Numerical Abstract Domain using Support Function.

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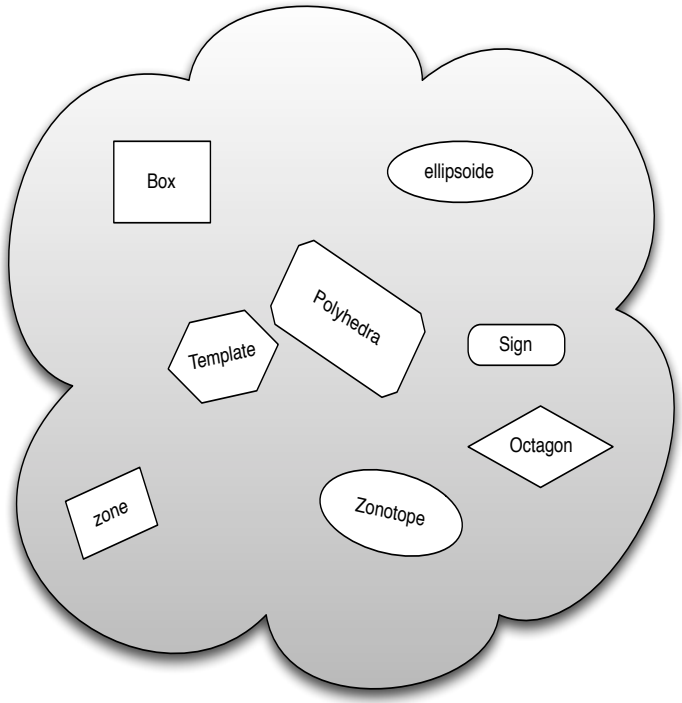
Introduction

Static analysis by abstract interpretation



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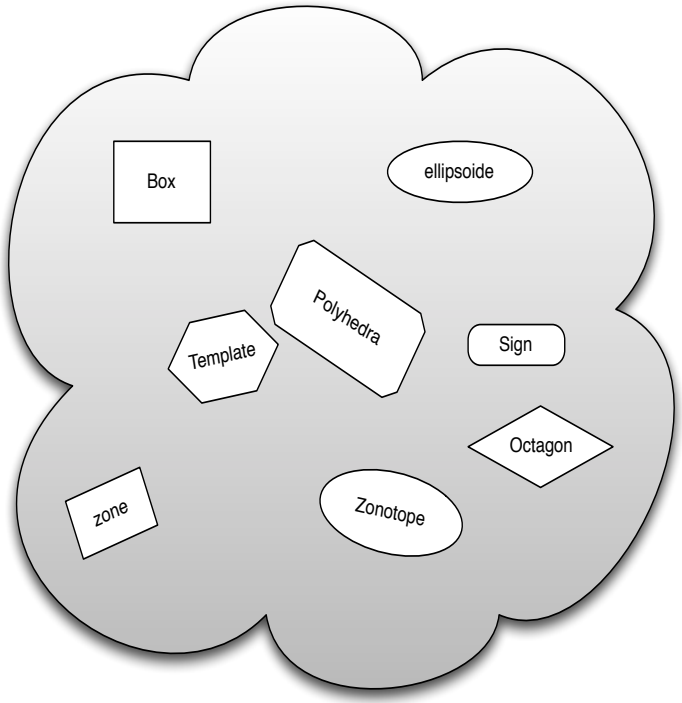

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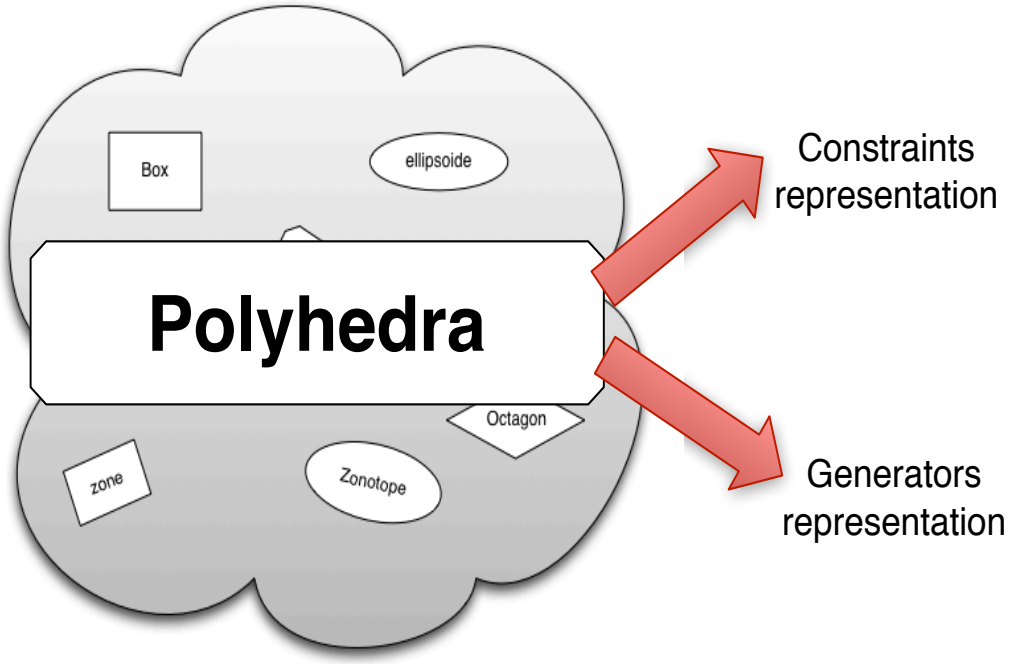

