Directed Topology and Concurrency

Uli Fahrenberg

LRDE, EPITA, France

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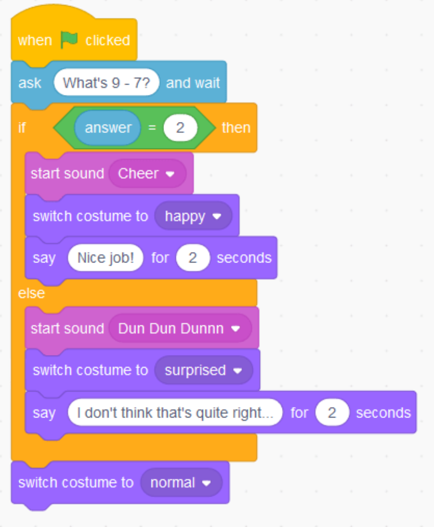
Algebraic View

A program is a sequence of instructions

- plus branches and loops

Kleene algebra:

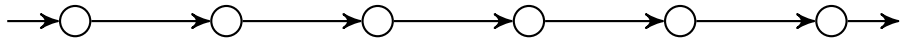
- set S with operations:
- concatenation $\otimes$
- choice $\oplus$
- repetition *
- idempotent semiring with unary * which computes fixed points



Geometric View

A program is a sequence of instructions

- ignoring branches and loops for now



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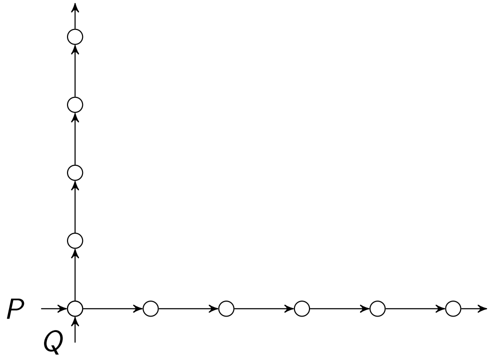

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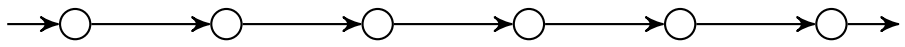

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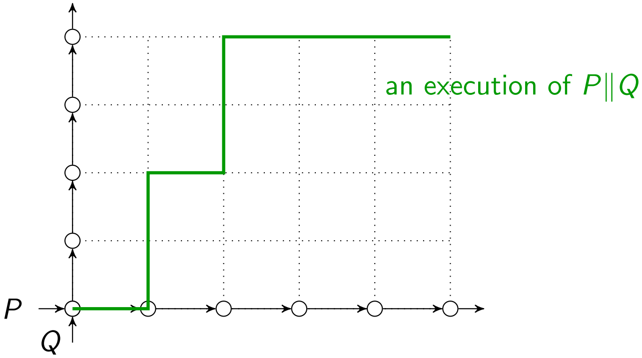

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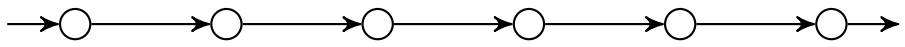

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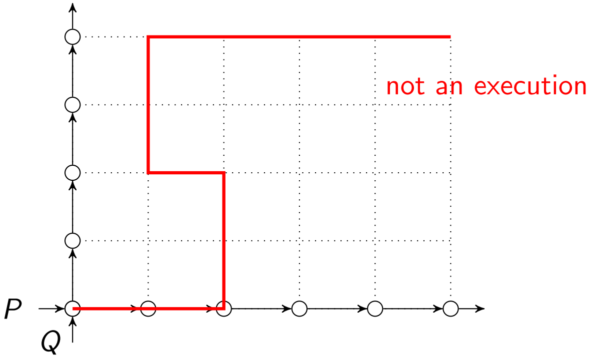

