Numerical Abstract Domain using Support Function.

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and
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19 juin 2012

An industriel problem

- The crash of Ariane 5 : caused by an overflow.
  $\Longrightarrow$ 700 Million euro of lost.

Program

Input : $S_0 \subseteq \mathbb{R}^n$
Input : $A \in \mathbb{R}^n \times \mathbb{R}^m$, $b \in \mathbb{R}^m$.
Input : $c \in \mathbb{R}^n$, $l \in \mathbb{R}$
$X \in S_0$
while $(\langle X, c \rangle \leq l)$ {
$\quad X = AX + b.$
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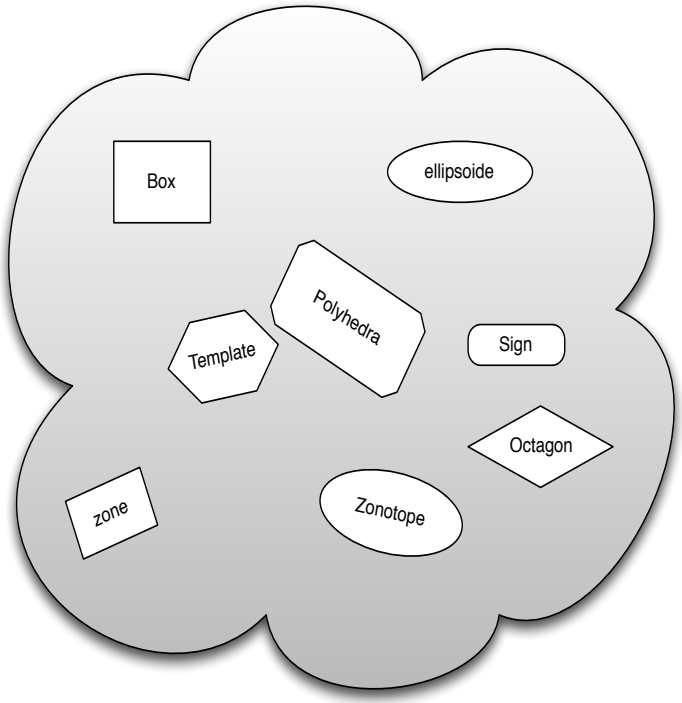
$$S_i = S_{i-1} \cup [(AS_{i-1} + b) \cap (\langle c, X \rangle \leq l)]$$

Static analysis by abstract interpretation



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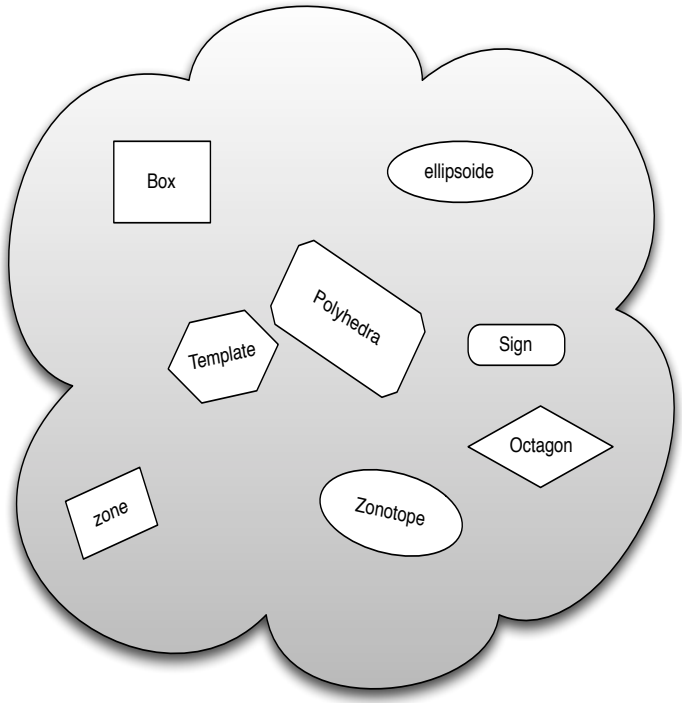
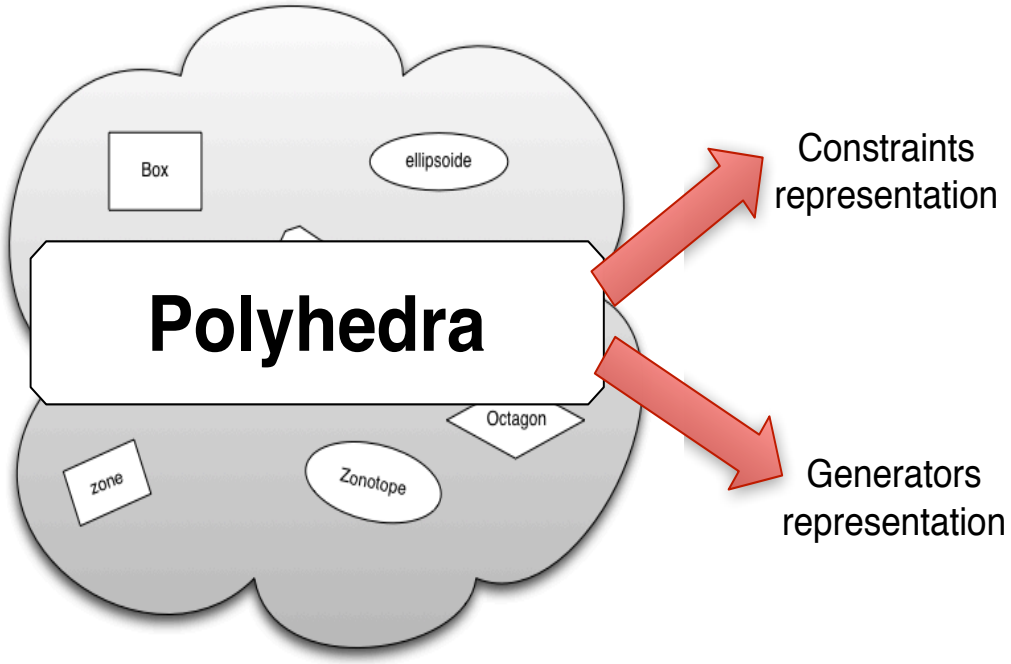
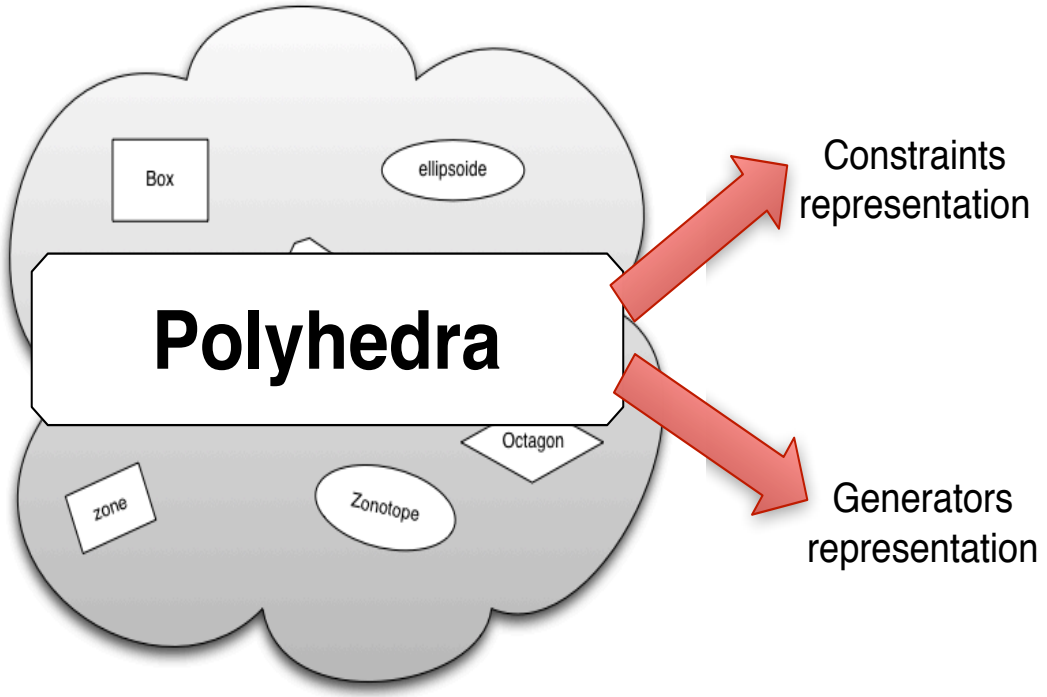
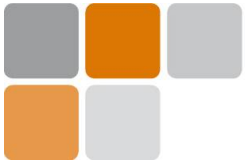



Box
ellipsoide
Polyhedra
Octagon
zone
Zonotope
Constraints representation
Generators representation