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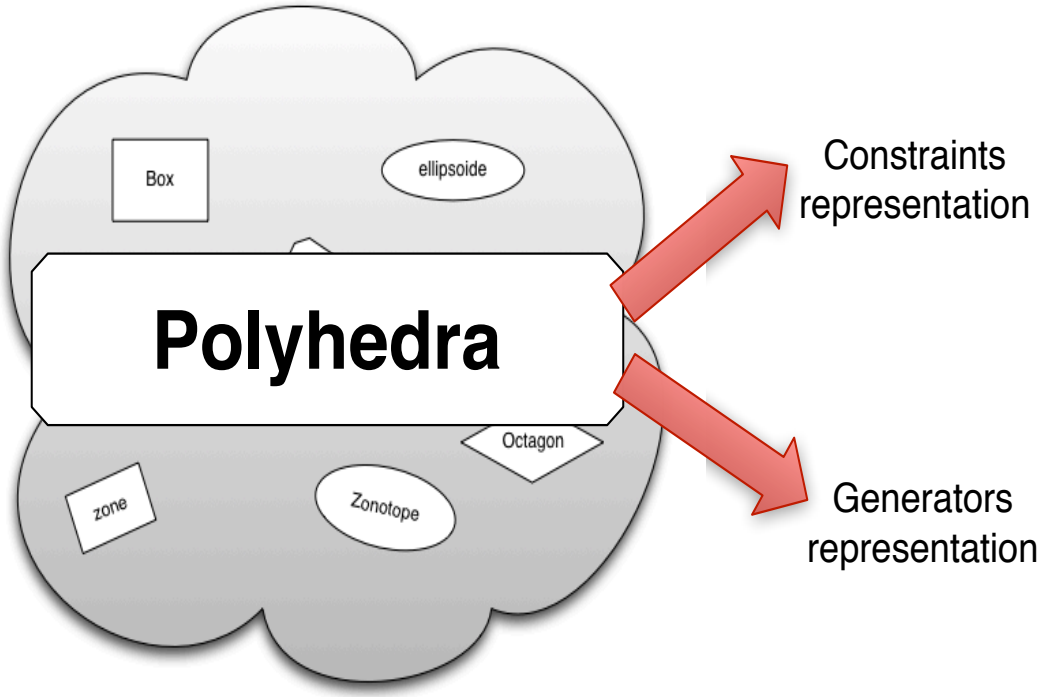
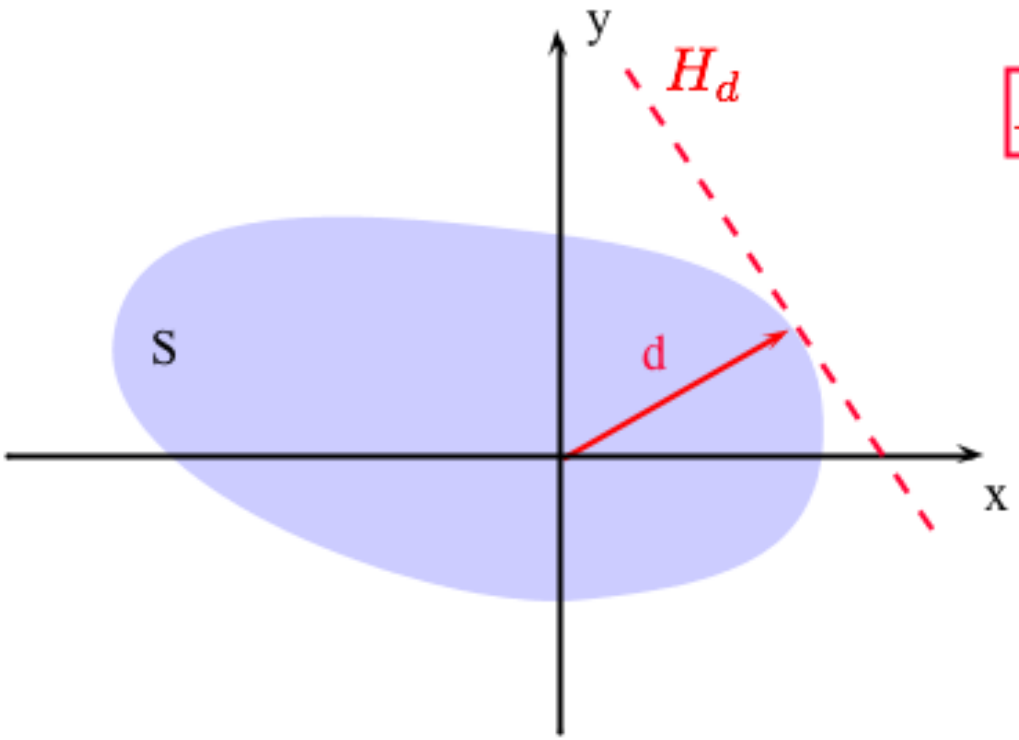


Support function

Definition

Let S be a closed convex set and δ_S its support function, such that :

$$\forall d \in \mathbb{R}^n, \delta_S(d) = sup\{\langle x, d\rangle : x \in S\}$$