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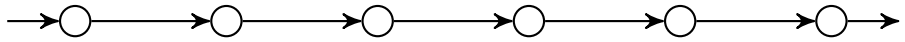

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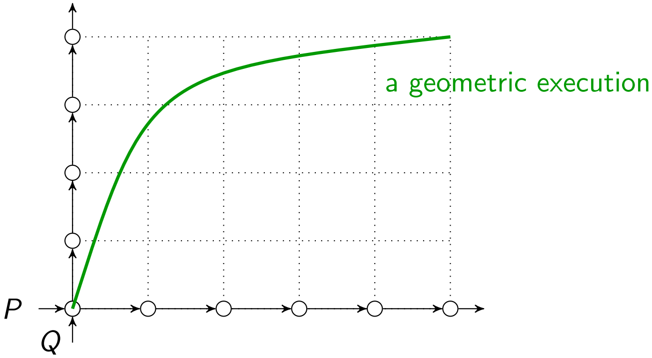

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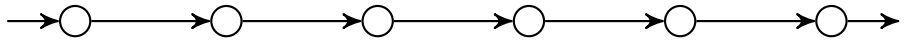

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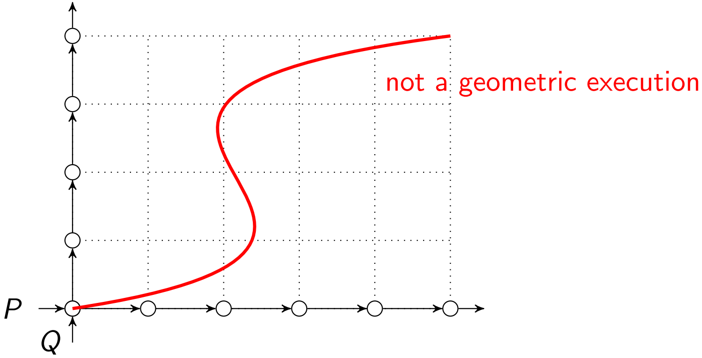
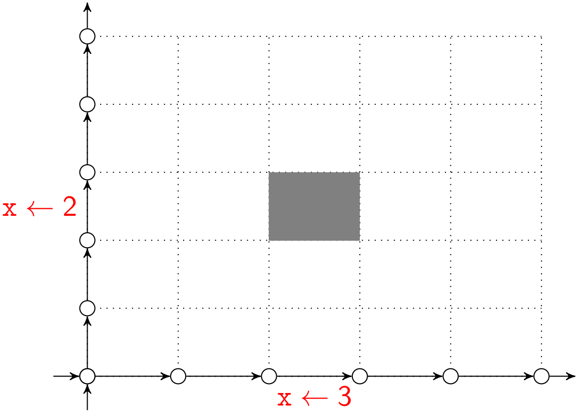


Holes

Adding mutual exclusion:



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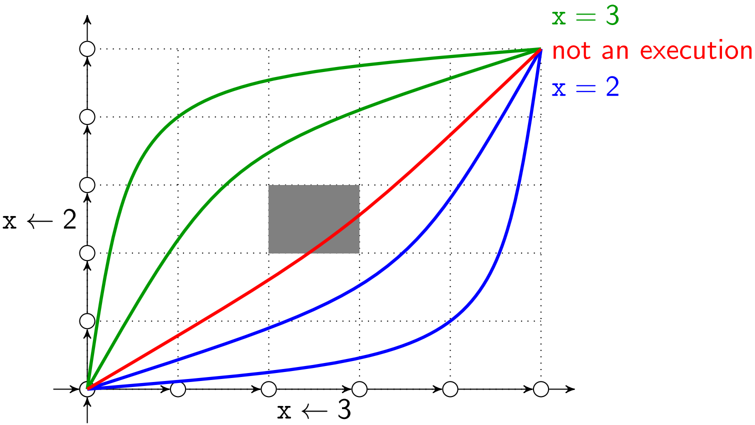
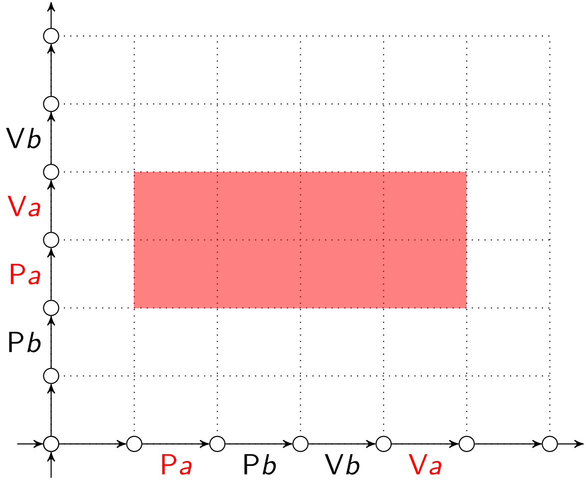