Static analysis by abstract interpretation



Support function

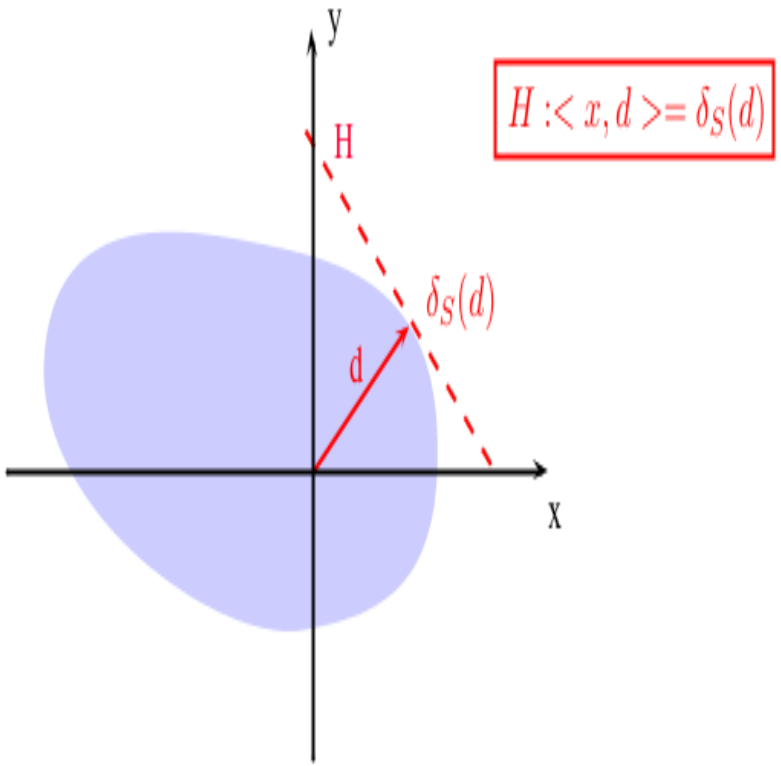


Outline

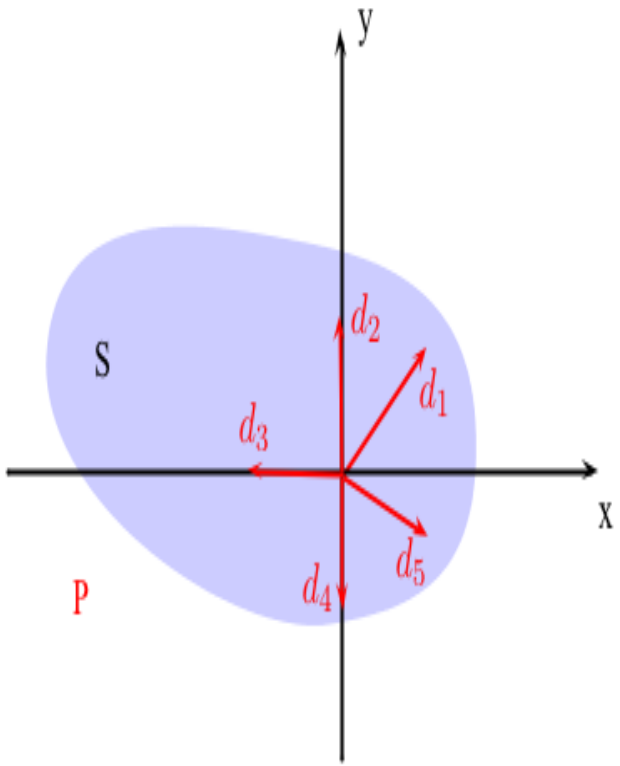
Definition

Let S be a convex set and δ_S its support function, such that :

$$\forall d \in \mathbb{R}^n, \delta_S(d) = sup\{\langle x, d\rangle : x \in S\}$$



Let $\Delta = \{d_1, d_2, d_3, d_4, d_5\}$ be a set of directions.



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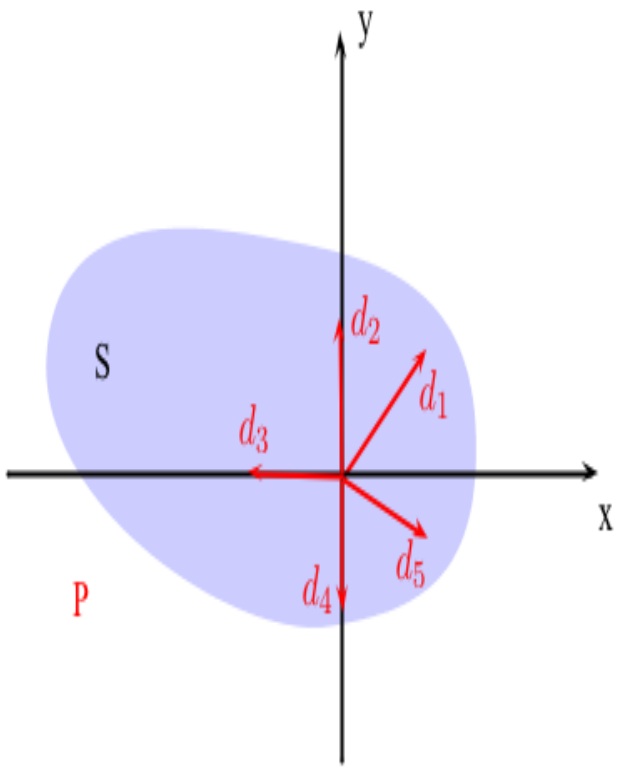

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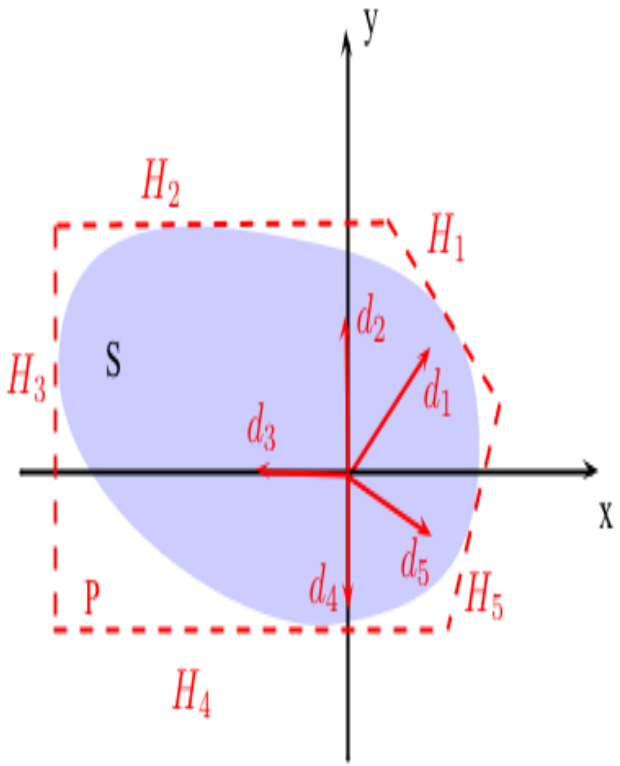

Property

Let S be a convex set, and $\Delta \subseteq \mathbb{R}^n$ be a set of directions.
We put

$$P = \bigcap_{d \in \Delta} \{x \in \mathbb{R}^n | \langle x, d \rangle \leq \delta_S(d)\}$$

Then

$$S \subseteq P$$

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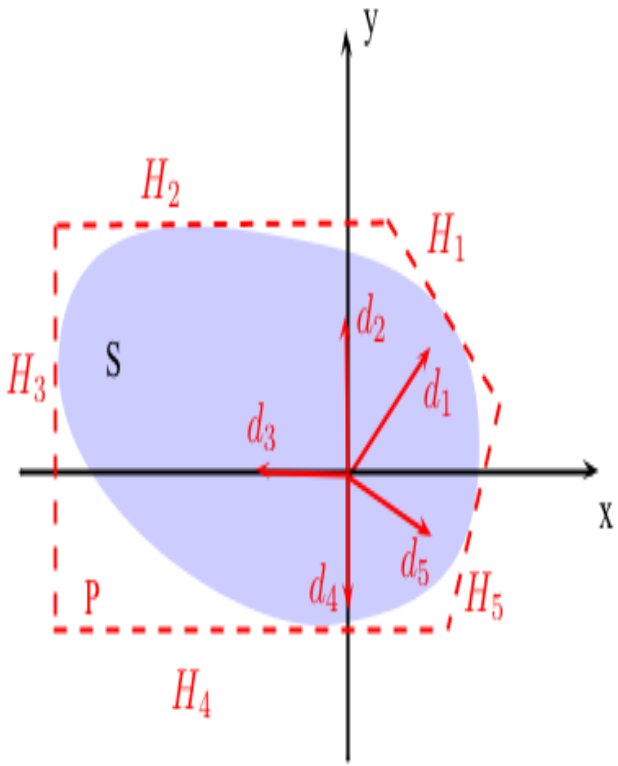

The special case of polyhedron

Let $\mathbb{P}$ be a polyhedron. If $\mathbb{P}$ is represented by :

- Linear system, $\delta_{\mathbb{P}}$ is obtained using Linear Programming.
- Generators (vertices) v_i, $\delta_{\mathbb{P}}(d) = sup\{\langle v_i, d\rangle : v_i \in \mathbb{P}\}$.