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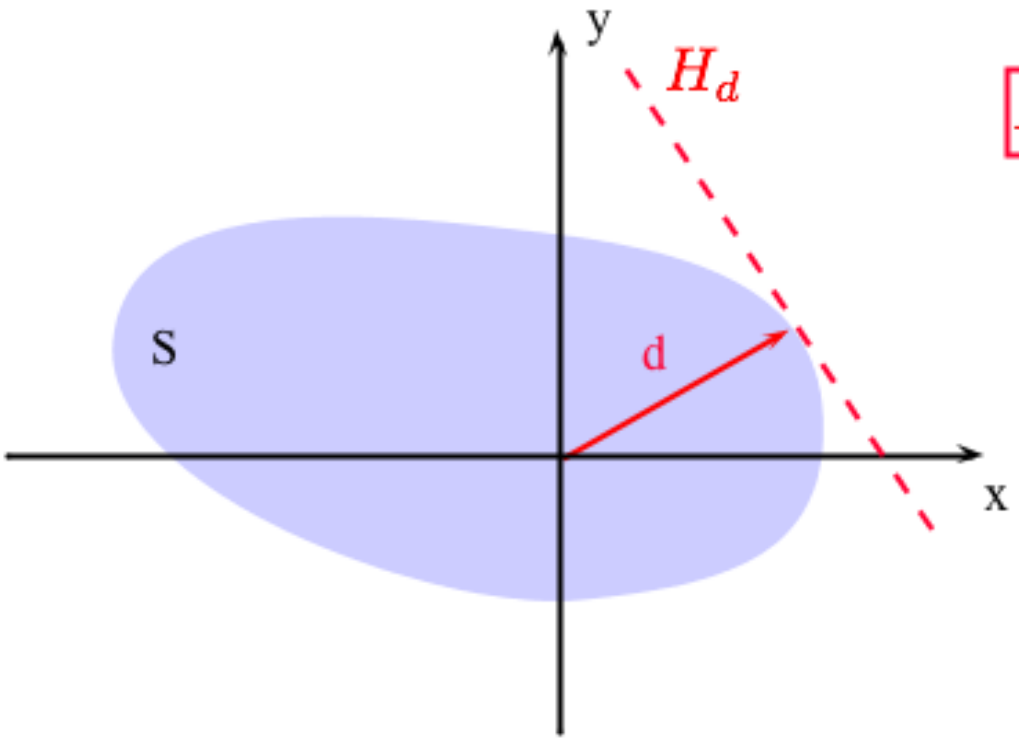
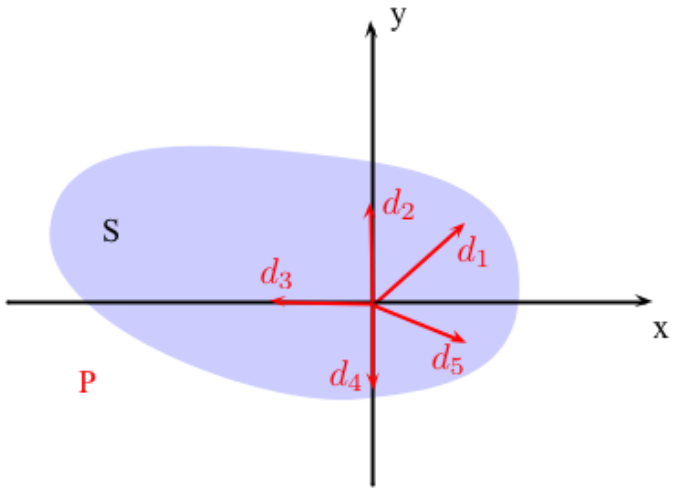


$$H_d : \langle x, d\rangle \leq \delta_S(d)$$

$$S = \cap_{d \in R^n} H_d$$

Let $\Delta = \{d_1, d_2, d_3, d_4, d_5\}$ be a set of directions.



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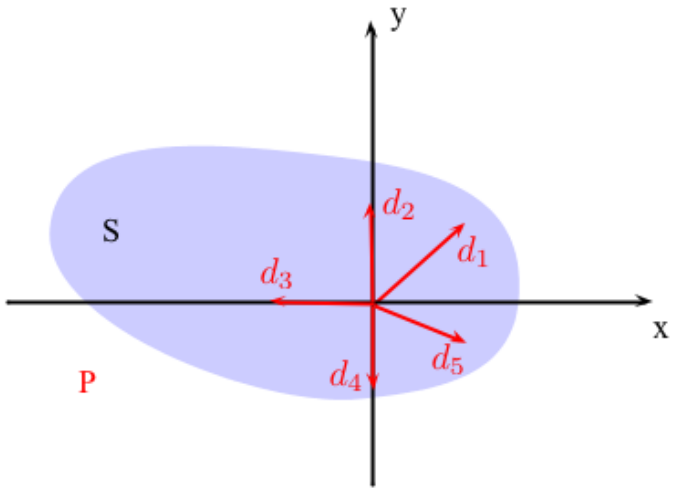

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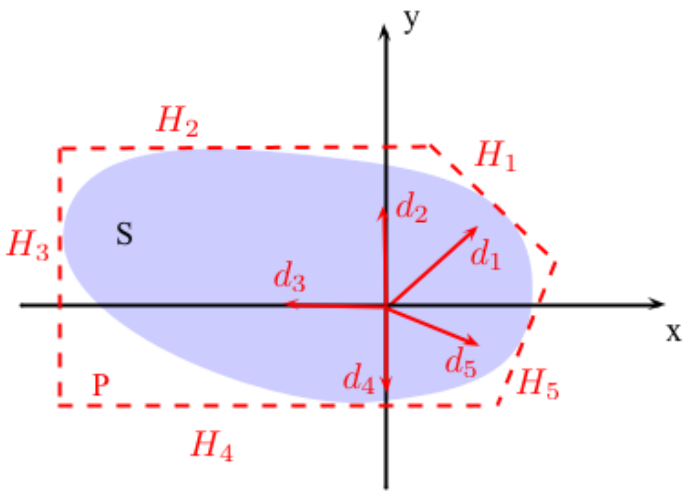


Property

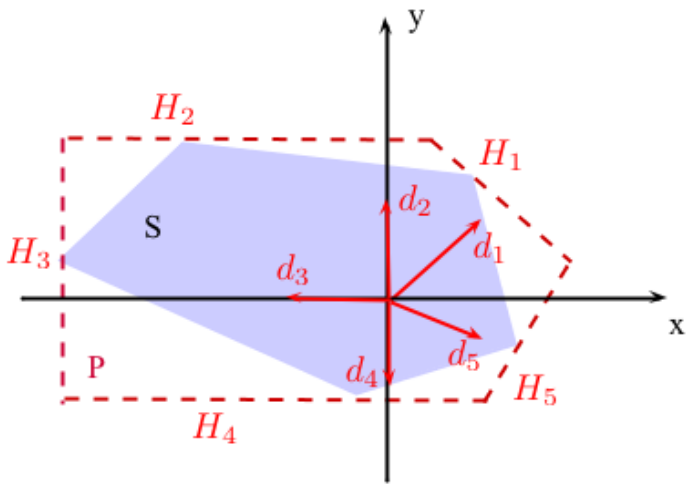
Let S be a closed convex set, and $\Delta \subseteq \mathbb{R}^n$ be a set of directions.
We put

$$P = \bigcap_{d \in \Delta} \{x \in \mathbb{R}^n | \langle x, d \rangle \leq \delta_S(d)\}$$

Then

$$S \subseteq P$$

Let $\Delta = \{d_1, d_2, d_3, d_4, d_5\}$ be a set of directions.