- homotopic paths ≙ equivalent executions

More Holes

Semaphores à la Dijkstra (P $\hat{=}$ acquire; V $\hat{=}$ release):



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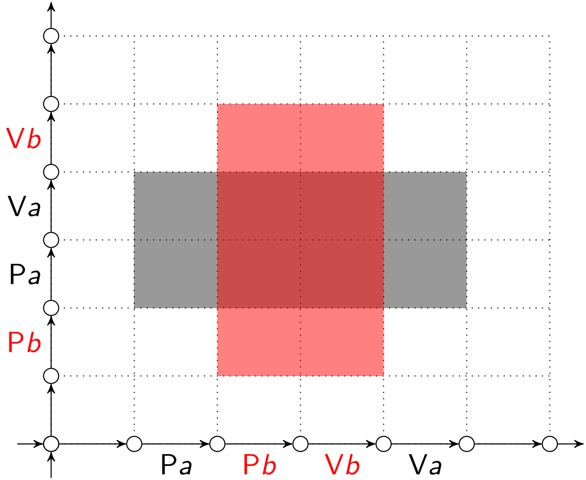

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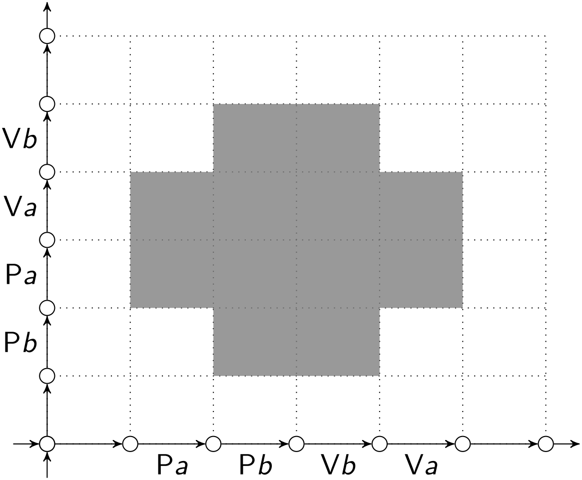