Properties

Let S, S' be two convex sets. We have :

- $\forall M \in \mathbb{R}^n \times \mathbb{R}^m, \delta_{MS}(d) = \delta_S(M^T d)$.
- $\forall \lambda \geq 0, \delta_{\lambda S}(d) = \lambda \delta_S(d)$.
- $\delta_{S \oplus S'}(d) = \delta_S(d) + \delta_{S'}(d)$.
- $\delta_{S \cup S'}(d) = \max(\delta_S(d), \delta_{S'}(d))$.
- $\delta_{S \cap S'}(d) \leq \min(\delta_S(d), \delta_{S'}(d))$.

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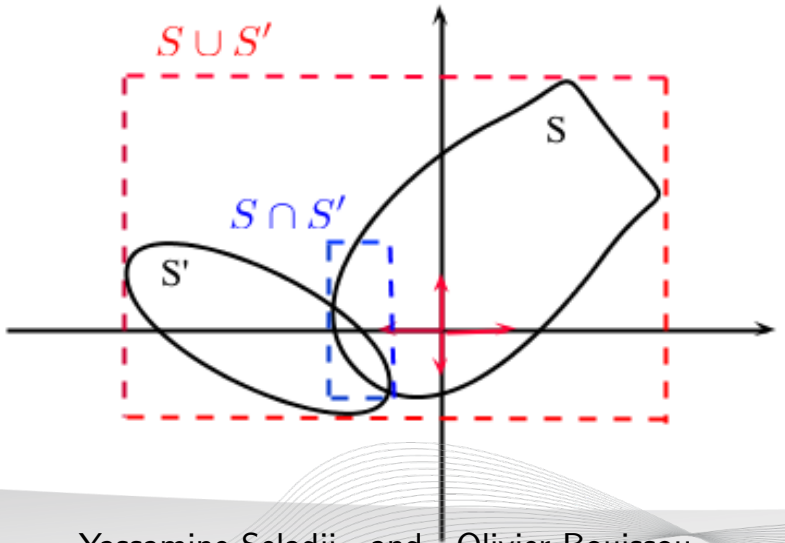

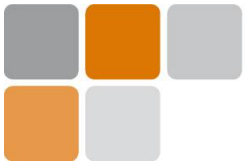

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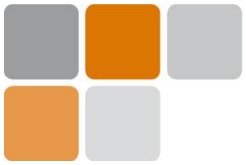

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The concretisation function

$$\begin{array}{rccl} \gamma_{\Delta} : & (\Delta \to \mathbb{R}_{\infty}) & \longrightarrow & P(\mathbb{R}^n) \\ & \Omega & \longrightarrow & \bigcap_{d \in \Delta} \{x \in \mathbb{R}^n \mid \langle x, d \rangle \leq \Omega(d)\} \end{array}$$

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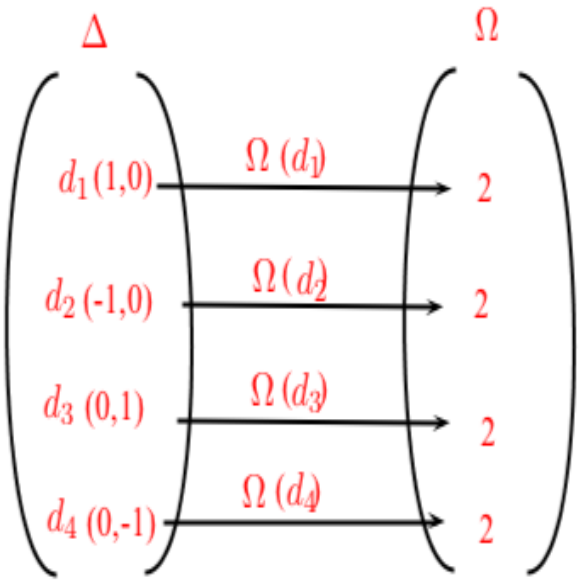
Example :



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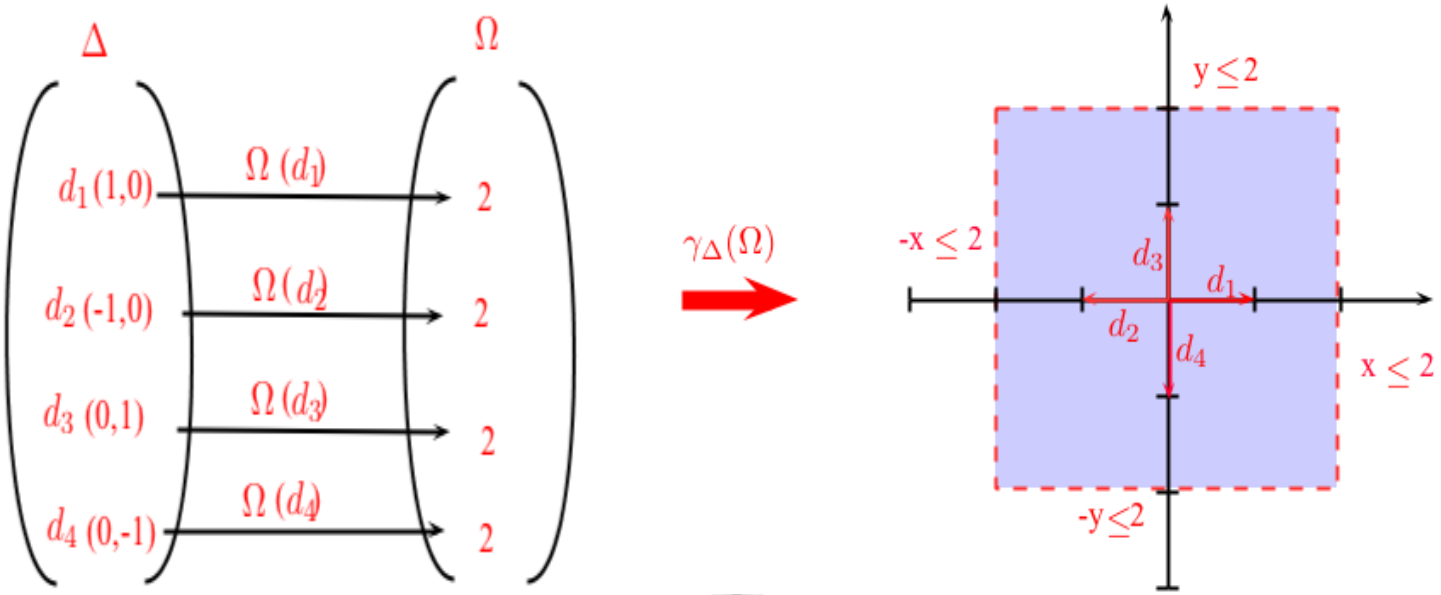

The abstraction function

$$\alpha_\Delta : \quad P(\mathbb{R}^n) \longrightarrow (\Delta \to \mathbb{R}_\infty)$$

$$S \longrightarrow \begin{cases} \lambda d.\ -\infty & \text{if } S = \emptyset \\ \lambda d.\ +\infty & \text{if } S = \mathbb{R}^n \\ \lambda d.\ \delta_S(d) & \text{otherwise} \end{cases}$$

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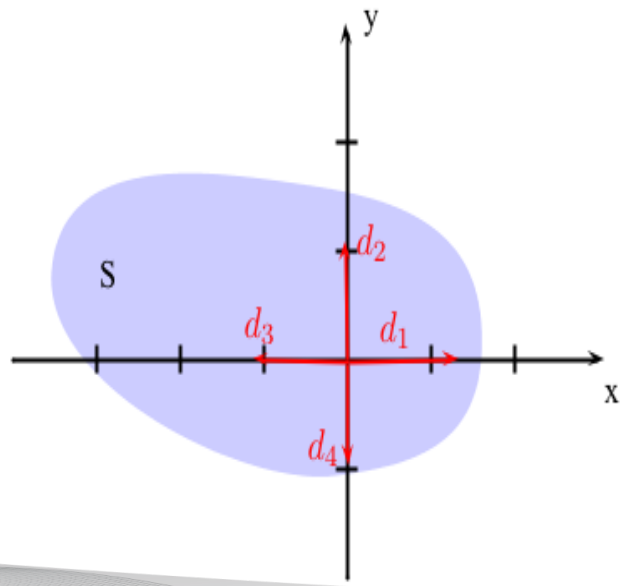

