From Discrete Morse Theory to Combinatorial Topological Dynamics

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Universidad de Buenos Aires and George Mason University

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Classical Morse Theory

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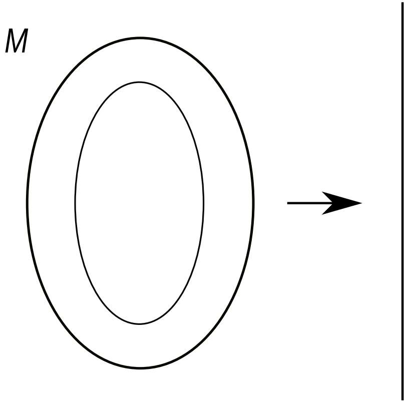

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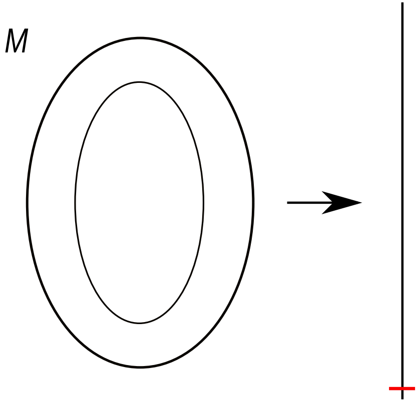

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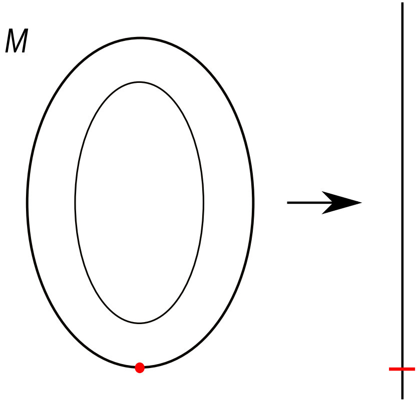

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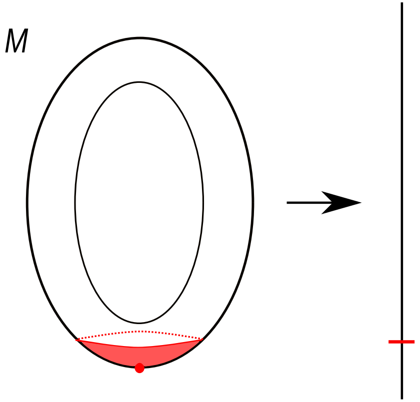

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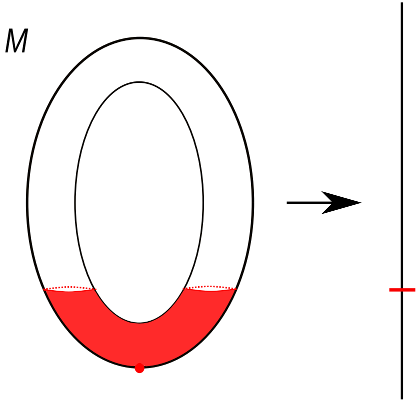

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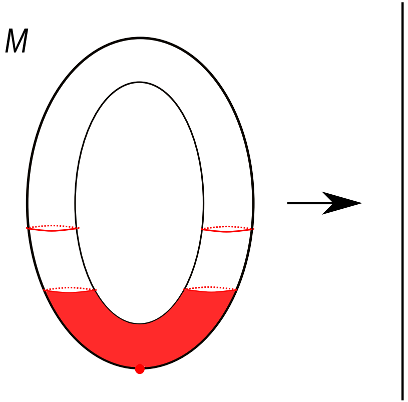

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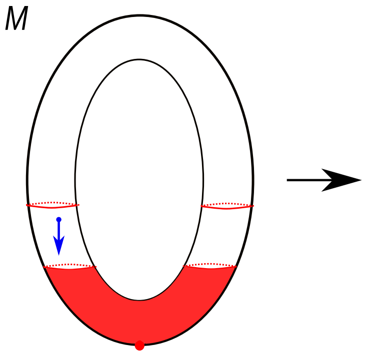

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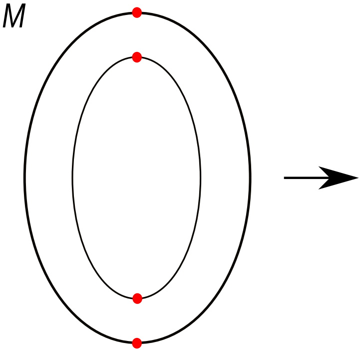