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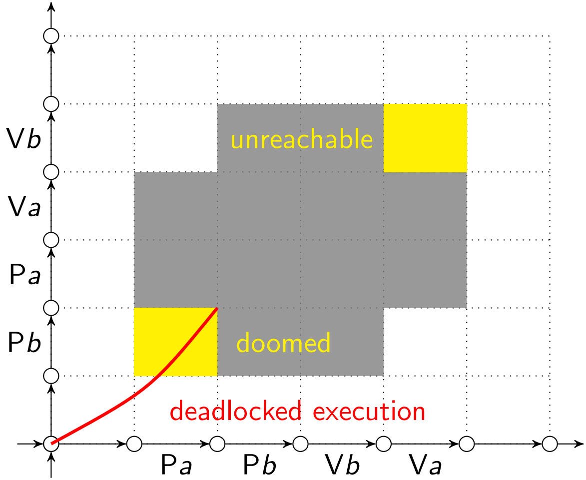
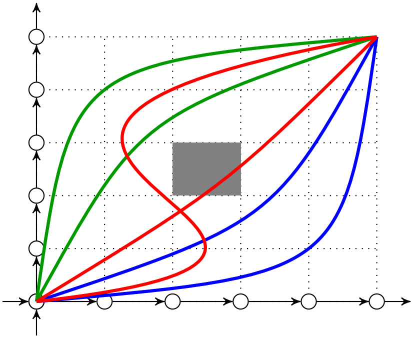
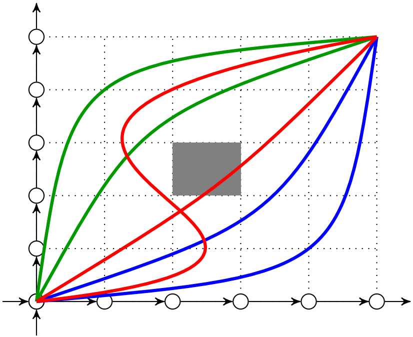


Summing Up



- A program is a topological space
- An execution is a path through said space
- Two executions are equivalent iff their paths are homotopic

Summing Up



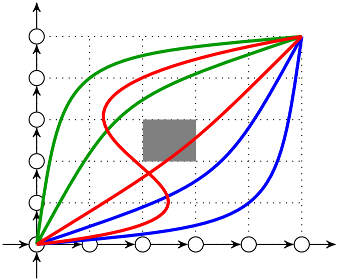
- A program is a directed topological space
- An execution is a directed path through said space
- Two executions are equivalent iff their dipaths are dihomotopic

Directed Spaces

Definition (po-space)

A partially ordered space is a topological space X together with a partial order $\leq$ on X such that $\leq \subseteq X \times X$ is *closed* in the product topology.
A morphism of po-spaces is a $\leq$-preserving continuous function.

- directed intervals; directed squares, cubes, etc.
- concatenation $\otimes$, branching $\oplus$
- no loops

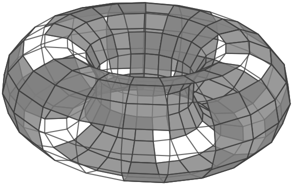


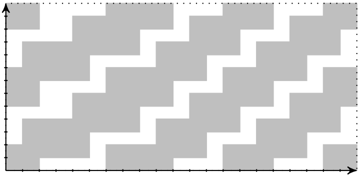
Directed Spaces

Definition (lpo-space)

A locally partially ordered space is a *Hausdorff* topological space X together with a relation $\leq$ on X in which any $x \in X$ has an open neighborhood $U \ni x$ such that the restriction of $\leq$ to U is a closed partial order.

A morphism of po-spaces is a continuous function which is *locally* $\leq$-preserving.





Directed Spaces

Definition (d-space)

A directed space is a topological space X together with a set $\vec{P}X$ of directed paths $I \to X$ such that

- all constant paths are directed,
- concatenations of directed paths are directed, and
- reparametrizations and restrictions of directed paths are directed.

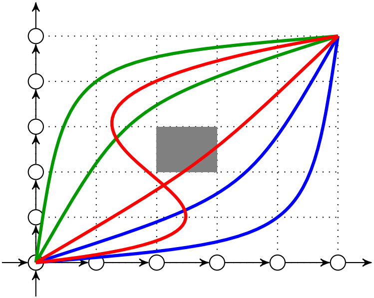
A morphism of d-spaces is a continuous function which preserves directed paths.