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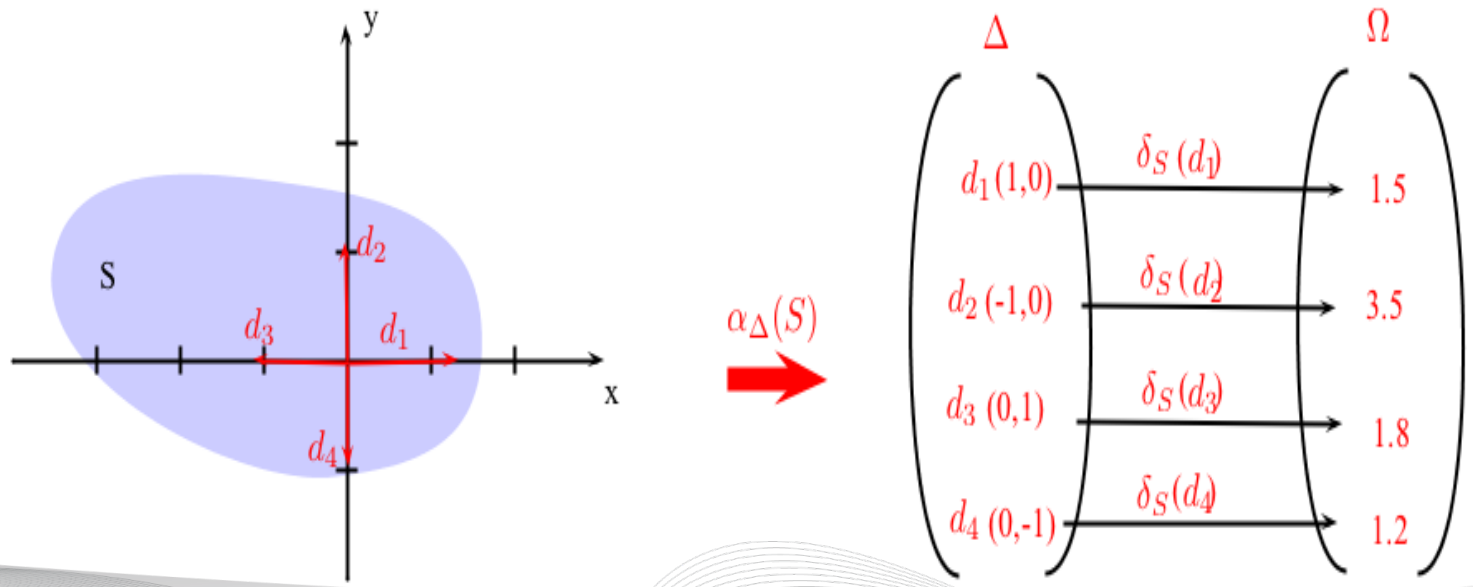

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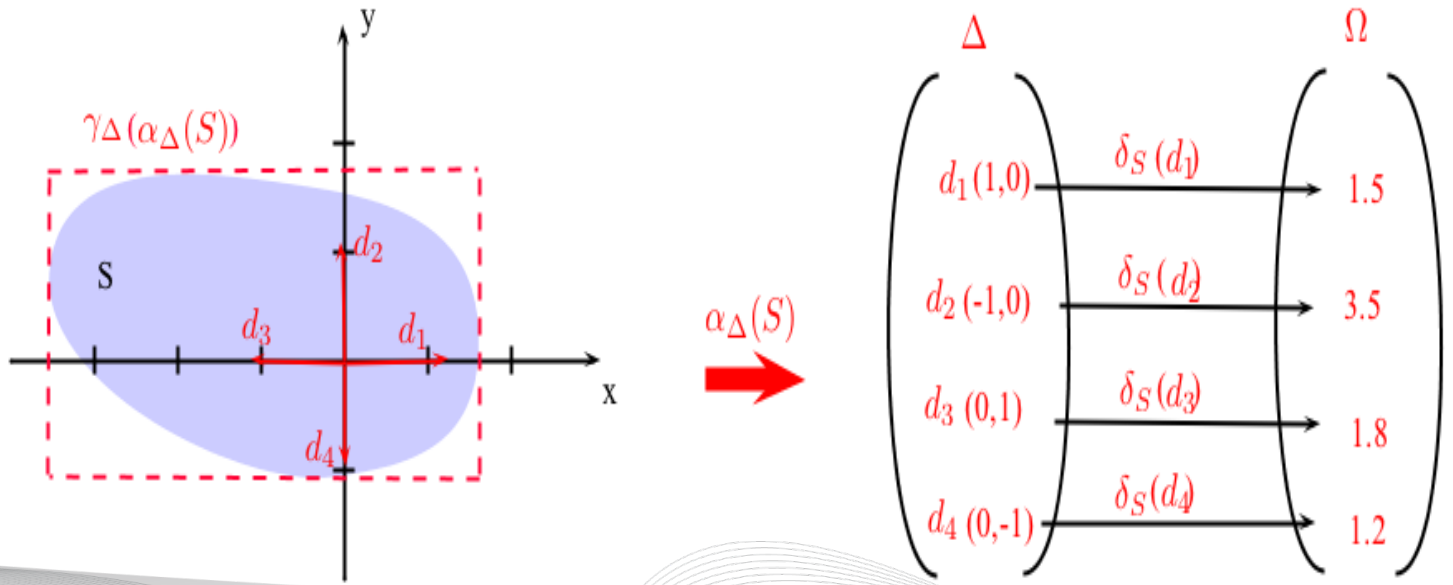

The complete lattice $\langle \mathbb{P}^{\sharp}_{\Delta}, \sqsubseteq, \bot, \top, \sqcup, \sqcap \rangle$ is defined by :

- An order relation : $\Omega_1 \sqsubseteq \Omega_2 \Leftrightarrow \gamma_{\Delta}(\Omega_1) \subseteq \gamma_{\Delta}(\Omega_2)$.
- A minimal element : $\bot = \lambda d.\ -\infty$.
- A maximal element : $\top = \lambda d.\ +\infty$.
- A join operator : $\Omega_1 \sqcup \Omega_2 = \lambda d.\ \max(\Omega_1(d), \Omega_2(d))$.
- A meet operator : $\Omega_1 \sqcap \Omega_2 = \lambda d.\ \min(\Omega_1(d), \Omega_2(d))$.

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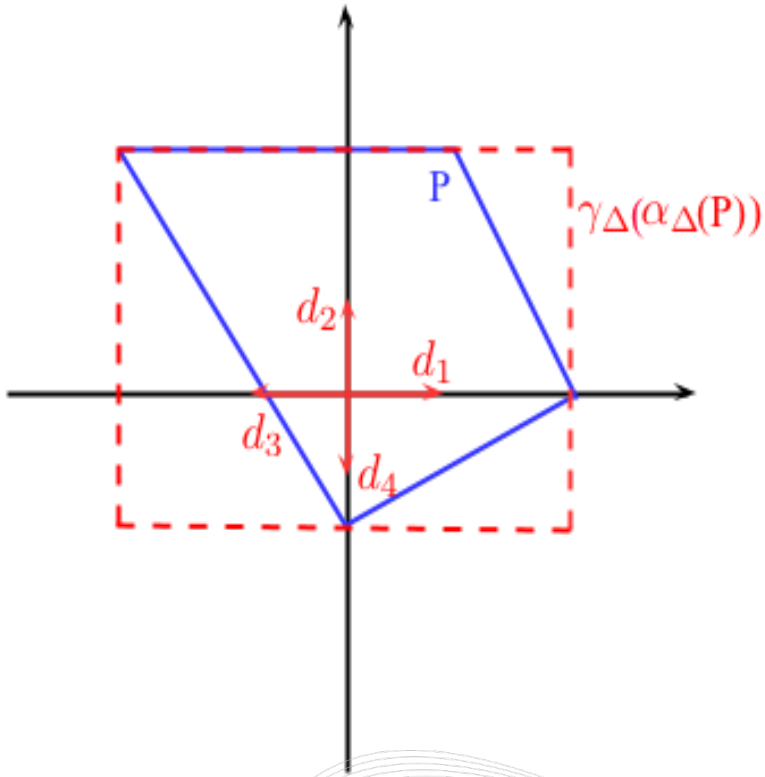
Notes :

$$\gamma_{\Delta}(\Omega_1 \sqcup \Omega_2) = \gamma_{\Delta}(\Omega_1) \sqcup \gamma_{\Delta}(\Omega_2).$$
$$\gamma_{\Delta}(\Omega_1 \sqcap \Omega_2) \sqsupseteq \gamma_{\Delta}(\Omega_1) \sqcap \gamma_{\Delta}(\Omega_2).$$

Property

Let P be a polyhedron and $\Omega \in \mathbb{P}^{\sharp}_{\Delta}$ such that $\Omega = \alpha_{\Delta}(P)$. We have that, $P \subseteq \gamma_{\Delta}(\Omega)$ where this over approximation is tight as the vertices of P touch the faces of $\gamma_{\Delta}(\Omega)$.