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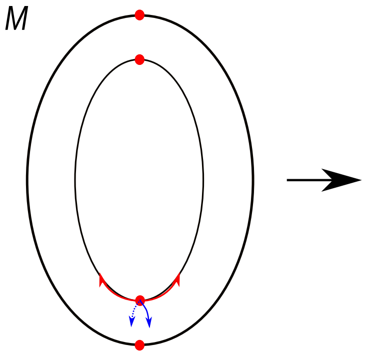

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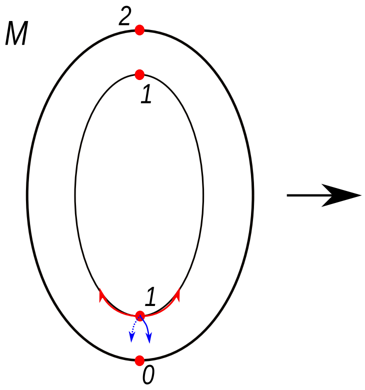

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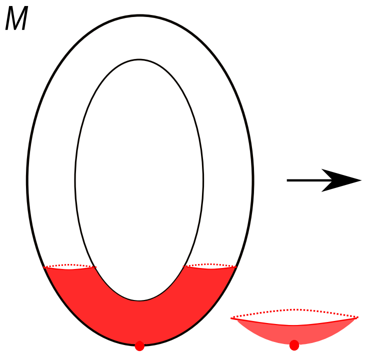

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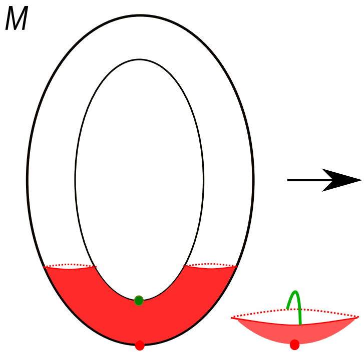

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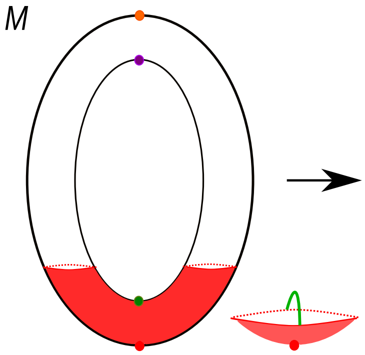

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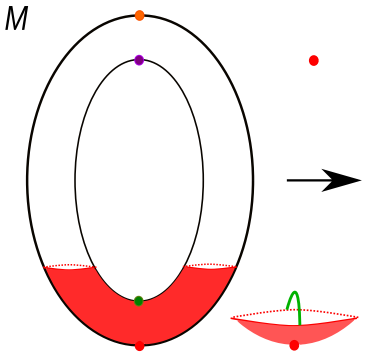

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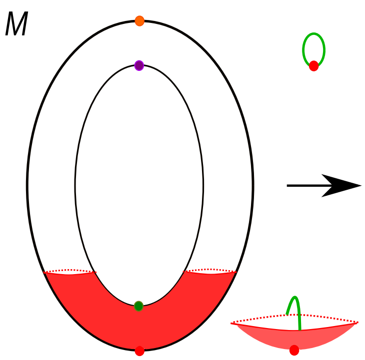

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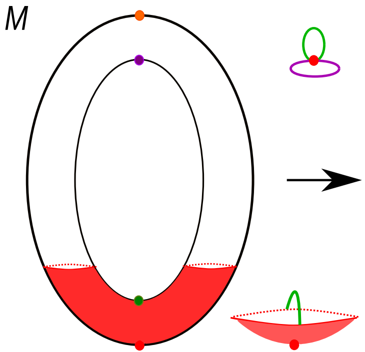

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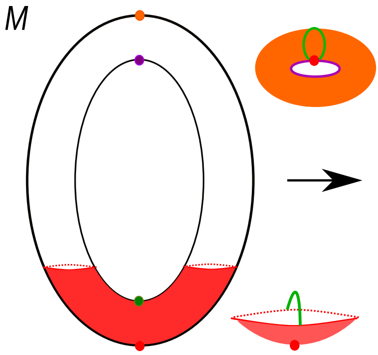