- po-spaces $\hookrightarrow$ lpo-spaces $\hookrightarrow$ d-spaces (not full)
- po-spaces are *loop-free*; lpo-spaces are *vortex-free*
- d-spaces are nice: axiomatize directly our objects of interest (dipaths); have good categorical properties

Directed Paths and Homotopies

- the directed interval $\vec{I}$:
  - $([0,1], \leq)$ (usual order): po-space; lpo-space
  - $([0,1], \vec{P}[0,1])$: all (weakly) increasing paths
- dipaths in X: morphisms $\vec{I} \to X$
  - for d-space $(X, \vec{P}X)$: dipaths $\hat{=}$ $\vec{P}X$
- a dihomotopy $H : I \times \vec{I} \to X$:
  - all $H(s, \cdot)$ dipaths
  - $H : I \times I \to X$ continuous
  - $H(\cdot, 0)$ and $H(\cdot, 1)$ constrained
  - (some variants exist)

Summing Up, Again

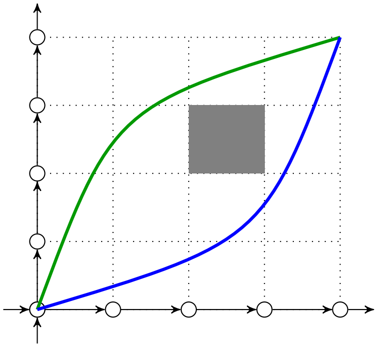


- A program is a directed topological space
  - po-space, lpo-space, d-space
  - (other models exist)
- An execution is a directed path through said space
- Two executions are equivalent iff their dipaths are dihomotopic

Combinatorial Model

Transition Systems?

"Programs are topological spaces"?!?



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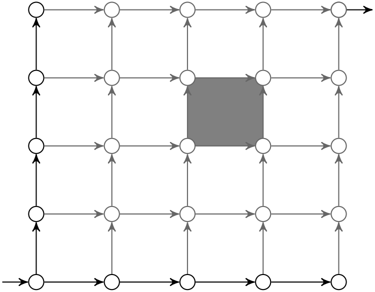

Transition Systems?

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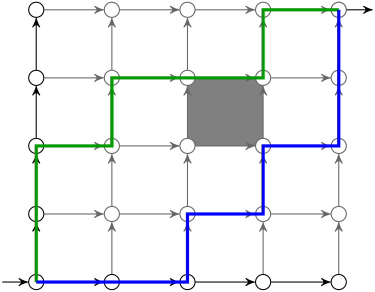
Programs are transition systems!

- have lost info on "forbidden squares"



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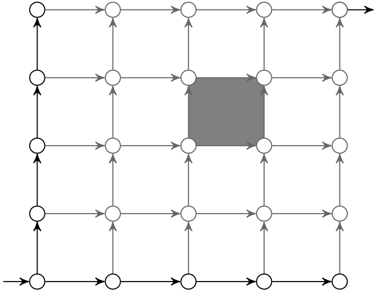
Higher-dimensional automata:

- transition systems
- plus info on concurrency



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Higher-dimensional automata:

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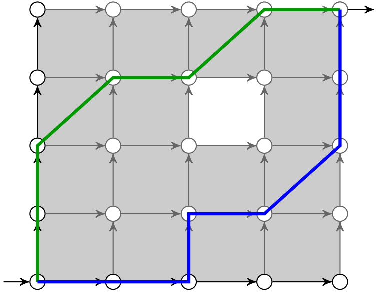
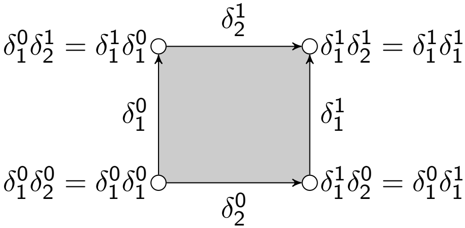


Higher-Dimensional Automata

Definition (pc-set)

A precubical set is a graded set $X = \{X_n\}_{n\geq 0}$ together with *face maps* $\delta_i^0, \delta_i^1 : X_n \to X_{n-1}$, for $i = 1, \dots, n$, satisfying $\delta_i^\nu \delta_j^\mu = \delta_{j-1}^\mu \delta_i^\nu$ for $i < j$.