The special case of polyhedron

Let S be a polyhedron. If S is represented by :

- Linear system, δ_S is obtained using Linear Programming.
- Generators (vertices) v_i, $\delta_S(d) = sup\{\langle v_i, d\rangle : v_i \in S\}$.

Properties

Let S, S' be two closed convex sets. We have :

- $\forall M \in \mathbb{R}^n \times \mathbb{R}^m, \delta_{MS}(d) = \delta_S(M^T d)$.
- $\delta_{S \oplus S'}(d) = \delta_S(d) + \delta_{S'}(d)$. $S \oplus S' = \{x + x' \mid x \in S,\ x' \in S'\}$
- $\delta_{S \cup S'}(d) = \max(\delta_S(d), \delta_{S'}(d))$.
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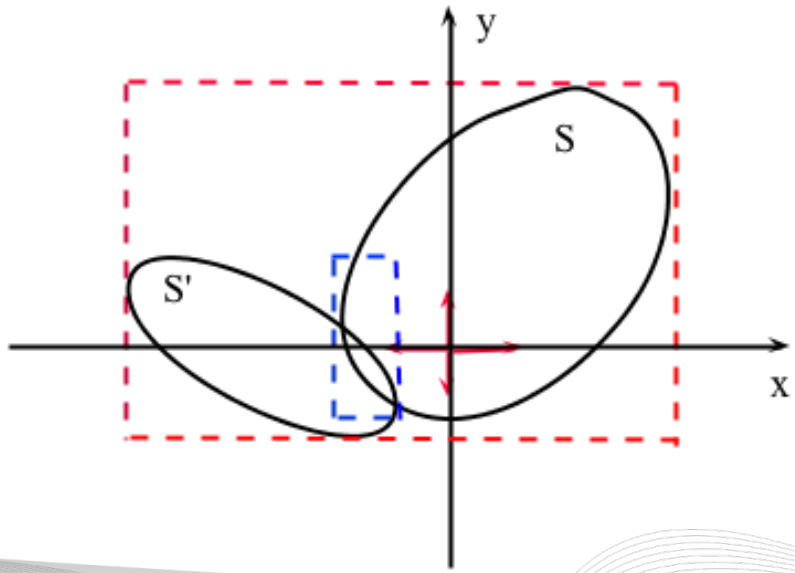

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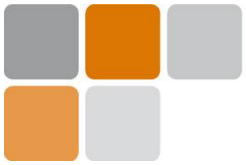

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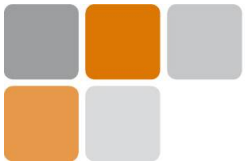

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The concretisation function

$$\begin{array}{rccl} \gamma_{\Delta} : & (\Delta \to \mathbb{R}_{\infty}) & \longrightarrow & P(\mathbb{R}^n) \\ & \Omega & \longrightarrow & \bigcap_{d \in \Delta} \{x \in \mathbb{R}^n \mid \langle x, d \rangle \leq \Omega(d)\} \end{array}$$

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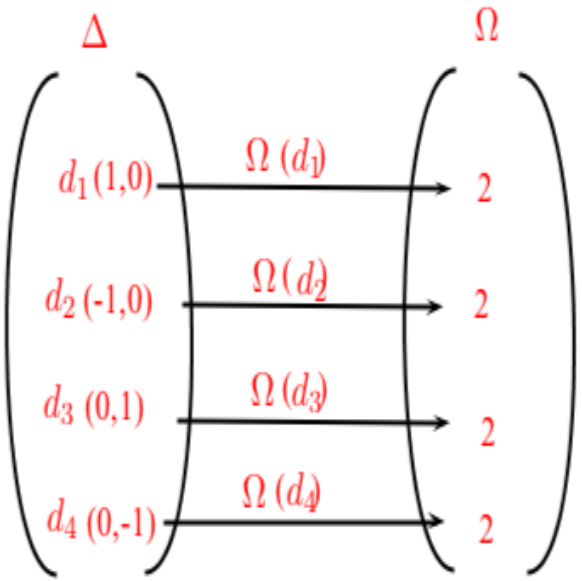

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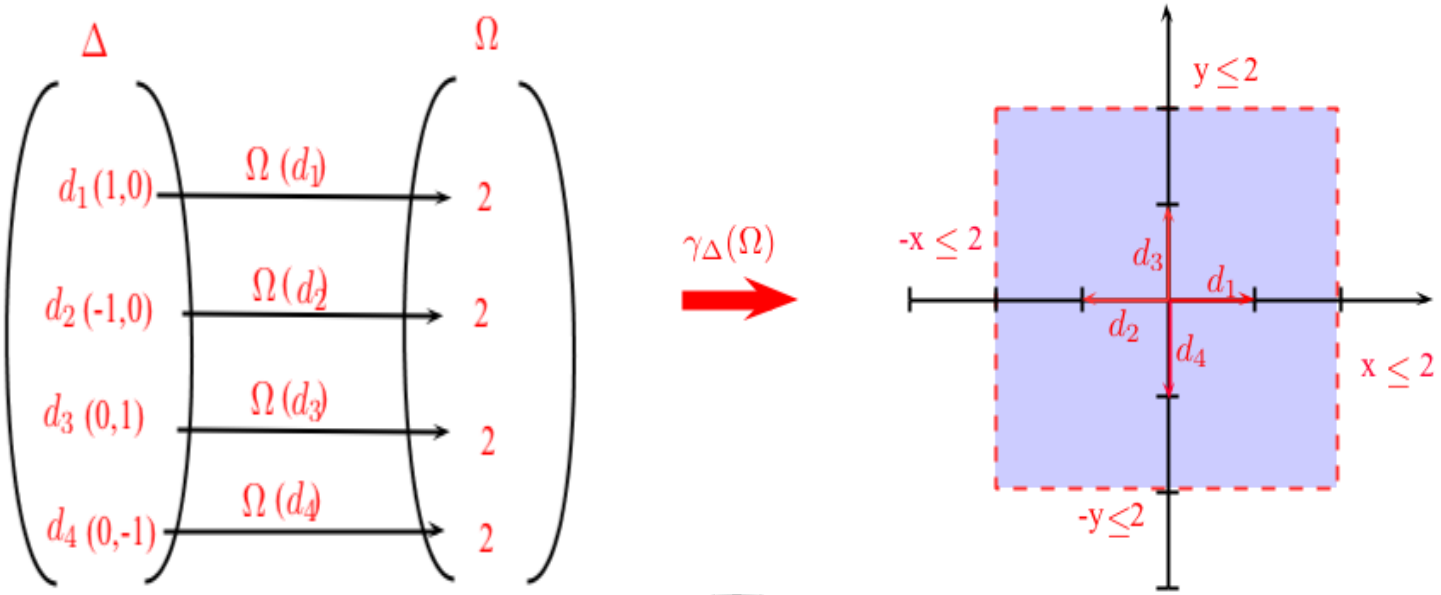