Program

Input : $\mathbb{P}_0$ a bounded polyhedron.

Input : $A \in \mathbb{R}^n \times \mathbb{R}^m$, $b \in \mathbb{R}^m$.

Input : $c \in \mathbb{R}^n$, $l \in \mathbb{R}$

$X \in \mathbb{P}_0$

while $(\langle X, c \rangle \leq l)$ {

$X = AX + b.$

}

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$$\Omega_i = \Omega_{i-1} \cup [(A\Omega_{i-1} + b) \cap (\langle c, X \rangle \leq l)]$$

- **Case 1 :** $\Omega = A\Omega_0 + b$

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Input : $\mathbb{P}_0$ a bounded polyhedron.
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The abstract element

$$\Omega \quad = \quad \lambda d.\delta_{A\mathbb{P}_0 \oplus b}(d)$$

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$$\begin{aligned} \Omega &= \lambda d.\delta_{A\mathbb{P}_0 \oplus b}(d) \\ &= \lambda d.\delta_{\mathbb{P}_0}(A^T d) + \langle b, d \rangle \end{aligned}$$

- **Case 2 :** $\Omega_i = \Omega_{i-1} \cup [(A\Omega_{i-1} + b) \cap (\langle c, X \rangle \not\# I)]$

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$\alpha_\Delta(\mathbb{P}_i) = \Omega_i$

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$$\delta_H(d) = \begin{cases} l & \text{if } d = \lambda c \\ +\infty & \text{otherwise} \end{cases}$$

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We put $\Delta \cup \{c\}$:

- $\Delta_1 = \{c\} \cup \{d \in \Delta | d = \lambda c, \lambda \geq 0\}$.
- $\Delta_2 = \Delta \setminus \Delta_1$

- **Case 3 :** $\Omega_i = \Omega_{i-1} \cup [(A\Omega_{i-1} + b) \cap (\langle c, X \rangle \leq l)]$
- For $d \in \Delta_2$: we use the same method as for **Case 2**.

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$$\Omega_i = \lambda d. \max\{\delta_{\mathbb{P}_0}(A^{Tj}d) + \sum_{k=1}^{j} \langle b, A^{T(k-1)}d \rangle, j = 0, .., i\}$$

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- For $d \in \Delta_1$, we have that :

$$\Omega_i(d) = \max(\delta_{\gamma_\Delta(\Omega_{i-1})}(d), \min(\delta_{\gamma_\Delta(\Omega_{i-1})}(A^T d) + \langle b, d \rangle, l))$$

Such that $\lambda d. \delta_{\mathbb{P}_i}(d) \leq \Omega_i(d)$.

$$\Omega_i(d) = \max\{\delta_{\mathbb{P}_0}(A^{Tj}d) + \sum_{k=1}^{j}\langle b, A^{T(k-1)}d\rangle, j = 0, .., i\}$$

Algorithm 1 Kleene Algorithm using support function.

Require: $\Delta \subset \mathbb{R}^n$, set of l directions. $\mathbb{P}_0$, The initial polyhedron

Require: $A \in \mathbb{R}^n \times \mathbb{R}^m, b \in \mathbb{R}^m$

1: $D = \Delta$, $\Omega = \delta_{\mathbb{P}_0}(\Delta)$
2: **repeat**
3: $\quad \Omega' = \Omega$
4: $\quad$ **for all** $i = 0, \ldots, (l-1)$ **do**
5: $\quad\quad \Theta[i] = \Theta[i] + \langle b, D[i]\rangle$
6: $\quad\quad D[i] = A^T D[i]$
7: $\quad\quad \Upsilon[i] = \delta_{\mathbb{P}_0}(d[i]) + \Theta[i]$
8: $\quad\quad \Omega[i] = \max(\Omega[i], \Upsilon[i])$
9: $\quad$ **end for**
10: **until** $\Omega \sqsubseteq \Omega'$

$$\Omega_i(d) = \max\{\delta_{\mathbb{P}_0}(A^{Tj}d) + \sum_{k=1}^{j}\langle b, A^{T(k-1)}d\rangle, j = 0, .., i\}$$

Algorithm 2 Kleene Algorithm using support function.

Require: $\Delta \subset \mathbb{R}^n$, set of l directions. $\mathbb{P}_0$, The initial polyhedron

Require: $A \in \mathbb{R}^n \times \mathbb{R}^m, b \in \mathbb{R}^m$

1: $D = \Delta$, $\Omega = \delta_{\mathbb{P}_0}(\Delta)$
2: **repeat**
3: $\Omega' = \Omega$
4: **for all** $i = 0, \ldots, (l-1)$ **do**
5: $\Theta[i] = \Theta[i] + \langle b, D[i]\rangle$
6: $D[i] = A^T D[i]$
7: $\Upsilon[i] = \delta_{\mathbb{P}_0}(d[i]) + \Theta[i]$
8: $\Omega[i] = \max(\Omega[i], \Upsilon[i])$
9: **end for**
10: **until** $\Omega \sqsubseteq \Omega'$

$$\Omega_i(d) = \max\{\delta_{\mathbb{P}_0}(A^{Tj}d) + \sum_{k=1}^{j}\langle b, A^{T(k-1)}d\rangle, j = 0, .., i\}$$

Algorithm 3 Kleene Algorithm using support function.

Require: $\Delta \subset \mathbb{R}^n$, set of l directions. $\mathbb{P}_0$, The initial polyhedron

Require: $A \in \mathbb{R}^n \times \mathbb{R}^m, b \in \mathbb{R}^m$

1: $D = \Delta$, $\Omega = \delta_{\mathbb{P}_0}(\Delta)$
2: **repeat**
3: $\quad\Omega' = \Omega$
4: $\quad$**for all** $i = 0, \ldots, (l-1)$ **do**
5: $\quad\quad\Theta[i] = \Theta[i] + \langle b, D[i]\rangle$
6: $\quad\quad D[i] = A^T D[i]$
7: $\quad\quad\Upsilon[i] = \delta_{\mathbb{P}_0}(d[i]) + \Theta[i]$
8: $\quad\quad\Omega[i] = \max(\Omega[i], \Upsilon[i])$
9: $\quad$**end for**
10: **until** $\Omega \sqsubseteq \Omega'$

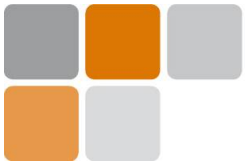
$$\Omega_i(d) = \max\{\delta_{\mathbb{P}_0}(A^{Tj}d) + \underbrace{\sum_{k=1}^{j}\langle b, A^{T(k-1)}d\rangle}, j = 0, .., i\}$$

Algorithm 4 Kleene Algorithm using support function.

Require: $\Delta \subset \mathbb{R}^n$, set of l directions. $\mathbb{P}_0$, The initial polyhedron

Require: $A \in \mathbb{R}^n \times \mathbb{R}^m, b \in \mathbb{R}^m$

1: $D = \Delta$, $\Omega = \delta_{\mathbb{P}_0}(\Delta)$
2: **repeat**
3: $\Omega' = \Omega$
4: **for all** $i = 0, \ldots, (l-1)$ **do**
5: $\Theta[i] = \Theta[i] + \langle b, D[i]\rangle$
6: $D[i] = A^T D[i]$
7: $\Upsilon[i] = \delta_{\mathbb{P}_0}(d[i]) + \Theta[i]$
8: $\Omega[i] = \max(\Omega[i], \Upsilon[i])$
9: **end for**
10: **until** $\Omega \sqsubseteq \Omega'$



The Algorithm doesn't guarantee the termination of the computation.

The Algorithm doesn't guarantee the termination of the computation.

- Solution 1 :

widening

$$\forall \Omega_1, \Omega_2 \in \mathbb{P}^{\sharp}_{\Delta},\ \Omega_1 \nabla_{\Delta} \Omega_2 = \lambda d. \begin{cases} \Omega_2(d) & \text{if } \Omega_1(d) = \Omega_2(d) \\ +\infty & \text{otherwise} \end{cases}$$

The Algorithm doesn't guarantee the termination of the computation.

