From Discrete Morse Theory to Combinatorial Topological Dynamics

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Morse inequalities: If $\alpha_k$ is the number of critical points of index $k$, $\alpha_k - \alpha_{k-1} + \ldots + (-1)^k \alpha_0 \geq b_k(M) - b_{k-1}(M) + \ldots + (-1)^k b_0(M)$. 
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$K$ a simplicial complex. A map $f : S_K \rightarrow \mathbb{R}$ is a discrete Morse function if for every $\sigma \in K$, 
\[ \#\{\tau \subseteq \sigma | f(\tau) \geq f(\sigma)\} \leq 1 \text{ and } \#\{\tau \supseteq \sigma | f(\tau) \leq f(\sigma)\} \leq 1. \]
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Applications: Topological combinatorics, Topological Data Analysis, Biology, Computer Science, etc.
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Gradient vector field: is the map
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Morse complex: For each $p \geq 0$ let $C_p$ be the free abelian group generated by the critical $p$-simplices. Define $\partial : C_{p+1} \rightarrow C_p$ by $\partial(\sigma) = \sum c_{\tau,\sigma} \tau$ where $c_{\tau,\sigma} = \sum_{\gamma \in \Gamma(\sigma,\tau)} m(\gamma)$, $m(\gamma) = \pm 1$ depending on orientations.
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Example: $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$. 
General definition of vector field in $K$: is map $V : A \to B$ for subsets $A, B \subseteq S_K$ such that
i) for every $\sigma \in A$, $V(\sigma) = \sigma$ or $\sigma$ is a codimension 1 face of $V(\sigma)$,
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\begin{center}
\begin{tikzpicture}
    \node (a) at (0,0) {$a$};
    \node (b) at (-1,-1) {$b$};
    \node (c) at (-1,-2) {$c$};
    \node (d) at (0,-2) {$d$};

    \draw (a) -- (b);
    \draw (a) -- (c);
    \draw (a) -- (d);
\end{tikzpicture}
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\textbf{Proof:} $\Leftarrow$) Suppose $f(x) \leq f(x')$. Then $U_{f(x)} \subseteq Y$ is open and so is $f^{-1}(U_{f(x)})$. Thus $x \in U_{x} \subseteq f^{-1}(U_{f(x)})$. Then $f(x) \in U_{f(x)}$, so $f(x) \leq f(x')$. 

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**Thm:** Two continuous maps $f, g : X \to Y$ between finite spaces are homotopic iff there is a sequence $f = f_0 \leq f_1 \geq f_2 \leq \ldots f_n = g$, where $h \leq h'$ means $h(x) \leq h'(x)$ for every $x \in X$. 
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