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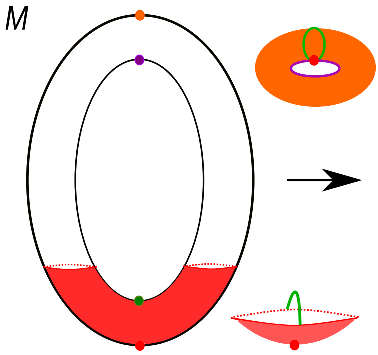

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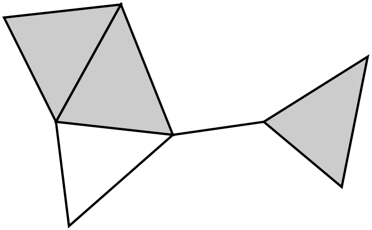

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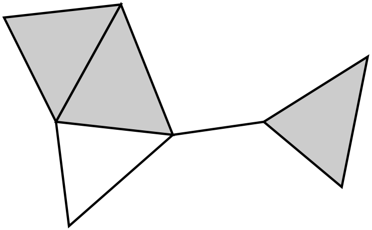

K a simplicial complex. A map $f : S_K \to \mathbb{R}$ is a discrete Morse function if for every $\sigma \in K$, $\sharp\{\tau \subsetneq \sigma | f(\tau) \geq f(\sigma)\} \leq 1$ and $\sharp\{\tau \supsetneq \sigma | f(\tau) \leq f(\sigma)\} \leq 1$.

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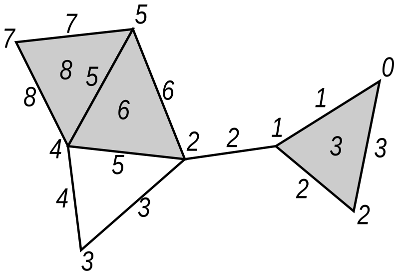

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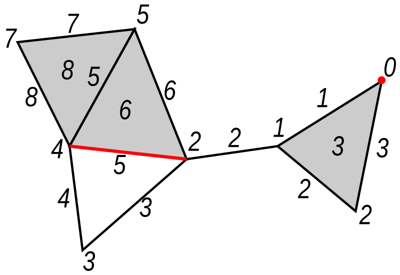

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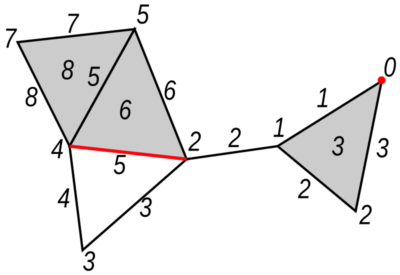

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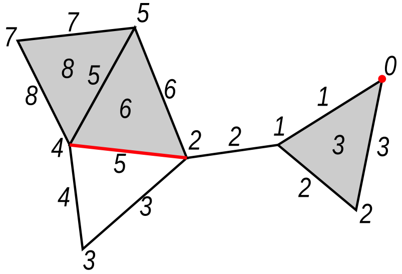

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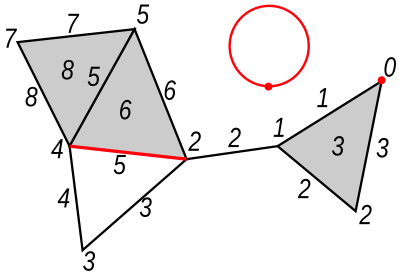

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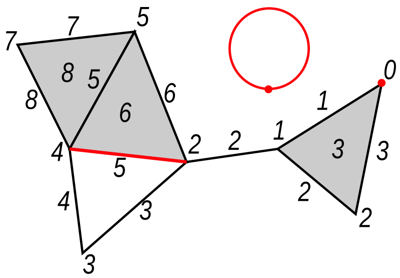

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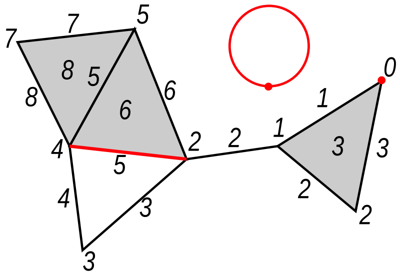

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Applications: Topological combinatorics

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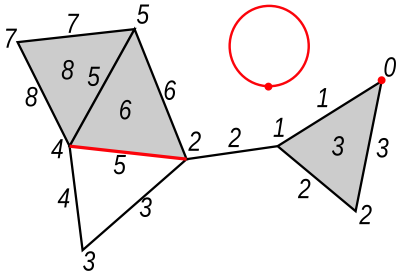

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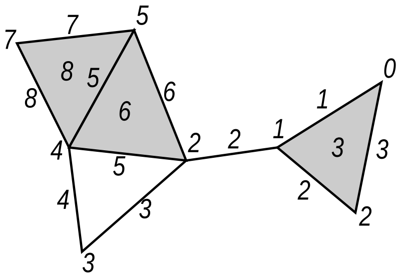
Applications: Topological combinatorics, Topological Data Analysis, Biology, Computer Science, etc.