The abstraction function

$$\alpha_\Delta : \quad P(\mathbb{R}^n) \longrightarrow (\Delta \to \mathbb{R}_\infty)$$

$$S \longrightarrow \begin{cases} \lambda d.\ -\infty & \text{if } S = \emptyset \\ \lambda d.\ +\infty & \text{if } S = \mathbb{R}^n \\ \lambda d.\ \delta_S(d) & \text{otherwise} \end{cases}$$

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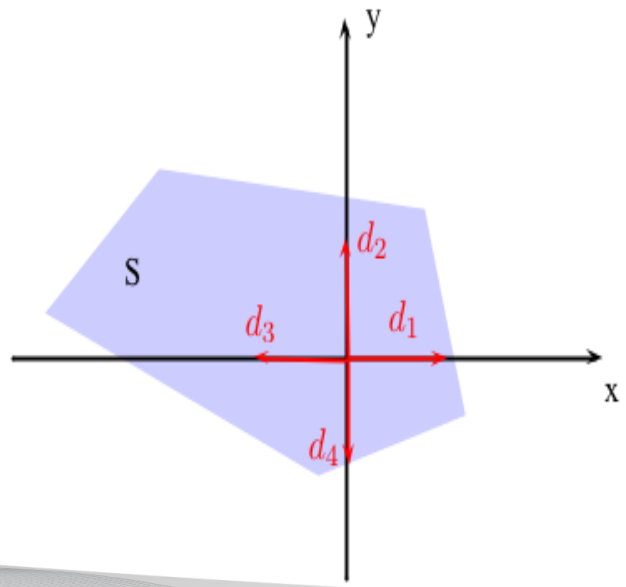

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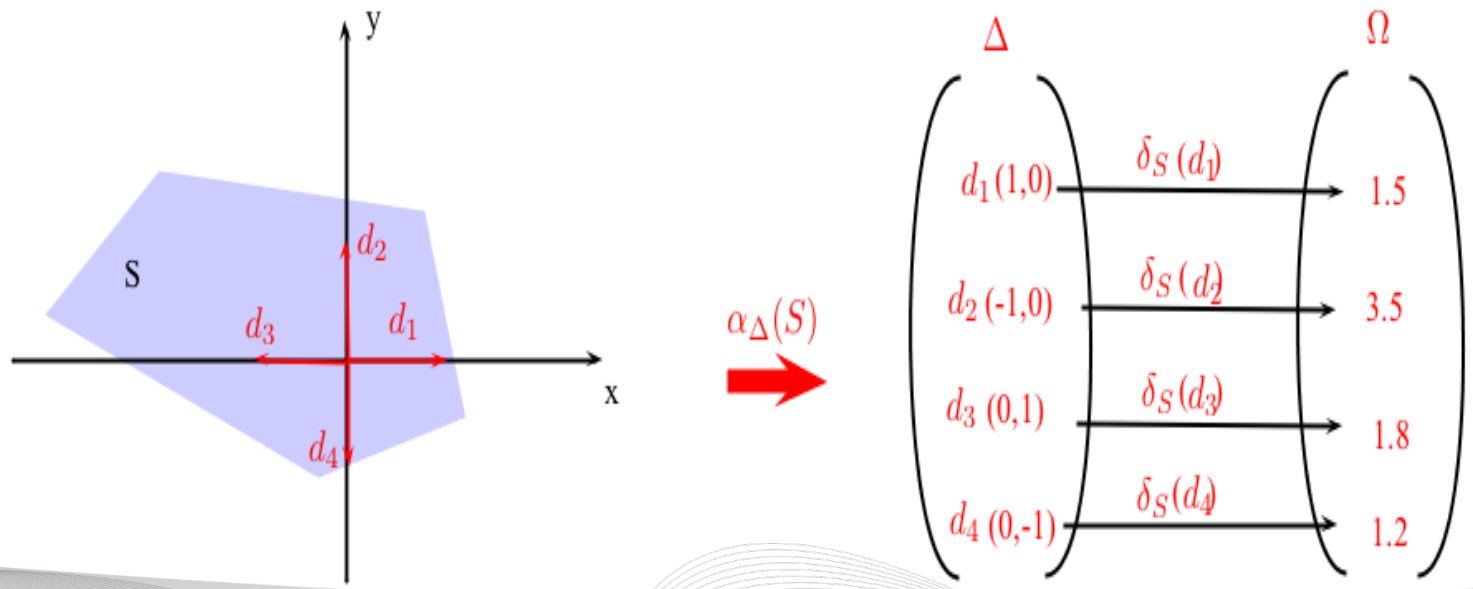