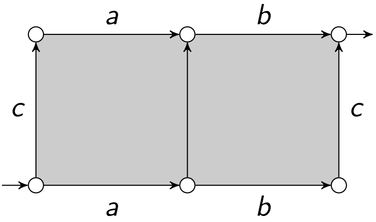


Definition (HDA)

A higher-dimensional automaton is a pc-set X together with a labeling $\lambda : X_1 \to \Sigma$ and specified start and accept cells $I, F \subseteq X$.

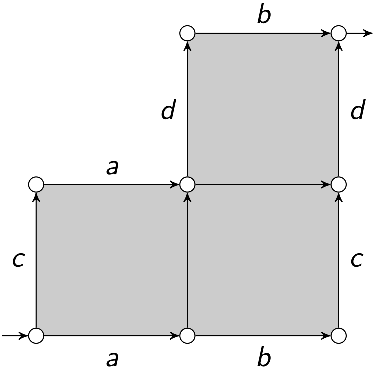
- also need $\lambda(\delta_i^0 x) = \lambda(\delta_i^1 x)$ for all $x \in X_2$

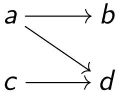
Examples



first a, then b; all in parallel with c: $ab \parallel c$

Examples





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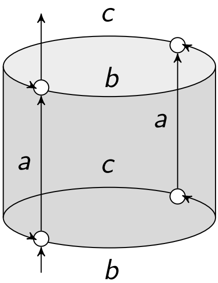


$$a \parallel (bc)^*$$

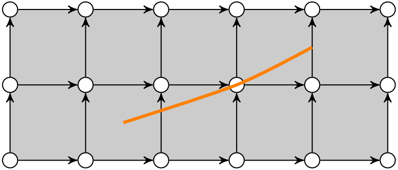
Geometric Realization

Definition

The geometric realization of a pc-set X is the d-space $|X| = \bigsqcup_{n\ge 0} X_n \times \vec{I}^n /_\sim$, where $\sim$ is the equivalence generated by $(\delta_i^\nu x, (t_1, \dots, t_{n-1})) \sim (x, (t_1, \dots, t_{i-1}, \nu, t_{i+1}, \dots, t_{n-1}))$.

- usual coend definition; left adjoint to *singular pc-set* functor
- actually, $|X|$ is an *lpo-space*

Dipaths in Geometric Realizations

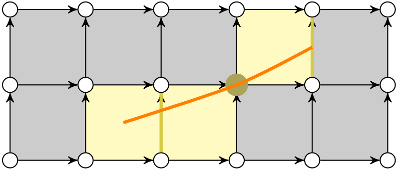


Let $p : \vec{I} \to |X|$ be a dipath in the geometric realization of pc-set X.

- let $C_p = \{x \in X \mid \operatorname{im}(p) \cap |x| \neq \emptyset\}$ – all cells touched by p
- organize C_p into a sequence $c_p = (x_1, \dots, x_m)$ s.t. $\forall i$:

$$x_i = \delta_+^0 x_{i+1} \qquad \text{or} \qquad x_{i+1} = \delta_+^1 x_i \quad \text{(iterated face maps)}$$

Dipaths in Geometric Realizations



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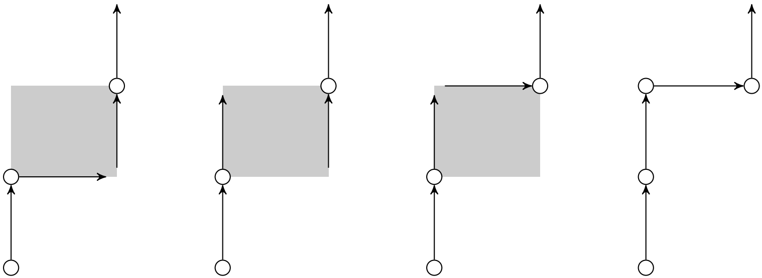
$$x_i = \delta^0_+ x_{i+1} \qquad \text{or} \qquad x_{i+1} = \delta^1_+ x_i \quad \text{(iterated face maps)}$$