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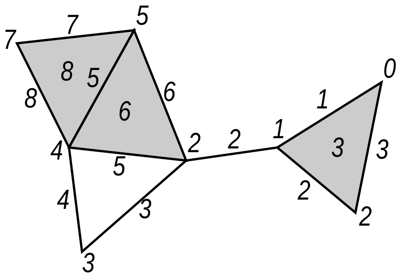

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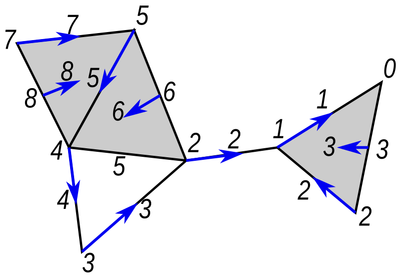

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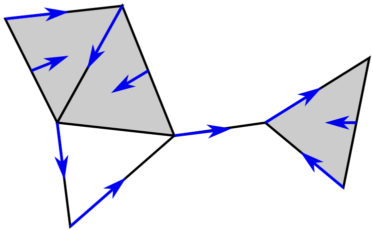

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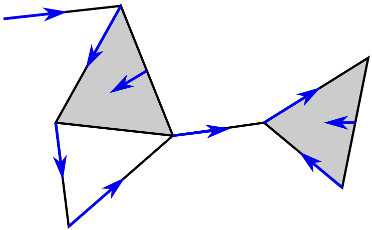

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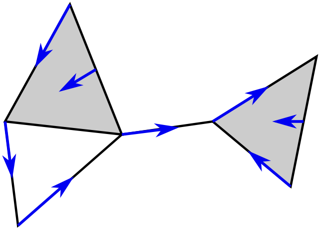

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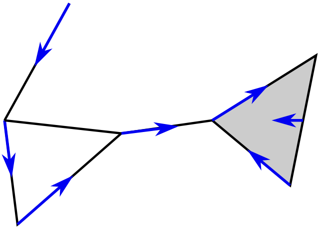

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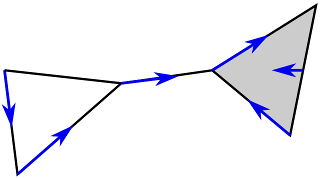

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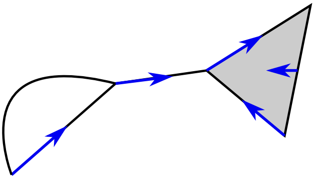

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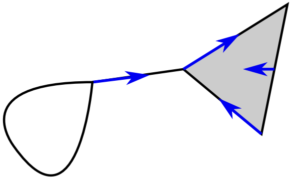

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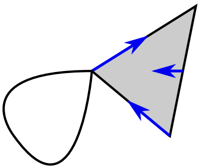

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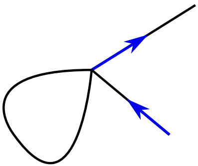

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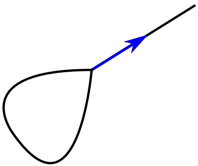

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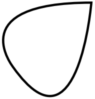

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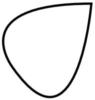

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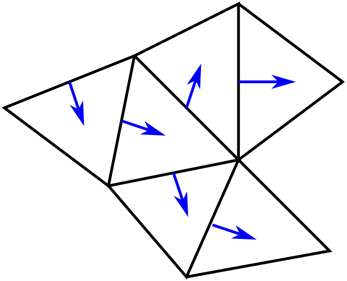

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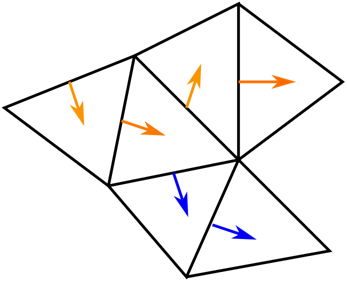