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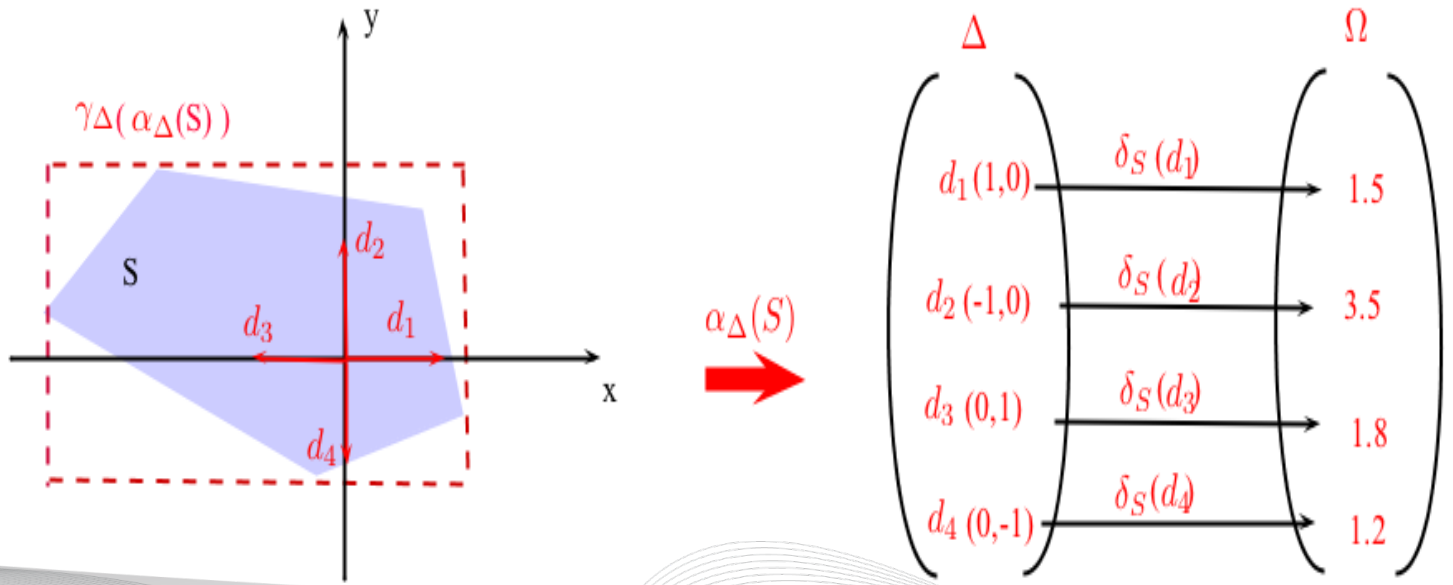

The complete lattice $\langle \mathbb{P}^{\sharp}_{\Delta}, \sqsubseteq, \bot, \top, \sqcup, \sqcap \rangle$ is defined by :

- An order relation : $\Omega_1 \sqsubseteq \Omega_2 \Leftrightarrow \gamma_{\Delta}(\Omega_1) \subseteq \gamma_{\Delta}(\Omega_2)$.
- A minimal element : $\bot = \lambda d.\ -\infty$.
- A maximal element : $\top = \lambda d.\ +\infty$.
- A join operator : $\Omega_1 \sqcup \Omega_2 = \lambda d.\ \max(\Omega_1(d), \Omega_2(d))$.
- A meet operator : $\Omega_1 \sqcap \Omega_2 = \lambda d.\ \min(\Omega_1(d), \Omega_2(d))$.

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Notes :

$$\gamma_{\Delta}(\Omega_1 \sqcup \Omega_2) = \gamma_{\Delta}(\Omega_1) \cup \gamma_{\Delta}(\Omega_2).$$
$$\gamma_{\Delta}(\Omega_1 \sqcap \Omega_2) \supseteq \gamma_{\Delta}(\Omega_1) \cap \gamma_{\Delta}(\Omega_2).$$

Program

Input : $\mathbb{P}_0$ a bounded polyhedron.
Input : $A \in \mathbb{R}^n \times \mathbb{R}^m$, $b \in \mathbb{R}^m$.
Input : $c \in \mathbb{R}^n$, $l \in \mathbb{R}$
$X \in \mathbb{P}_0$
while $(\langle X, c \rangle \leq l)$ {
$\quad X = AX + b.$
}

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$$\Omega_i = \Omega_{i-1} \sqcup [(A\Omega_{i-1} + b) \sqcap (\langle c, X \rangle \leq l)]$$

- **Case 1 :** $\Omega_i = \Omega_{i-1} \sqcup [(A\Omega_{i-1} + b) \sqcap (\langle c, X \rangle \# I)]$

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The first abstract element

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Fixpoint computation using Kleene iteration

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The i^{th} abstract element

$$\Omega_i = \lambda d.\, \max\{\delta_{\mathbb{P}_0}(A^{Tj} d) + \sum_{k=1}^{j} \langle b, A^{T(k-1)} d \rangle, j = 0, .., i\}$$

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$$\alpha_\Delta(\mathbb{P}_i) = \Omega_i$$

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$$\delta_H(d) = \begin{cases} I & \text{if } d = \lambda c, \lambda \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

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$$H : \langle X, c \rangle \leq l$$

$$\delta_H(d) = \begin{cases} l & \text{if } d = \lambda c, \lambda \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

We put $\Delta \cup \{c\}$:

- $\Delta_1 = \{c\} \cup \{d \in \Delta | d = \lambda c, \lambda \geq 0\}$.
- $\Delta_2 = \Delta \setminus \Delta_1$

- **Case 2 :** $\Omega_i = \Omega_{i-1} \sqcup [(A\Omega_{i-1} + b) \sqcap (\langle c, X \rangle \leq l)]$
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- For $d \in \Delta_1$, we have that :

$$\Omega_i = \lambda d. \max(\delta_{\gamma_\Delta(\Omega_{i-1})}(d), \min(\delta_{\gamma_\Delta(\Omega_{i-1})}(A^T d) + \langle b, d \rangle, l))$$

Such that $\lambda d. \delta_{\mathbb{P}_i}(d) \leq \Omega_i(d)$.

Fixpoint computation using Kleene iteration

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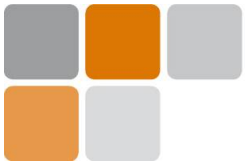
Algorithm 1 Kleene Algorithm using support function.

