Approximating Discrete Dynamical Systems

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Outline

- Preliminaries and motivation.
- Vietoris-like maps and multivalued maps.
- Lefschetz fixed point theorem.
- Approximating Discrete Dynamical Systems.
- Localization of finite spaces at Vietoris-like maps.
Preliminaries and motivation

Definition

An Alexandroff space is a topological space for which arbitrary intersections of open sets are still open.
### Preliminaries and motivation

#### Definition

An Alexandroff space is a topological space for which arbitrary intersections of open sets are still open.

#### Theorem (Alexandroff, 1937)

The category of Alexandroff $T_0$-spaces is isomorphic to the category of partially ordered sets.

**Objects:**
- *Alexandroff $T_0$-spaces*
- *Partially ordered sets*

**Morphisms:**
- *Continuous maps*
- *Order-preserving maps*
Given an Alexandroff space $X$ and $x \in X$, $U_x$ denotes the intersection of all the open sets which contain $x$. Let $x, y \in X$, $x \leq y$ if and only if $U_x \subseteq U_y$ ($U_y \subseteq U_x$).
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**Example.** Finite topological $T_0$-spaces. We call them from now on just finite spaces.
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**Example.** Finite topological $T_0$-spaces. We call them from now on just finite spaces.

**Example.** Let $X = \{A, B, C, D\}$ and $\tau = \{X, \emptyset, \{A\}, \{B\}, \{A, B\}, \{C, A, B\}, \{D, A, B\}\}$. Then $U_A = \{A\}$, $U_B = \{B\}$, $U_C = \{C, A, B\}$ and $U_D = \{D, A, B\}$, which yields $A < C, D$ and $B < C, D$.

**Proposition**

Let $f, g : X \to Y$ be continuous maps between finite spaces. Then $f$ is homotopic to $g$ if and only if there exists a finite sequence of continuous maps $f_1, ..., f_n : X \to Y$ such that $f(x) = f_1(x) \leq f_2(x) \geq ... \leq f_n(x) = g(x)$ for every $x \in X$. 
**Hasse diagrams.** Let $X$ be a finite space. The Hasse diagram of $X$ is a directed graph. The vertices are the points of $X$ and there is an edge between two points $x$ and $y$ if and only if $x < y$ and there is no $z$ satisfying $x < z < y$. 
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**Example.**

![Hasse diagram example](image-url)
Order complex. Given a finite space $X$, the order complex of $X$, denoted by $\mathcal{K}(X)$, is the simplicial complex whose simplices are the non-empty chains of $X$. 
Preliminaries and motivation

**Face poset.** Given a simplicial complex $L$, the face poset of $L$, denoted by $\mathcal{X}(L)$, is the poset of simplices of $K$ ordered by inclusion.
Theorem (McCord, 1966)

There exists a correspondence that assigns to each Alexandroff $T_0$-space a simplicial complex $\mathcal{K}(X)$ and a weak homotopy equivalence $f_X : |\mathcal{K}(X)| \to X$. Each continuous map $\varphi : X \to Y$ of Alexandroff $T_0$-spaces is also a simplicial map $\mathcal{K}(\varphi) : \mathcal{K}(X) \to \mathcal{K}(Y)$, and $\varphi \circ f_X = f_Y \circ \mathcal{K}(\varphi)$.
Theorem (McCord, 1966)

There exists a correspondence that assigns to each simplicial complex \( K \) an Alexandroff \( T_0 \)-space \( \mathcal{X}(K) \) and a weak homotopy equivalence \( f_K : \lvert K \rvert \to \mathcal{X}(K) \). Furthermore, to each simplicial map \( \psi : K \to L \) is assigned a continuous map \( \mathcal{X}(\psi) : \mathcal{X}(K) \to \mathcal{X}(L) \) such that \( \mathcal{X}(\psi) \circ f_K \) is homotopic to \( f_L \circ \lvert \psi \rvert \).

\[
\begin{align*}
\mathcal{X}(K) & \xrightarrow{\mathcal{X}(\psi)} \mathcal{X}(L) \\
|K| & \xrightarrow{\psi} |L|
\end{align*}
\]
Finite barycentric subdivision. Given a finite space $X$, the finite barycentric subdivision of $X$ is defined as $\mathcal{X}(\mathcal{K}(X))$. We denote by $X^n$ the $n$-th finite barycentric subdivision of $X$. 
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There is a natural map $h : X^1 \to X$ given by $h(x_1 < \ldots < x_n) = x_n$. Then, we can consider $h_{n,m} : X_m \to X_n$ for every $m \geq n$. 
Given a simplicial complex $K$, $X^0$ denotes $\mathcal{X}(K)$. Therefore, there is a natural inverse sequence of finite spaces.

$$X^0 \leftarrow X^1 \leftarrow X^2 \leftarrow X^3 \leftarrow \cdots$$
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**Example.** Let us consider the unit interval $I$.