$\Rightarrow$ the track of p

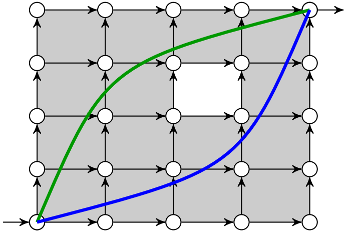
- any track c gives rise to dipath p_c (non-unique) with $c_{p_c} = c$
- if $c_p = c_q$, then p and q are *dihomotopic*

Homotopy of Tracks



- generated by local replacements
- dipaths p, q are dihomotopic iff c_p and c_q are homotopic
- tracks c, d are homotopic iff p_c and p_d are dihomotopic

Summing Up



- precubical sets / higher-dimensional automata: combinatorial models of directed spaces
- natural extension of transition systems; also used in computer science
- closely linked to directed spaces via geometric realization:
  - dipaths $\hat{=}$ tracks $\hat{=}$ executions
  - dihomotopy $\hat{=}$ track homotopy $\hat{=}$ equivalence of executions

Invariants

Fundamental Category

- topological space $X \rightsquigarrow$ fundamental group $\pi(X)$
  - all information may be reduced to loops (and base points)
- *directed* spaces: no reduction to loops!

$\Rightarrow$ directed space $X \rightsquigarrow$ fundamental category $\vec{\Pi}X$

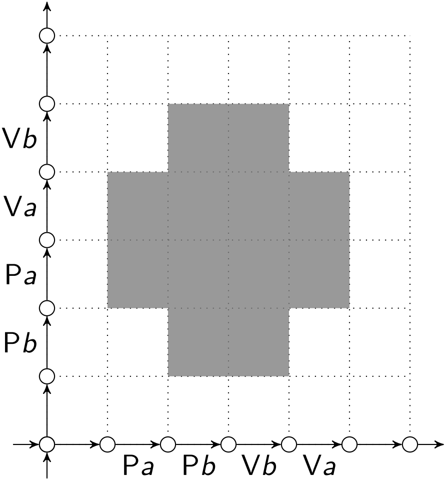
  - objects points of X
  - morphisms dihomotopy classes of dipaths
  - inspired by fundamental *groupoids* of topological spaces

- (higher dihomotopy invariants: unclear!)

Component Category

The fundamental category $\vec{\Pi}X$ is *huge*!

- objects: points of X ...
- but some points are equivalent



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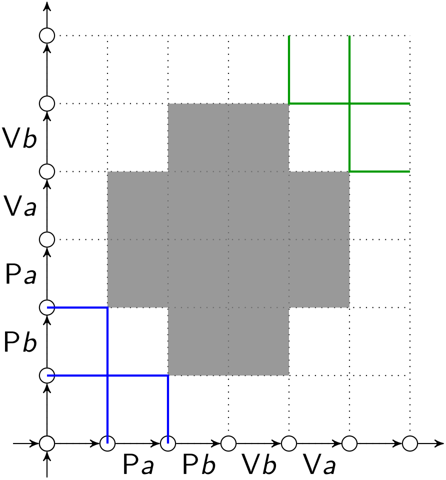


Definition

$\alpha \in \vec{\Pi}X(x, y)$ is a future weak iso if $\forall z \in X$ s.t. $\vec{\Pi}X(y, z) \neq \emptyset$, α induces an iso between $\vec{\Pi}X(y, z)$ and $\vec{\Pi}X(x, z)$.