Require: $\Delta \subset \mathbb{R}^n$, set of l directions. $\mathbb{P}_0$, The initial polyhedron

Require: $A \in \mathbb{R}^n \times \mathbb{R}^m, b \in \mathbb{R}^m$

1: $D = \Delta$, $\Omega = \delta_{\mathbb{P}_0}(\Delta)$
2: **repeat**
3: $\quad\Omega' = \Omega$
4: $\quad$**for all** $i = 0, \ldots, (l-1)$ **do**
5: $\quad\quad\Theta[i] = \Theta[i] + \langle b, D[i]\rangle$
6: $\quad\quad D[i] = A^T D[i]$
7: $\quad\quad\Upsilon[i] = \delta_{\mathbb{P}_0}(D[i]) + \Theta[i]$
8: $\quad\quad\Omega[i] = \max(\Omega[i], \Upsilon[i])$
9: $\quad$**end for**
10: **until** $\Omega \sqsubseteq \Omega'$



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- Solution 2 :

The accelerated Kleene iteration

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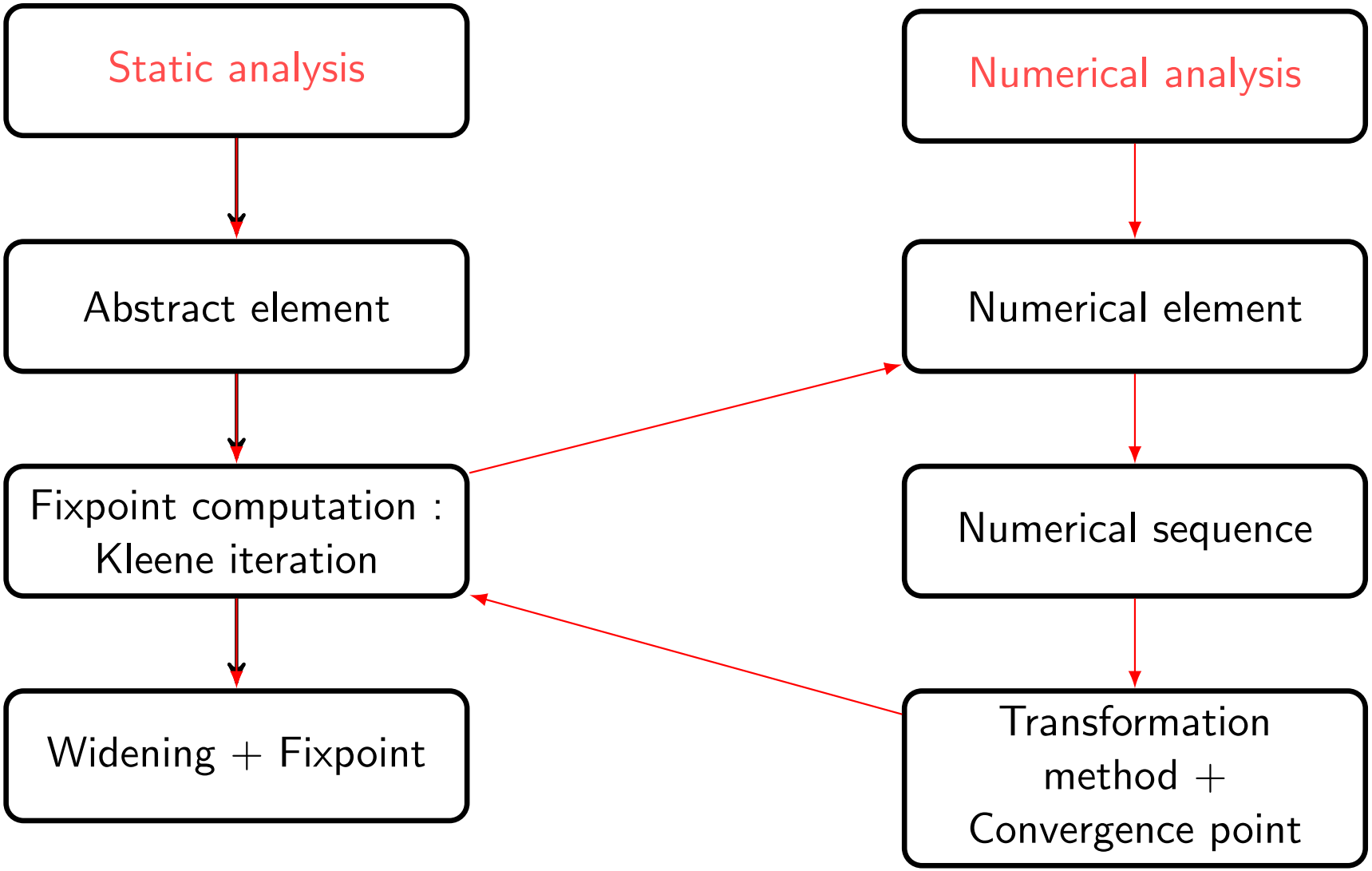

The Aitken-Δ^2 method

Let (S_n) be the initial sequence and (S'_n) its accelerated version

s.t : $\forall n \in, S'_{n+1} = \dfrac{S_{n+1} - S_n}{S_{n+2} - 2S_{n+1} + S_n}$.

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Example : $S_n = 1 + \dfrac{1}{n+1}, \forall n \geq$ with $\lim_{n \to +\infty} S_n = 1$

ε_n^0	ε_n^2	ε_n^4	ε_n^6	ε_n^8
2.0000000				
1.5000000				
1.3333333				
1.2500000				
1.2000000				
1.1666667				
1.1428571				
1.1250000				
1.1111111				
1.1000000				

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ε_n^0	ε_n^2	ε_n^4	ε_n^6	ε_n^8
2.0000000				
1.5000000	1.2500000			
1.3333333				
1.2500000				
1.2000000				
1.1666667				
1.1428571				
1.1250000				
1.1111111				
1.1000000				

The Aitken-Δ^2 method

Let (S_n) be the initial sequence and (S'_n) its accelerated version

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1.1428571	1.0714287			
1.1250000				
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1.1666667	1.0833333			
1.1428571	1.0714287			
1.1250000	1.0624998			
1.1111111				
1.1000000				

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1.1428571	1.0714287			
1.1250000	1.0624998			
1.1111111	1.0555557			
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1.5000000	1.2500000			
1.3333333	1.1666667	1.1111109		
1.2500000	1.1249999	1.0833337		
1.2000000	1.1000001	1.0666663		
1.1666667	1.0833333	1.0555556		
1.1428571	1.0714287	1.0476161		
1.1250000	1.0624998	1.0416761		
1.1111111	1.0555557			
1.1000000				

