DIAGRAMMATIC SETS &
TOPOLOGICALLY SOUND REWRITING

Amar Hadžihasanović
TALTECH

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Main reference:
anXiv: 2007.14505

These slides on:
10c.ee/~amar
Central Idea of Higher-Dimensional Rewriting

\[
\text{REWRITE SYSTEMS} = \text{DIRECTED CELL COMPLEXES}
\]
CELL COMPLEX

→ Space "assembled from n-balls, glued by their (n-1)-sphere boundaries"

Minimal REGULAR cell structure on n-balls:

1. n-dimensional cell,
2. k-dimensional cells for k < n
DIRECTED
\CELL COMPLEX

\Rightarrow \text{Space "assembled from directed \(n\)-balls, glued by their \((n-1)\)-sphere boundaries"}

Minimal \REGULAR cell structure on \(n\)-balls:

1. \(n\)-dimensional cell,
2. \(k\)-dimensional cells for \(k < n\)
A directed \(n\)-ball has its boundary subdivided into an \textit{INPUT} half and an \textit{OUTPUT} half, each of which is a directed \((n-1)\)-ball.

A particular kind of orientation, inspired by higher category theory
Abstract Rewrite Systems
~ Directed Graphs
~ Directed 1D cell complexes

Rewrite sequence ~ Directed homotopy
String Rewrite Systems

- Rewriting paths in graphs
- Directed 2D cell complexes
3D Term Rewriting Systems

Diagram rewriting in PROC(D)5

Directed 3D cell complexes

\[ \mu(\mu(x, y), z) \Rightarrow \mu(x, \mu(y, z)) \]
To model a (directed) cell complex, we need:

1. Models of \( n \)-cells & their boundaries

2. Models of "gluing maps", specifying how cells can be put together
Models could be...

- POINT-SET
- COMBINATORIAL
- ALGEBRAIC
- LOGICAL/SYNTACTIC
Example: POLYGRAPHHS

- An $n$-cell is modelled by the (free $n$-category on) the $n$-globe

- It can be glued along any functor of strict $w$-categories
3 PROBLEMS FOR A HIGHER-DIMENSIONAL REWRITING THEORY

#0  EXPRESSIVENESS

#1  TOPOLOGICAL SOUNDNESS

#2  HIGHER-CATEGORICAL SEMANTICS
A HDRT has to adequately address the existing practice of rewriting theory.
#1  **Topological Soundness**

A directed cell complex also presents a (topological) cell complex.

A well-formed rewrite sequence induces a (cellular) homotopy in the presented space.
Higher-dimensional rewrite systems should admit a suitably wide class of "semantic universes" in which they can be interpreted (e.g. when used to present higher algebraic theories).
HDRSs can be interpreted in higher categories, but they themselves aren't necessarily higher categories.
How do polygraphs do?

#0

Great!

#1

No sound interpretation of all gluing maps (due to "strict Eckmann-Hilton")

#2

So-so...

Semantics only in strict higher categories
Notice that #1 fails because polygraphs are **too expressive** (too many "cell complexes")

while #2 fails because strict w-categories are **not expressive enough** (too few "semantic universes")
"CELL SHAPES" FOR HIGHER CATEGORIES:

- GLOBES
- ORIENTED SIMPLICES
- ORIENTED CUBES
- OPETOPES

#0

#1  GREAT!

#2  GREAT!
A trade-off:

Oriented simplices - opetopes - polygraph

Expressiveness

Oriented cubes - sound - unsound
GOAL:

A large, expressive class of directed balls & maps of directed balls, which is still topologically sound
A classical result of combinatorial topology:

A regular CW complex is uniquely (up to cellular homeomorphism) described by its face poset.

Face posets of regular CW balls are combinatorial models of balls.
Encode a DIRECTED BALL as its ORIENTED FACE POSET.

- Source/Input
- Target/Output
Def: A finite poset $P$ is graded if $\forall x \in P$, all maximal descending chains under $x$ have the same length.

\begin{align*}
&x_k \\
&z_k \\
&y_k \\
&y_{k-1} \\
&y_{k-2} \\
&y_0
\end{align*}

\begin{align*}
&w_k \\
&w_{k-1} \\
&w_{k-2} \\
&w_0
\end{align*}

\text{Rank / Dimension of } x : k+1
Def: An oriented graded poset is a graded poset together with an edge-labelling of its Hasse diagram in \{ -, + \}.

We work mainly with (downwards) closed subsets of an o.g. poset (which inherit an o.g. poset structure).
**Def** Boundaries of a closed subset $U$ 

$k \in \mathbb{N} \quad \alpha \in \{-, +\}$

$$
\Delta^\alpha_k U := \left\{ x \in U \mid \dim(x) = k \right\} \quad \forall y \in U \quad y \rightarrow_{\alpha} y \quad x \rightarrow_{\alpha} x
$$

$$
\text{Max}_j U := \left\{ x \in U \mid \dim(x) = j \right\} \quad \text{and} \quad x \text{ is maximal}
$$

$$
\mathcal{E}^\alpha_k U := \text{cl}(\Delta^\alpha_k U) \cup \bigcup_{j < k} \text{cl}(\text{Max}_j U)
$$
Notation: for $x \in P$, $\mathcal{G}_k^x := \mathcal{G}_k \cap \{x\}$

Def A map $f : P \rightarrow Q$ of o.g. posets is a function satisfying
\[ f(\mathcal{G}_k^x) = \mathcal{G}_k^x f(x) \]
for all $x \in P$, $k \in \mathbb{N}$, $x \in \{-, +\}$.