$I\quad X^0\quad h_{0,1}\quad X^1\quad h_{1,2}\quad X^2\quad h_{2,3}\quad \cdots$
Preliminaries and motivation

Theorem (Clader, 2009)

Let $K$ be a compact simplicial complex. The inverse limit of $(X^n, h_{n,n+1})$ contains a homeomorphic copy of $K$, which is a strong deformation retract.

Remark. The same result also holds for compact metric spaces.
Preliminaries and motivation

Theorem (Clader, 2009)

Let $K$ be a compact simplicial complex. The inverse limit of $(X^n, h_{n,n+1})$ contains a homeomorphic copy of $K$, which is a strong deformation retract.

Remark. The same result also holds for compact metric spaces.
A dynamical system for a topological space $X$ consists of a triad $(\mathbb{T}, X, \varphi)$, where $\mathbb{T}$ is usually $\mathbb{Z}$ or $\mathbb{R}$ and $\varphi : \mathbb{T} \times X \to X$ is a continuous function satisfying

1. $\varphi(0, x) = x$ for every $x \in X$.
2. $\varphi(t + s, x) = \varphi(t, \varphi(s, x))$ for all $s, t \in \mathbb{T}$ and $x \in X$. 
Preliminaries and motivation

Main Idea:
Preliminaries and motivation

Proposition

Let $A$ be a finite space.
- If $(\mathbb{R}, A, \varphi)$ is a continuous dynamical system, then $\varphi$ is trivial.
- If $(\mathbb{Z}, A, \varphi)$ is a discrete dynamical system, there exists $n \in \mathbb{N}$ such that $\varphi^n = id$. 
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**Proposition**

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Consider **Multivalued maps** to define dynamical systems.
We say that a topological space $X$ is **acyclic** if the homology groups in all dimensions of $X$ are isomorphic to the corresponding homology groups of a point.

**Definition**

Given a continuous map $f : X \to Y$ between two finite spaces, we say that $f$ is a Vietoris-like map if for every chain $y_1 < y_2 < \ldots < y_n$ in $Y$ we get that $\bigcup_{i=1}^{n} f^{-1}(y_i)$ is acyclic.
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**Example.** Every homeomorphism is a Vietoris-like map. Indeed, $f : X \to X$ is a Vietoris-like map if and only if $f$ is a homeomorphism.
Theorem

If $f : X \rightarrow Y$ is a Vietoris-like map, then $f$ induces isomorphisms in all homology groups.
Vietoris-like maps and multivalued maps

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Some properties of Vietoris-like maps

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous maps between finite spaces.

- If $f$ and $g$ are Vietoris-like maps, then $g \circ f : X \rightarrow Z$ is a Vietoris-like map.
- If $f$ and $g \circ f$ are Vietoris-like maps, then $g$ is a Vietoris-like map.
- The 2-out-of-3 property does not hold for Vietoris-like maps.
Vietoris-like maps and multivalued maps

**Definition**

Let $F : X \rightrightarrows Y$ be a multivalued map between finite spaces. We say that $F$ is a Vietoris-like multivalued map if the projection $p$ onto the first coordinate from the graph of $\Gamma(F)$ is a Vietoris-like map.
Vietoris-like maps and multivalued maps

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Let \( F : X \rightarrow Y \) be a multivalued map between finite spaces. We say that \( F \) is a Vietoris-like multivalued map if the projection \( p \) onto the first coordinate from the graph of \( \Gamma(F) \) is a Vietoris-like map.

**Remark.** \( F_* : H_*(X) \rightarrow H_*(Y) \) is given by \( q_* \circ p_*^{-1} \), where \( q : \Gamma(F) \rightarrow Y \) is the projection onto the second coordinate.
Vietoris-like maps and multivalued maps

**Examples**

- Let $f : X \to Y$ be a continuous map. If we consider $f$ as a multivalued map, then $f$ is a Vietoris-like multivalued map since $p : \Gamma(f) \to X$ is a homeomorphism. Moreover, $f_* = q_* \circ p_*^{-1}$.

- If $f : X \to Y$ is a Vietoris-like map, then $F : Y \to X$ given by $F(y) = f^{-1}(y)$ is a Vietoris-like multivalued map.
A Coincidence theorem and consequences

**Lefschetz number.** Let $f : X \rightarrow X$ be a continuous map, where $X$ is a finite space. The lefschetz number of $f$ is given by

$$
\Lambda(f) = \sum_{i=0} \left( -1 \right)^i tr(f_* : H_i(X) \rightarrow H_i(X)),
$$

where $tr$ denotes the trace and $f_*$ denotes the linear map induced by $f$ on the torsion-free part of the homology of $X$.

**Theorem**

Let $f, g : X \rightarrow Y$ be continuous maps between finite spaces, where $f$ is a Vietoris-like map. If $\Lambda(g_* \circ f_*^{-1}) \neq 0$, then there exists $x \in X$ such that $f(x) = g(x)$.
Lefschetz fixed point theorem for multivalued maps

Let $X$ be a finite space. If $F : X \mapsto X$ is a Vietoris-like multivalued map and $\Lambda(F_* = q_* \circ p_*^{-1}) \neq 0$, then there exists $x \in X$ with $x \in F(x)$. 