- Solution 1 :

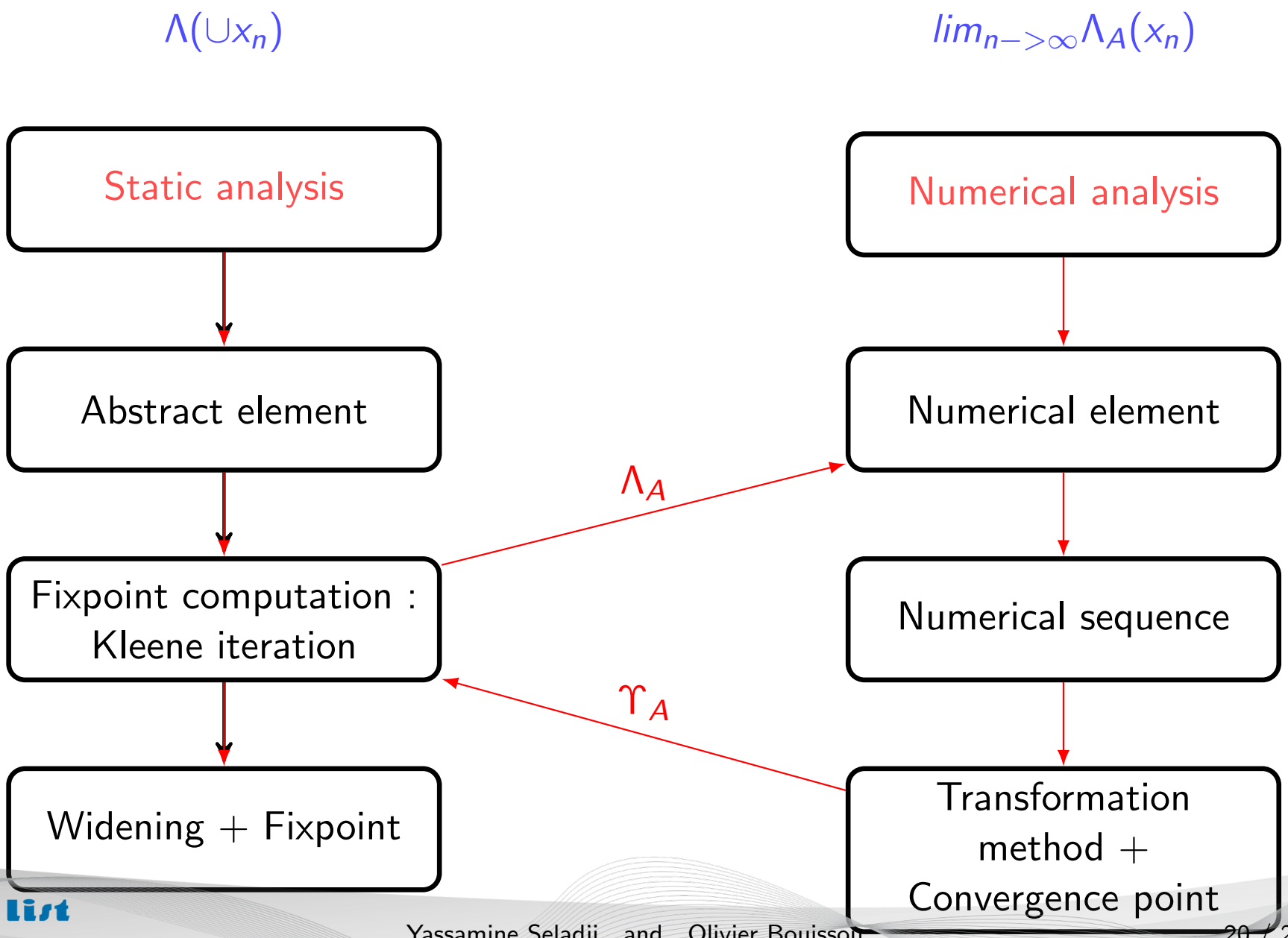
widening

$$\forall \Omega_1, \Omega_2 \in \mathbb{P}^{\sharp}_{\Delta},\ \Omega_1 \nabla_{\Delta} \Omega_2 = \lambda d. \begin{cases} \Omega_2(d) & \text{if } \Omega_1(d) = \Omega_2(d) \\ +\infty & \text{otherwise} \end{cases}$$

- Solution 2 :

The accelerated Kleene iteration

The accelerated Kleene iteration



The accelerated Kleene iteration

Algorithm 5 The accelerated Kleene iteration

1: **repeat**
2: $\quad \overrightarrow{X_i} := \overrightarrow{X_{i-1}} \sqcup F(\overrightarrow{X_{i-1}})$
3: $\quad \overrightarrow{y_i} := \text{Accelerate}\left(\Lambda_A(\overrightarrow{X_0}), \ldots, \Lambda_A(\overrightarrow{X_i})\right)$
4: $\quad$ **if** $||\overrightarrow{y_i} - \overrightarrow{y_{i-1}}|| \leq \delta$ **then**
5: $\quad\quad \overrightarrow{X_i} := \overrightarrow{X_i} \sqcup \Upsilon_A(\overrightarrow{y_i})$
6: $\quad$ **end if**
7: **until** $\overrightarrow{X_i} \sqsubseteq \overrightarrow{X_{i-1}}$

The accelerated Kleene iteration

- The method to accelerated the numerical sequences convergence.

Algorithm 6 The accelerated Kleene iteration

1: **repeat**
2: $\overrightarrow{X_i} := \overrightarrow{X_{i-1}} \sqcup F(\overrightarrow{X_{i-1}})$
3: $\overrightarrow{y_i} :=$ Accelerate $(\Lambda_A(X_0), \ldots, \Lambda_A(\overrightarrow{X_i}))$
4: **if** $||\overrightarrow{y_i} - \overrightarrow{y_{i-1}}|| \leq \delta$ **then**
5: $\overrightarrow{X_i} := \overrightarrow{X_i} \sqcup \Upsilon_A(\overrightarrow{y_i})$
6: **end if**
7: **until** $\overrightarrow{X_i} \sqsubseteq \overrightarrow{X_{i-1}}$

The accelerated Kleene iteration

- The method to accelerated the numerical sequences convergence.

Algorithm 7 The accelerated Kleene iteration

1: **repeat**
2: $\overrightarrow{X_i} := \overrightarrow{X_{i-1}} \sqcup F(\overrightarrow{X_{i-1}})$
3: $\overrightarrow{y_i} :=$ Accelerate $(\Lambda_A(X_0), \ldots, \Lambda_A(\overrightarrow{X_i}))$
4: **if** $||\overrightarrow{y_i} - \overrightarrow{y_{i-1}}|| \leq \delta$ **then**
5: $\overrightarrow{X_i} := \overrightarrow{X_i} \sqcup \Upsilon_A(\overrightarrow{y_i})$
6: **end if**
7: **until** $\overrightarrow{X_i} \sqsubseteq \overrightarrow{X_{i-1}}$

- The *Extraction* function

The accelerated Kleene iteration

- The method to accelerated the numerical sequences convergence.

Algorithm 8 The accelerated Kleene iteration

1: **repeat**
2: $\overrightarrow{X_i} := \overrightarrow{X_{i-1}} \sqcup F(\overrightarrow{X_{i-1}})$
3: $\overrightarrow{y_i} :=$ Accelerate $(\Lambda_A(X_0), \ldots, \Lambda_A(\overrightarrow{X_i}))$
4: **if** $||\overrightarrow{y_i} - \overrightarrow{y_{i-1}}|| \leq \delta$ **then**
5: $\overrightarrow{X_i} := \overrightarrow{X_i} \sqcup \Upsilon_A(\overrightarrow{y_i})$
6: **end if**
7: **until** $\overrightarrow{X_i} \sqsubseteq \overrightarrow{X_{i-1}}$

- The *Extraction* function
- The *Combination* function

The accelerated Kleene iteration

Example

```
while (1) {
xn1 = -0.4375 * x1 + 0.0625 * x2 + 0.2652 * x3 + 0.1 * u1 ;
xn2 = 0.0625 * x1 + 0.4375 * x2 + 0.2652 * x3 + 0.1 * u2 ;
xn3 = -0.2652 * x1 + 0.2652 * x2 + 0.375 * x3 + 0.1 * u3 ;
x1 = xn1 ;x2 = xn2 ;x3 = xn3 ;
} [-5.197505,8.873306]
```

Itération :3

x1	$\Upsilon_A(\overrightarrow{y})$	$\|y_i - y_{i-1}\|$	**Seuils** ($\delta = 10^{-3}$)
[1.000000,2.000000]	[1.000000,2.000000]		
[-0.447300,5.716500]			
[-2.291255,6.573381]			