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1.2000000	1.1000001	1.0666663	1.0500028	
1.1666667	1.0833333	1.0555556	1.0416545	
1.1428571	1.0714287	1.0476161	1.0357504	
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1.1666667	1.0833333	1.0555556	1.0416545	1.0334257
1.1428571	1.0714287	1.0476161	1.0357504	
1.1250000	1.0624998	1.0416761		
1.1111111	1.0555557			
1.1000000				

Accelerated Kleene Algorithm using support function

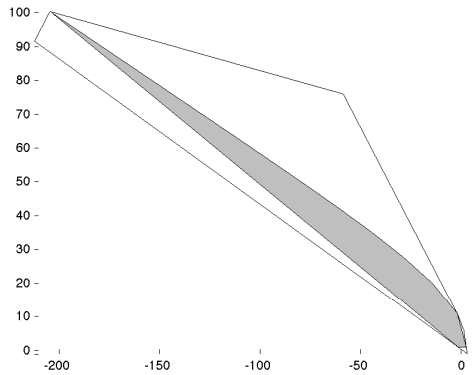
Algorithm 2 Accelerated Kleene Algorithm using support function.

Require: $\Delta \subset \mathbb{R}^n$, $\mathbb{P}_0$, $A \in \mathbb{R}^n \times \mathbb{R}^m$, $b \in \mathbb{R}^m$

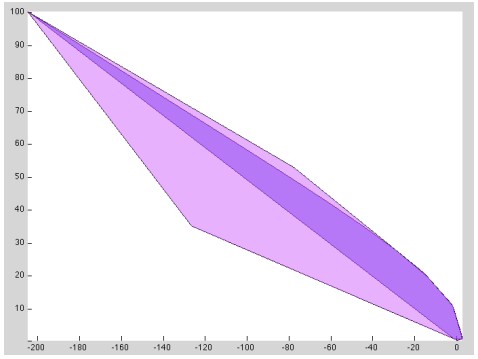
$D = \Delta$, $\Omega = \delta_{\mathbb{P}_0}(\Delta)$
repeat
  $\Omega' = \Omega$, $Y' = Y$
  for all $i = 0, \ldots, (l-1)$ **do**
    $\Theta[i] = \Theta[i] + \langle b, D[i] \rangle$
    $D[i] = A^T D[i]$
    $\Upsilon[i] = \delta_{\mathbb{P}_0}(d[i]) + \Theta[i]$
    $Y[i] = \text{Accelerate}\,(\Upsilon[0], \ldots, \Upsilon[i])$
    if $||Y[i] - Y'[i]|| \leq \varepsilon$ **then**
      $\Omega[i] = \max(\Omega[i], Y[i])$
    else
      $\Omega[i] = \max(\Omega[i], \Upsilon[i])$
    end if
  end for
until $\Omega \sqsubseteq \Omega'$

Kleene Algorithm using support function

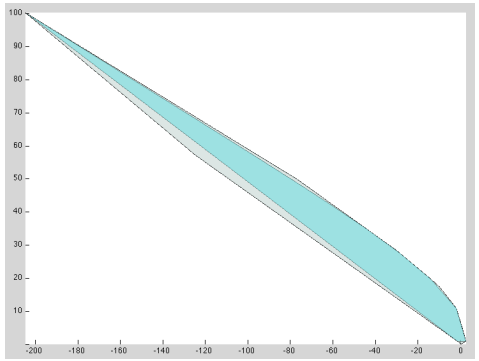
```
begin
    while (0<=10) do
    xn = 0.5 *x - y - 2.5;
     yn = 0.9 *y + 10;
     x = xn; y = yn;
    done;
end
```



8 directions
(0.044 seconds)



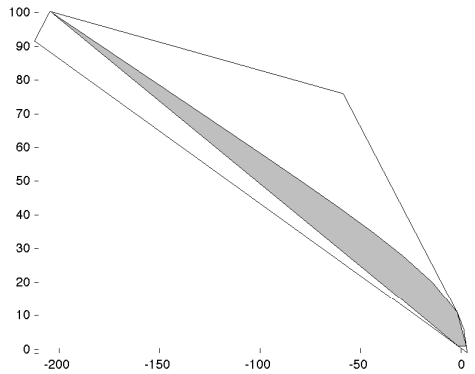
50 directions
(0.34 seconds)



100 directions
(0.7 seconds)

Accelerated Kleene Algorithm using support function

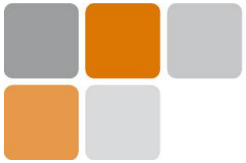
```
begin
    while (0<=10) do
    xn = 0.5 *x - y - 2.5;
     yn = 0.9 *y + 10;
     x = xn; y = yn;
    done;
end
```



8 directions

- Kleene iteration using support function : 200 iterations
- Accelerated Kleene iteration using support function : 11 iterations

1 Le Guernic, C., Girard, A. : Reachability analysis of linear systems using support functions. Nonlinear Analysis : Hybrid Systems (2010)



Related work

1 Le Guernic, C., Girard, A. : Reachability analysis of linear systems using support functions. Nonlinear Analysis : Hybrid Systems (2010)

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Similarities :

- Abstract domain based on a static choice of directions set.
- The same definition of inclusion, meet and join operators.

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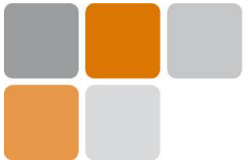
Similarities :

- Abstract domain based on a static choice of directions set.
- The same definition of inclusion, meet and join operators.

Differences :

- We have the property that : $\alpha_{\Delta}(\mathbb{P}_i) = \Omega_i$, which is not true for the template abstract domain.
- Our domain reduces the use of linear programming but it's less general than the template abstract domain.



Conclusion and future work

Conclusion

- We develop a new numerical abstract domain based on support function.
- Our abstract domain depends on a set of finite directions.
- In the case where the fixpoint computation terminates, it is obtained in a polynomial time.

Perspectives

- Implements this domain on APRON.
- According to the program to analyse, defines a relevant set of directions.