Remark. Not every multivalued map may be expressed as a composition of Vietoris-like multivalued maps.
Lefschetz fixed point theorem

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Theorem

Let $F : X \rightarrow X$ be a multivalued map, where $X$ is a finite space. Suppose that $F = G_n \circ \cdots \circ G_0$, where $G_i : Y_i \rightarrow Y_{i+1}$, $Y_0 = Y_{n+1} = X$, $Y_i$ is a finite space and $G_i$ is a Vietoris-like multivalued map. If $\Lambda(G_n* \circ \cdots \circ G_0*) \neq 0$, then there exists a point $x \in X$ such that $x \in F(x)$.
**Lefschetz fixed point theorem**

**Lefschetz fixed point theorem for multivalued maps**

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**Theorem**

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**Remark.** Not every multivalued map may be expressed as a composition of Vietoris-like multivalued maps.
Recall that given a finite space $X^0$ we may consider the following inverse sequence

$$
X^0 \xleftarrow{h_{0,1}} X^1 \xleftarrow{h_{1,2}} X^2 \xleftarrow{h_{2,3}} X^3 \xleftarrow{h_{3,4}} \cdots X^n \xleftarrow{h_{n,n+1}} X^{n+1} \xleftarrow{h_{n+1,n+2}} \cdots
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$$

**Proposition**

Let $X$ be a finite space and $m \geq n$. Then $h_{n,m} : X^m \to X^n$ is a Vietoris-like map which induces the identity in homology.
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**Corollary**

Let $X$ be a finite space and $m \geq n$. Then $H_{m,n} : X^n \to X^m$, defined by $H(x) = h^{-1}(x)$, is a Vietoris-like multivalued map which induces the identity in homology.
Given a continuous map \( f : |K| \to |K| \), there is a natural inverse sequence induced by \( f \) (use simplicial approximation theorem).

\[
\begin{align*}
X^0 & \leftarrow f_{0,1} X^1 & f_{1,2} & f_{2,3} & X^3 & \cdots
\end{align*}
\]
Given a continuous map $f : |K| \to |K|$, there is a natural inverse sequence induced by $f$ (use simplicial approximation theorem).

Therefore, we have

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\begin{array}{cccccc}
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\bullet & F_1 & \bullet & F_2 & \bullet & F_3 \\
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\end{array}
$$

where $F_{n+1} = H_{n+1,n} \circ f_{n,n+1}$. 
Proposition

If $\Lambda(f) \neq 0$, then there exists a point $x_{n+1} \in X^{n+1}$ such that $x_{n+1} \in F_{n+1}(x_{n+1})$ for every $n \in \mathbb{N}$. 
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Theorem

If $f : |K| \to |K|$ is a continuous map, where $K$ is a simplicial complex, then $f$ has a fixed point if and only if there exist a finite approximative sequence for $f$, $(X^n, h_{n,n+1})$, a sequence $\{x_{n+1}\}_{n \in \mathbb{N}}$ and $m \in \mathbb{N}$ such that $x_{n+1} \in X^{n+1}$, $x_n = h_{n,n+1}(x_{n+1})$ for every $n \in \mathbb{N}$ and $x_{n+1} \in F_{n+1}(x_{n+1})$ for every $n + 1 \geq m$. 
Example. Let $f : S^1 \to S^1$ be given by $f(x, y) = (x, -y)$.

Goal. Generalize these results to compact metric spaces using more geometrical constructions.
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Main Idea: Enclose Vietoris-like multivalued maps in a category to get other dynamical invariants.
Localization of finite spaces at Vietoris-like maps

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**Definition**

Let $X$ and $Y$ be finite spaces. We say that $X \xleftarrow{p} Z \xrightarrow{q} Y$ is a span or a diagram if $p$ is a Vietoris-like map.
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**Examples.** Continuous maps and Vietoris-like multivalued maps.
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**Examples.** Continuous maps and Vietoris-like multivalued maps.

**Steps**

1. Define the composition of spans. Solution: pull-backs.
2. Define an equivalence relation between spans. Solution: define a new notion of homotopy that generalizes the usual notion of homotopy for single valued maps in the category of finite spaces.
Localization of finite spaces at Vietoris-like maps

Category of finite spaces
- Objects: finite topological $T_0$-spaces
- Morphisms: continuous maps

Homotopical category of finite spaces
- Objects: finite topological $T_0$-spaces
- Morphisms: homotopy classes of continuous maps

Localization of the category of finite topological spaces at the class of Vietoris-like maps
- Objects: finite topological $T_0$-spaces
- Morphisms: homotopy classes of spans
1. Let $X$ and $Y$ be finite models of $S^n$. In the usual category it is not possible to get that every integer number may be realized as the topological degree of a continuous map $f : X \to Y$.

2. In the localized category of finite spaces the above result is possible.


5. Chocano, P. J *Category of fractions and localization on finite spaces*  
   In preparation.

6. E. Clader. *Inverse limits of finite topological spaces*. Homology,  

7. M.C. McCord. *Singular homology and homotopy groups of finite  
Thanks for your attention! Any questions?