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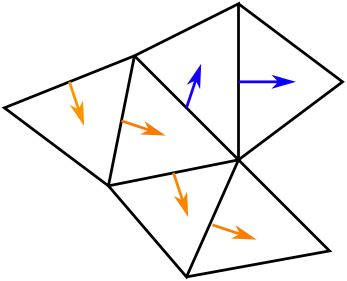

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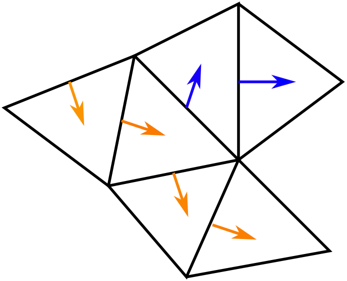

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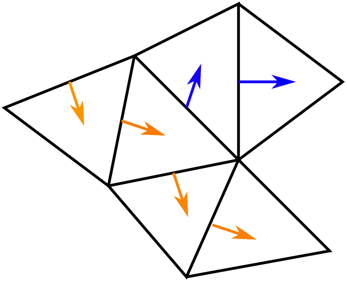

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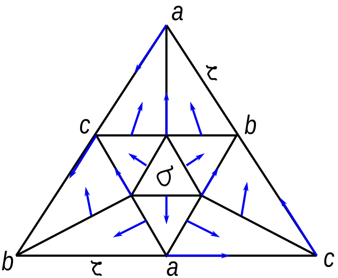
Example: $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$.



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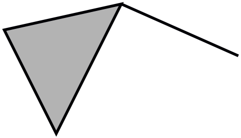

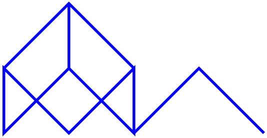

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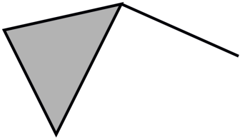

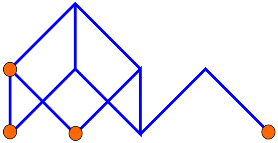

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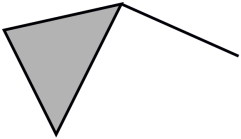

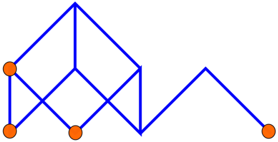

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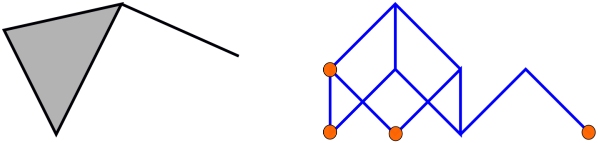

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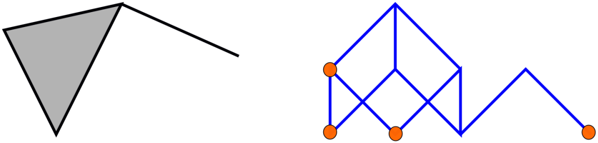

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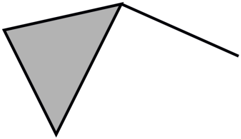

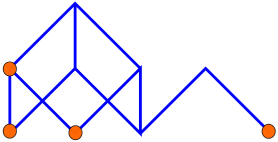

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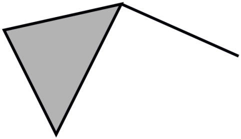

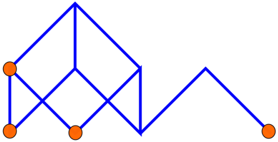

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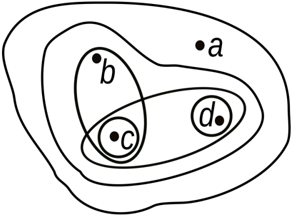

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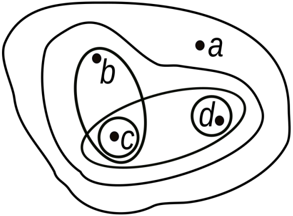

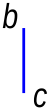

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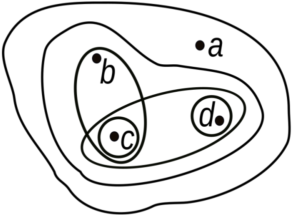

