n^- \top = U$, $\partial_n^+ \top = V$

Example :

$U = \bullet \longrightarrow \bullet \longrightarrow \bullet$

$V = \bullet \longrightarrow \bullet$



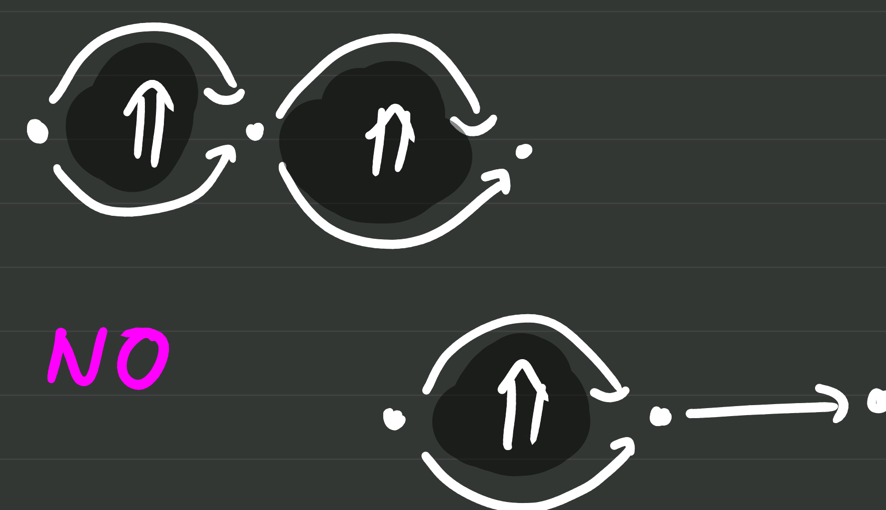
$U \Rightarrow V =$ 

For all $U \in \mathcal{R}$, $k < \dim(U)$,

$$\partial_{k-1} U \subseteq \partial^-_k U \cap \partial^+_k U .$$

Def U is **ROUND** if these are equalities.

YES



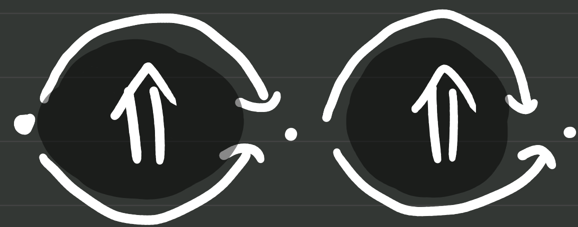
NO

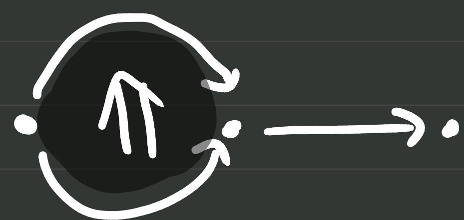
3. (PASTE):

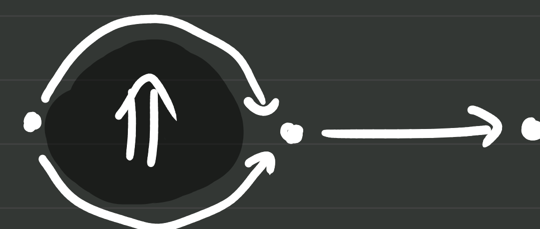
Glue U, V along the (unique!) isomorphism of $\partial_k^+ U$, $\partial_k^- V$.

$$\begin{tikzcd}
\partial_k^+ U \arrow[r, hook, "\sim"] \arrow[d, hook] & \partial_k^- V \arrow[r, hook] & V \arrow[d, hook] \\
U \arrow[rr, hook] & & U \#_k V \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Example:

 as

1.  $\#_1$ 

2.  $\#_1$ 

3.  $\#_0$ 

In this model,

DIRECTED BALL

||

ROUND, REGULAR MOLECULE

J.W. with Diana Kessler:

An implementation of these data structures as a Python library, rewal (to be released soon!)

THE SHAPE CATEGORY $\odot$
(ATOM)

- Objects : Regular atoms
- Morphisms : Maps of o.g. posets

Factor as surjective (co-degeneracies)
followed by injective (co-faces)

$\odot\underline{\mathrm{Set}} := [\odot^{op}, \underline{\mathrm{Set}}]$

$\odot$ contains the following as full subcategories:

- The category of SIMPLICES;
- The REFLEXIVE GLOBE category;
- The category of CUBES WITH CONNECTIONS;
- The category of POSITIVE OPETOPES WITH CONTRACTIONS.