The accelerated Kleene iteration

Example

```
while (1) {
xn1 = -0.4375 * x1 + 0.0625 * x2 + 0.2652 * x3 + 0.1 * u1 ;
xn2 = 0.0625 * x1 + 0.4375 * x2 + 0.2652 * x3 + 0.1 * u2 ;
xn3 = -0.2652 * x1 + 0.2652 * x2 + 0.375 * x3 + 0.1 * u3 ;
x1 = xn1 ;x2 = xn2 ;x3 = xn3 ;
} [-5.197505,8.873306]
```

Itération :3

x1	$\Upsilon_A(\overrightarrow{y})$	$\|y_i - y_{i-1}\|$	**Seuils** ($\delta = 10^{-3}$)
[1.000000,2.000000]	[1.000000,2.000000]		
[-0.447300,5.716500]			
[-2.291255,6.573381]	[6.280857,6.830145]	max(5.280857,4.830145)	

The accelerated Kleene iteration

Example

```
while (1) {
xn1 = -0.4375 * x1 + 0.0625 * x2 + 0.2652 * x3 + 0.1 * u1 ;
xn2 = 0.0625 * x1 + 0.4375 * x2 + 0.2652 * x3 + 0.1 * u2 ;
xn3 = -0.2652 * x1 + 0.2652 * x2 + 0.375 * x3 + 0.1 * u3 ;
x1 = xn1 ;x2 = xn2 ;x3 = xn3 ;
} [-5.197505,8.873306]
```

Itération :5

x1	$\Upsilon_A(\overrightarrow{y})$	$\|y_i - y_{i-1}\|$	**Seuils** ($\delta = 10^{-3}$)
[1.000000,2.000000]	[1.000000,2.000000]		
[-0.447300,5.716500]			
[-2.291255,6.573381]	[6.280857,6.830145]	max(5.280857,4.830145)	
[-3.038029,7.492492]			
[-3.695558,7.871720]	[-4.663282,8.463092]	max(10.944139,2.367053)	

The accelerated Kleene iteration

Example

```
while (1) {
xn1 = -0.4375 * x1 + 0.0625 * x2 + 0.2652 * x3 + 0.1 * u1 ;
xn2 = 0.0625 * x1 + 0.4375 * x2 + 0.2652 * x3 + 0.1 * u2 ;
xn3 = -0.2652 * x1 + 0.2652 * x2 + 0.375 * x3 + 0.1 * u3 ;
x1 = xn1 ;x2 = xn2 ;x3 = xn3 ;
} [-5.197505,8.873306]
```

Itération :7

x1	$\Upsilon_A(\overrightarrow{y})$	$\|y_i - y_{i-1}\|$	**Seuils** ($\delta = 10^{-3}$)
[1.000000,2.000000]	[1.000000,2.000000]		
[-0.447300,5.716500]			
[-2.291255,6.573381]	[6.280857,6.830145]	max(5.280857,4.830145)	
[-3.038029,7.492492]			
[-3.695558,7.871720]	[-4.663282,8.463092]	max(10.944139,2.367053)	
[-4.084503,8.185954]			
[-4.386442,8.369221]	[-5.189239,8.851176]	max(0.525957,0.388084)	

The accelerated Kleene iteration

Example

```
while (1) {
xn1 = -0.4375 * x1 + 0.0625 * x2 + 0.2652 * x3 + 0.1 * u1 ;
xn2 = 0.0625 * x1 + 0.4375 * x2 + 0.2652 * x3 + 0.1 * u2 ;
xn3 = -0.2652 * x1 + 0.2652 * x2 + 0.375 * x3 + 0.1 * u3 ;
x1 = xn1 ;x2 = xn2 ;x3 = xn3 ;
} [-5.197505,8.873306]
```

Itération :9

x1	$\Upsilon_A(\overrightarrow{y})$	$\|y_i - y_{i-1}\|$	**Seuils** ($\delta = 10^{-3}$)
[1.000000,2.000000]	[1.000000,2.000000]		
[-0.447300,5.716500]			
[-2.291255,6.573381]	[6.280857,6.830145]	max(5.280857,4.830145)	
[-3.038029,7.492492]			
[-3.695558,7.871720]	[-4.663282,8.463092]	max(10.944139,2.367053)	
[-4.084503,8.185954]			
[-4.386442,8.369221]	[-5.189239,8.851176]	max(0.525957,0.388084)	
[-4.594886,8.508279]			
[-4.751445,8.603235]	[-5.197089,8.872567]	max(0.00785,0.021391)	

The accelerated Kleene iteration

Example

```
while (1) {
xn1 = -0.4375 * x1 + 0.0625 * x2 + 0.2652 * x3 + 0.1 * u1 ;
xn2 = 0.0625 * x1 + 0.4375 * x2 + 0.2652 * x3 + 0.1 * u2 ;
xn3 = -0.2652 * x1 + 0.2652 * x2 + 0.375 * x3 + 0.1 * u3 ;
x1 = xn1 ;x2 = xn2 ;x3 = xn3 ;
} [-5.197505,8.873306]
```

Itération :11

x1	$\Upsilon_A(\overrightarrow{y})$	$\|y_i - y_{i-1}\|$	**Seuils** ($\delta = 10^{-3}$)
[1.000000,2.000000]	[1.000000,2.000000]		
[-0.447300,5.716500]			
[-2.291255,6.573381]	[6.280857,6.830145]	max(5.280857,4.830145)	
[-3.038029,7.492492]			
[-3.695558,7.871720]	[-4.663282,8.463092]	max(10.944139,2.367053)	
[-4.084503,8.185954]			
[-4.386442,8.369221]	[-5.189239,8.851176]	max(0.525957,0.388084)	
[-4.594886,8.508279]			
[-4.751445,8.603235]	[-5.197089,8.872567]	max(0.00785,0.021391)	
[-4.865385,8.673918]			
[-4.950393,8.725095]	[-5.197506,8.873307]	max(0.000417,0.00074)	[-5.197506,8.873307]

The accelerated Kleene iteration

Example

```
while (1) {
xn1 = -0.4375 * x1 + 0.0625 * x2 + 0.2652 * x3 + 0.1 * u1 ;
xn2 = 0.0625 * x1 + 0.4375 * x2 + 0.2652 * x3 + 0.1 * u2 ;
xn3 = -0.2652 * x1 + 0.2652 * x2 + 0.375 * x3 + 0.1 * u3 ;
x1 = xn1 ;x2 = xn2 ;x3 = xn3 ;
} [-5.197505,8.873306]
```

Itération : 12

x1	$\Upsilon_A(\overrightarrow{y})$	$\|y_i - y_{i-1}\|$	**Seuils** ($\delta = 10^{-3}$)
[1.000000,2.000000]	[1.000000,2.000000]		
[-0.447300,5.716500]			
[-2.291255,6.573381]	[6.280857,6.830145]	max(5.280857,4.830145)	
[-3.038029,7.492492]			
[-3.695558,7.871720]	[-4.663282,8.463092]	max(10.944139,2.367053)	
[-4.084503,8.185954]			
[-4.386442,8.369221]	[-5.189239,8.851176]	max(0.525957,0.388084)	
[-4.594886,8.508279]			
[-4.751445,8.603235]	[-5.197089,8.872567]	max(0.00785,0.021391)	
[-4.865385,8.673918]			
[-4.950393,8.725095]	[-5.197506,8.873307]	max(0.000417,0.00074)	[-5.197506,8.873307]
[-5.197506,8.873307]			

The accelerated Kleene iteration

Example

```
while (1) {
xn1 = -0.4375 * x1 + 0.0625 * x2 + 0.2652 * x3 + 0.1 * u1 ;
xn2 = 0.0625 * x1 + 0.4375 * x2 + 0.2652 * x3 + 0.1 * u2 ;
xn3 = -0.2652 * x1 + 0.2652 * x2 + 0.375 * x3 + 0.1 * u3 ;
x1 = xn1 ;x2 = xn2 ;x3 = xn3 ;
} [-5.197505,8.873306]
```

Itération : 13

x1	$\Upsilon_A(\overrightarrow{y})$	$\|y_i - y_{i-1}\|$	**Seuils** ($\delta = 10^{-3}$)
[1.000000,2.000000]	[1.000000,2.000000]		
[-0.447300,5.716500]			
[-2.291255,6.573381]	[6.280857,6.830145]	max(5.280857,4.830145)	
[-3.038029,7.492492]			
[-3.695558,7.871720]	[-4.663282,8.463092]	max(10.944139,2.367053)	
[-4.084503,8.185954]			
[-4.386442,8.369221]	[-5.189239,8.851176]	max(0.525957,0.388084)	
[-4.594886,8.508279]			
[-4.751445,8.603235]	[-5.197089,8.872567]	max(0.00785,0.021391)	
[-4.865385,8.673918]			
[-4.950393,8.725095]	[-5.197506,8.873307]	max(0.000417,0.00074)	[-5.197506,8.873307]
[-5.197506,8.873307]			
[-5.197506,8.873307]	[-5.197089,8.872567]	max(0.000417,0.00074)	[-5.197089,8.872567]