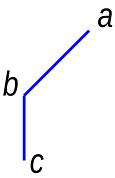

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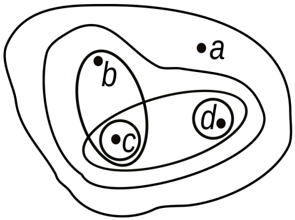

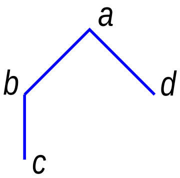

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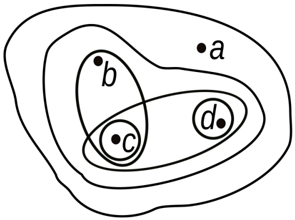

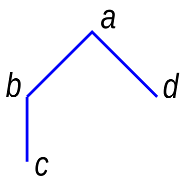

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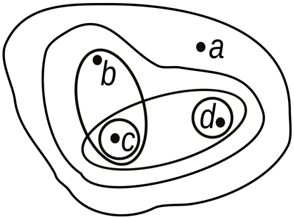

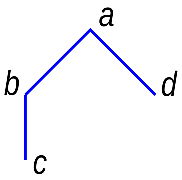

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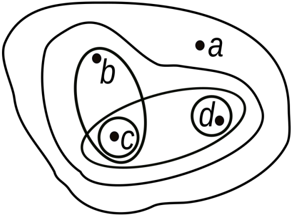

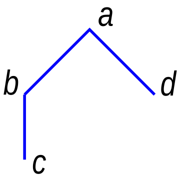

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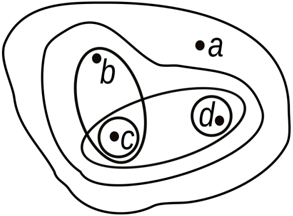

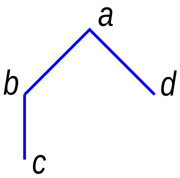

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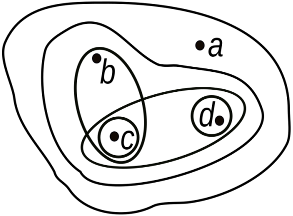

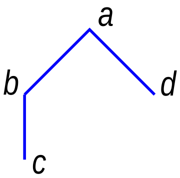

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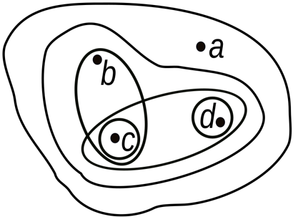

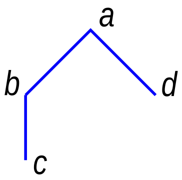

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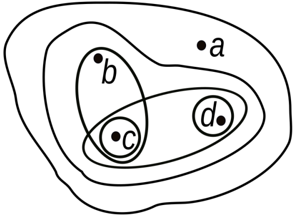

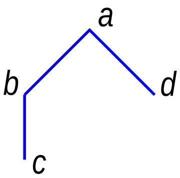

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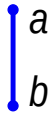
Connectivity: The Sierpiński space 

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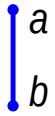
Connectivity: The Sierpiński space 

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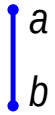
Connectivity: The Sierpiński space 

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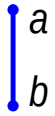
Connectivity: The Sierpiński space 

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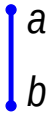
Connectivity: The Sierpiński space 

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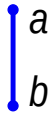
Connectivity: The Sierpiński space 

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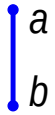
Connectivity: The Sierpiński space 

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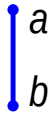
Connectivity: The Sierpiński space 

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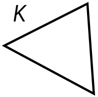

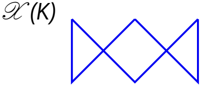

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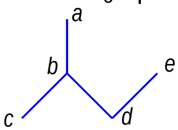

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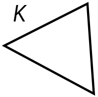

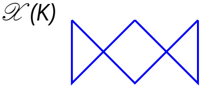

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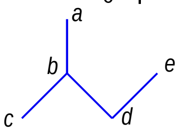

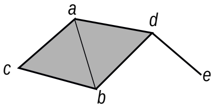

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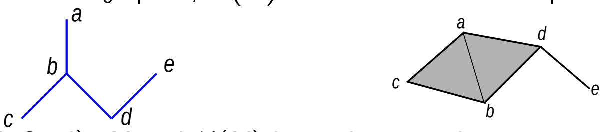

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