Prop A map of o.g. posets is order-preserving, and dimension-non-increasing.
The class $R$ of regular molecules:

1. (Point) The terminal o.g. poset
   $\bullet \in R$

2. (Atom) If $U, V \in R$,
   a) $\dim(U) = \dim(V) = n$,
   b) $\mathbb{2}^n_{\alpha}, U \approx \mathbb{2}^n_{\alpha}, V$ for all $\alpha \in \{-, +\}$,
   c) $U, V$ are round,
   then $U \Rightarrow V \in R$

3. (Paste) If $U, V \in R$,
   $\mathbb{2}^+_k U \approx \mathbb{2}^-_k V$, then $U \#_k V \in R$
2. (ATOM):

- Glue $U, V$ along the (unique!) isomorphism of their boundaries

$$\exists_{n,n} U \sim \exists_{n,n} V \hookrightarrow V$$

$$U \overset{T}{\longrightarrow} e(U \Rightarrow V)$$

- Add a greatest element $T$ with $\exists_n T = U$, $\exists^+ n T = V$
Example:

\( U = \bullet \rightarrow \bullet \rightarrow \bullet \)

\( V = \bullet \rightarrow \bullet \)

\( U \Rightarrow V = \bullet \rightarrow \bullet \rightarrow \bullet \)
For all $U \in \mathbb{R}^d$, $k < \dim(u)$,
$\exists \ U \in \mathbb{R}^d$ \ $\exists_{k-1} U 
eq \cap_{k} U$.

**Def** $U$ is **ROUND** if these are equalities.

![Diagram with arrows indicating YES and NO conditions]
3. \textbf{(PASTE)}:

Glue $U, V$ along the (unique!) isomorphism of $\mathfrak{e}_k^+ U$, $\mathfrak{e}_k^- V$.

\[
\begin{array}{ccc}
\mathfrak{e}_k^+ U & \xrightarrow{\sim} & \mathfrak{e}_k^- V & \xrightarrow{} & V \\
\downarrow & & & & \downarrow \\
U & \xrightarrow{} & U \#_k V
\end{array}
\]
Example:

1. \[ \Rightarrow \quad \#_1 \quad \Rightarrow \]

2. \[ \Rightarrow \quad \Rightarrow \quad \Rightarrow \quad \#_1 \quad \Rightarrow \]

3. \[ \Rightarrow \quad \Rightarrow \quad \Rightarrow \quad \Rightarrow \quad \#_0 \]
In this model, a directed ball, \( \Rightarrow \), is round, regular molecule.
J.W. with Diana Kessler:

An implementation of these data structures as a Python library, rewal
(to be released soon!)
THE SHAPE CATEGORY \( \odot \)

- **Objects**: Regular atoms
- **Morphisms**: Maps of o.g. posets

Factor as surjective (co-degeneracies)
Followed by injective (co-faces)

\[ \odot \mathsf{Set} := [\mathcal{O}^{\text{op}}, \mathsf{Set}] \]
0 contains the following as full subcategories:

- The category of simplices,
- The reflexive globe category,
- The category of cubes with connections,
- The category of positive ope-topes with contractions.
is closed under

• All **DIRECTION-REVERSING** dualities (like **CUBES, GLOBES**);

• **SUSPENSIONS** (like **GLOBES**);

• **GRAY PRODUCTS** (like **CUBES**);

• **JOINS** (like **SIMPLICIES**).
Regular molecules & their maps can be Yoneda-embedded in $\mathbb{OSet}$.

**Terminology:**
- $U$ molecule, $X$ diagrammatic set
  - A **DIAGRAM** in $X$ of SHAPE $U$ is a morphism $U \xrightarrow{} X$.
  - It is **COMPOSABLE** if $U$ has spherical boundary.
  - It is a **CELL** if $U$ is an atom.
Expressiveness

Very similar to polygraphs; the main restriction is the "spherical boundary" constraint on cell shapes → No "strictly degenerate" boundaries!

However, DEGENERACIES give access to "weak units & unitors" that can be used to "regularise" shapes.
Example:

can be replaced with
#1 Topological Soundness

\[
\begin{align*}
\circ & \xrightarrow{\text{Forget}} \mathbf{Pos} & \xrightarrow{\text{Name}} \mathbf{sSet} & \xrightarrow{\text{Realise}} \mathbf{Space} \\
\downarrow \text{Yoneda} & & \downarrow \text{Lany} & \\
\circ \mathbf{Set} & \xrightarrow{\text{So}} \circ \mathbf{Set}
\end{align*}
\]
Prop If $U$ is an $n$-dim. atom,
  - $|U| \simeq D^n$
  - $|\partial U| \simeq S^{n-1}$.

(Almost) Corollary If $X$ is a "cell complex" with generating cells $\{x_i: U_i \to X\}_{i \in I}$, then $|X|$ is a CW complex with generating cells $\{|x_i|: |U_i| \to |X|\}_{i \in I}$.
# Higher-Categorical Semantics

Idea (shared with complicial & operadic):

A diagrammatic set is a higher category if every composable diagram is equivalent to a single cell (its weak composite).

The equivalence is exhibited by a higher diagram (compositor).
Conjecture Diagrammatic sets with weak composites are equivalent to other models of (∞,∞)-cats (in the "comductive" sense)
FURTHER WORK

- "THE SMASH PRODUCT OF MONOIDAL THEORIES": A topological construction applied to presentations of higher algebraic theories

(arXiv: 2101.10361)
QUESTION FOR YOU:

Can this help in...

- Directed topology?
  (Use for higher directed cells?)

- "Undirected" Topology?
  (Algebraic grip on pasting, "given" orientation, etc)
MERCI POUR VOTRE ATTENTION!

http://10c.ee/~amar

amar@cs.10c.ee