- quotient by future weak isos
- (or by past weak isos, or by weak isos)

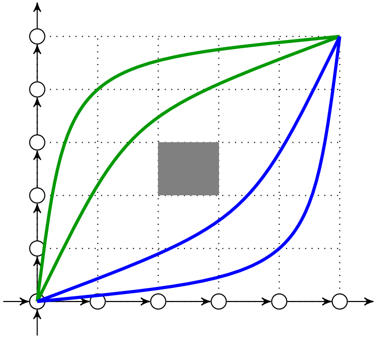
Path Spaces

Idea: objects of interest are not *points* of X, but *dipaths* in $\vec{P}X$

- equip $\vec{P}X$ with compact-open topology
- $\vec{P}X$ is standard topological space
  $\Rightarrow$ can use tools!

Usually consider traces: dipaths modulo direparametrizations: $\vec{T}X$

- if X is precubical, then
  $\vec{P}X \simeq \vec{T}X$
  $\simeq$ prod*simplicial* complex
  $\simeq$ prod*permutahedral* complex
  $\simeq$ nerve of *cube chain category*

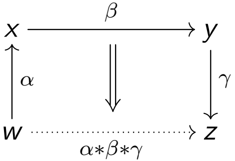


Directed Homology

¯_(ツ)_/¯

Natural Homology

- Trace space $\vec{T}X$ forms a (topological) category: objects points; morphisms $\vec{T}X(x,y)$ traces from x to y
- Idea: $\vec{H}_n X(x,y) = H_{n-1} \vec{T}X(x,y)$
- These combine into a *natural system* of abelian groups:
- Let $F\vec{T}X$ be the *factorization category* of $\vec{T}X$
  - objects traces
  - morphisms *extensions*



- $\vec{H}_n$ maps traces $\beta \in \vec{T}X(x,y)$ to $H_{n-1}\vec{T}X(x,y)$
- and extensions $(\alpha * \_ * \gamma)$ to $H_{n-1}(\alpha * \_ * \gamma)$

Natural Homology

Huge but manageable:

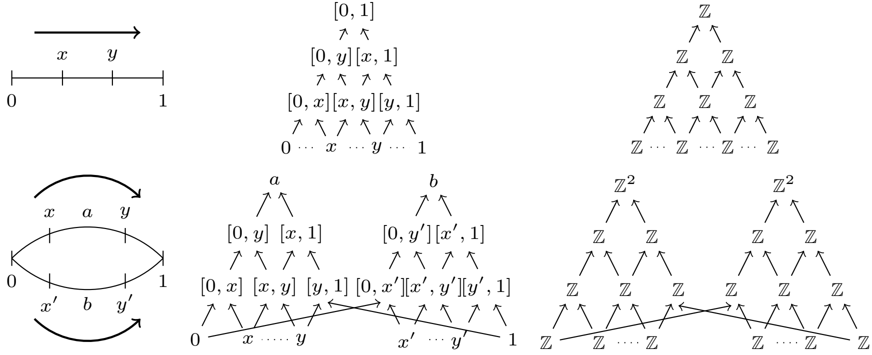


Fig. 2: Natural homology of two simple pospaces

Conclusion

- Programs are directed topological spaces
  - po-spaces $\hookrightarrow$ lpo-spaces $\hookrightarrow$ d-spaces
  - (lpo-spaces $\to$ po-spaces: delooping / universal dicover)
  - executions are dipaths; equivalence of executions is dihomotopy
- Programs are precubical sets
  - higher-dimensional automata
  - executions are tracks; equivalence of executions is track homotopy
  - strong link to spaces via geometric realization
- Invariants
  - fundamental category, component category
  - path spaces
  - natural homology

Current Work

- Languages of higher-dimensional automata
- Path spaces via cube chains and discrete Morse theory
- Model categories for directed topology
- Directed homology via persistent homology
- Higher-dimensional Kleene algebra
- ...

Incomplete (!) Bibliography

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