$\odot$ is closed under

- All DIRECTION-REVERSING dualities (like CUBES, GLOBES);
- SUSPENSIONS (like GLOBES);
- GRAY PRODUCTS (like CUBES);
- JOINS (like SIMPLICES)

Regular molecules & their maps can be Yoneda-embedded in $\odot\underline{\mathbf{Set}}$.

Terminology:

U molecule, X diagrammatic set

- A DIAGRAM in X of SHAPE U is a morphism $U \longrightarrow X$.
- It is COMPOSABLE if U has spherical boundary.
- It is a CELL if U is an atom.

$\boxed{\#0}$ EXPRESSIVENESS

Very similar to polygraphs; the main restriction is the "spherical boundary" constraint on cell shapes

$\mapsto$ No "strictly degenerate" boundaries!

However, DEGENERACIES give access to "WEAK UNITS & UNITORS" that can be used to "regularise" shapes.

Example:

$$
\begin{tikzcd}
x \arrow[r, bend left=60, "h"] \arrow[r, bend right=60, "f"'] \arrow[r, phantom, "\alpha\Uparrow"] & y \arrow[r, bend left=60, "k"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, "\Uparrow\beta"] & z
\end{tikzcd}
\quad \Rrightarrow \quad
\begin{tikzcd}
& y \arrow[dr, "k"] & \\
x \arrow[ur, "h"] \arrow[dr, "f"'] \arrow[rr, phantom, "\gamma\Uparrow"] & & z \\
& y \arrow[ur, "g"'] &
\end{tikzcd}
$$

can be replaced with

$$
\begin{tikzcd}
& y \arrow[dr, "k"] \arrow[dd, "\varepsilon_y"] & \\
x \arrow[ur, "h"] \arrow[dr, "f"'] \arrow[r, phantom, "\Uparrow p^*\alpha"] & & z \arrow[l, phantom, "\Uparrow q^*\beta"'] \\
& y \arrow[ur, "g"'] &
\end{tikzcd}
\quad \Rrightarrow \quad
\begin{tikzcd}
& y \arrow[dr, "k"] & \\
x \arrow[ur, "h"] \arrow[dr, "f"'] \arrow[rr, phantom, "\gamma\Uparrow"] & & z \\
& y \arrow[ur, "g"'] &
\end{tikzcd}
$$

$\boxed{\#1}$ TOPOLOGICAL SOUNDNESS

$$\begin{tikzcd}
\odot \arrow[r, "\text{Forget}"] \arrow[d, "\text{Yoneda}"] & \underline{\mathrm{Pos}} \arrow[r, "\overset{\text{Nerve}}{N}"] & s\underline{\mathrm{Set}} \arrow[r, "\overset{\text{Realise}}{|{-}|}"] & \underline{\mathrm{Spaces}} \arrow[dlll, bend left, "S_\odot"] \arrow[dlll, phantom, "\perp"] \\
\odot\underline{\mathrm{Set}} \arrow[urrr, bend left=15, "\mathrm{Lan}_y\,|{-}|"] & & &
\end{tikzcd}$$

Prop If U is an n-dim. atom,

- $|U| \cong D^n$
- $|\partial U| \cong S^{n-1}$.

(Almost) **Corollary** If X is a "cell complex" with generating cells $\{x_i : U_i \longrightarrow X\}_{i\in I}$, then $|X|$ is a CW complex with generating cells $\{|x_i| : |U_i| \longrightarrow |X|\}_{i\in I}$.

$\boxed{\#2}$ HIGHER-CATEGORICAL SEMANTICS

Idea (shared with complicial & opetopic):

A diagrammatic set is a higher category if every composable diagram is equivalent to a single cell (its weak composite).

The equivalence is exhibited by a higher diagram (compositor).

Conjecture Diagrammatic sets with weak composites are equivalent to other models of (∞,∞)-cats (in the "coinductive" sense)

FURTHER WORK

- "THE SMASH PRODUCT OF MONOIDAL THEORIES": A topological construction applied to presentations of higher algebraic theories

  (arXiv: 2101.10361)

QUESTION FOR YOU:

Can this help in ...

- Directed topology?
  (Use for higher directed cells?)

- "Undirected" Topology?
  (Algebraic grip on pasting, "given" orientation, etc)

MERCI POUR VOTRE ATTENTION !

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