The accelerated Kleene iteration

Example

```
while (1) {
xn1 = -0.4375 * x1 + 0.0625 * x2 + 0.2652 * x3 + 0.1 * u1 ;
xn2 = 0.0625 * x1 + 0.4375 * x2 + 0.2652 * x3 + 0.1 * u2 ;
xn3 = -0.2652 * x1 + 0.2652 * x2 + 0.375 * x3 + 0.1 * u3 ;
x1 = xn1 ;x2 = xn2 ;x3 = xn3 ;
} [-5.197505,8.873306]
```

Itération : 15

x1	$\Upsilon_A(\overrightarrow{y})$	$\|y_i - y_{i-1}\|$	**Seuils** ($\delta = 10^{-3}$)
[1.000000,2.000000]	[1.000000,2.000000]		
[-0.447300,5.716500]			
[-2.291255,6.573381]	[6.280857,6.830145]	max(5.280857,4.830145)	
[-3.038029,7.492492]			
[-3.695558,7.871720]	[-4.663282,8.463092]	max(10.944139,2.367053)	
[-4.084503,8.185954]			
[-4.386442,8.369221]	[-5.189239,8.851176]	max(0.525957,0.388084)	
[-4.594886,8.508279]			
[-4.751445,8.603235]	[-5.197089,8.872567]	max(0.00785,0.021391)	
[-4.865385,8.673918]			
[-4.950393,8.725095]	[-5.197506,8.873307]	max(0.000417,0.00074)	[-5.197506,8.873307]
[-5.197506,8.873307]			
[-5.197506,8.873307]	[-5.197089,8.872567]	max(0.000417,0.00074)	[-5.197089,8.872567]
[-5.197567,8.873341]			
[-5.197616,8.873348]	[-5.198638,8.849716]	max(0.001549,0.022851)	

The accelerated Kleene iteration

Example

```
while (1) {
xn1 = -0.4375 * x1 + 0.0625 * x2 + 0.2652 * x3 + 0.1 * u1 ;
xn2 = 0.0625 * x1 + 0.4375 * x2 + 0.2652 * x3 + 0.1 * u2 ;
xn3 = -0.2652 * x1 + 0.2652 * x2 + 0.375 * x3 + 0.1 * u3 ;
x1 = xn1 ;x2 = xn2 ;x3 = xn3 ;
} [-5.197505,8.873306] ↷ [-5.197598,8.873362]
```

Itération : 16

x1	$\Upsilon_A(\overrightarrow{y})$	$\|y_i - y_{i-1}\|$	Seuils ($\delta = 10^{-3}$)
[1.000000,2.000000]	[1.000000,2.000000]		
[-0.447300,5.716500]			
[-2.291255,6.573381]	[6.280857,6.830145]	max(5.280857,4.830145)	
[-3.038029,7.492492]			
[-3.695558,7.871720]	[-4.663282,8.463092]	max(10.944139,2.367053)	
[-4.084503,8.185954]			
[-4.386442,8.369221]	[-5.189239,8.851176]	max(0.525957,0.388084)	
[-4.594886,8.508279]			
[-4.751445,8.603235]	[-5.197089,8.872567]	max(0.00785,0.021391)	
[-4.865385,8.673918]			
[-4.950393,8.725095]	[-5.197506,8.873307]	max(0.000417,0.00074)	[-5.197506,8.873307]
[-5.197506,8.873307]			
[-5.197506,8.873307]	[-5.197089,8.872567]	max(0.000417,0.00074)	[-5.197089,8.872567]
[-5.197567,8.873341]			
[-5.197616,8.873348]	[-5.198638,8.849716]	max(0.001549,0.022851)	
[-5.197598,8.873362]			

Accelerated Kleene Algorithm using support function

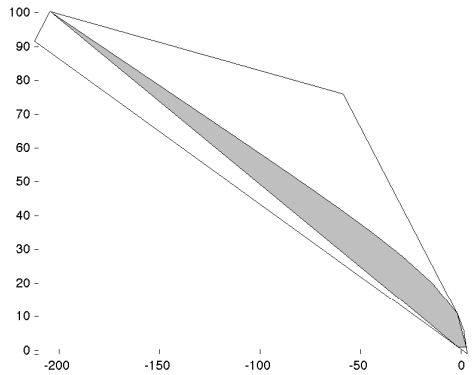
Algorithm 9 Accelerated Kleene Algorithm using support function.

Require: $\Delta \subset \mathbb{R}^n$, $\mathbb{P}_0$, $A \in \mathbb{R}^n \times \mathbb{R}^m$, $b \in \mathbb{R}^m$

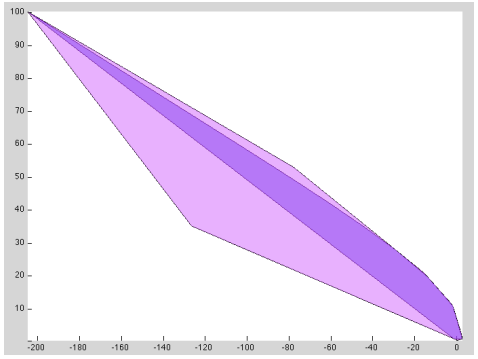
$D = \Delta$, $\Omega = \delta_{\mathbb{P}_0}(\Delta)$
repeat
  $\Omega' = \Omega$, $Y' = Y$
  for all $i = 0, \ldots, (l-1)$ **do**
    $\Theta[i] = \Theta[i] + \langle b, D[i] \rangle$
    $D[i] = A^T D[i]$
    $\Upsilon[i] = \delta_{\mathbb{P}_0}(d[i]) + \Theta[i]$
    $Y[i] :=$ Accelerate $(\Upsilon[0], \ldots, \Upsilon[i])$
    if $||Y[i] - Y'[i]|| \leq \varepsilon$ **then**
      $\Omega[i] = \max(\Omega[i], Y[i])$
    else
      $\Omega[i] = \max(\Omega[i], \Upsilon[i])$
    end if
  end for
until $\Omega \sqsubseteq \Omega'$

Kleene Algorithm using support function

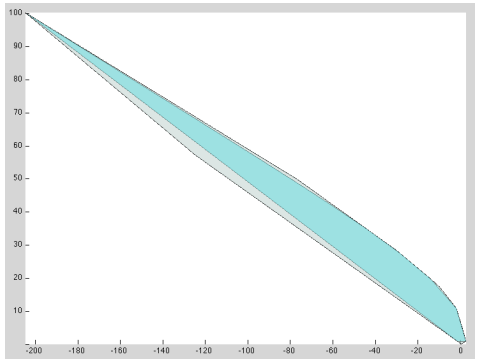
```
begin
    while (0<=10) do
    xn = 0.5 *x - y - 2.5;
     yn = 0.9 *y + 10;
     x = xn; y = yn;
    done;
end
```



8 directions
(0.044 seconde)



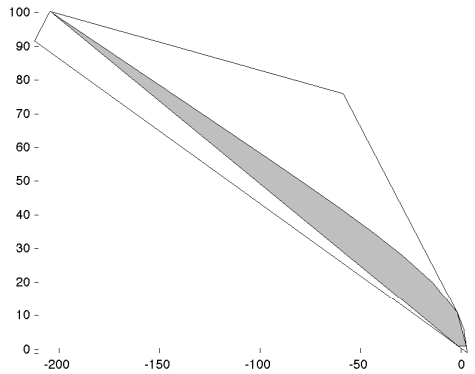
50 directions
(0.34 seconde)



100 directions
(0.7 seconde)

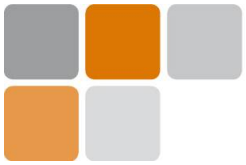
Accelerated Kleene Algorithm using support function

```
begin
    while (0<=10) do
    xn = 0.5 *x - y - 2.5;
     yn = 0.9 *y + 10;
     x = xn; y = yn;
    done;
end
```



8 directions

- Kleene iteration using support function : 200 iterations
- Accelerated Kleene iteration using support function : 11 iterations



S. Sankaranarayanan, H. Sipma, and Z. Manna. Scalable analysis of linear systems using mathematical programming. In VMCAI. Springer, 2005.

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Similarities :

- Abstract domain based on a static choice of directions set.
- The same definition of inclusion, meet and join operators.

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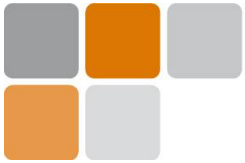
Similarities :

- Abstract domain based on a static choice of directions set.
- The same definition of inclusion, meet and join operators.

Differences :

- The fixpoint computation uses Linear Programming.
- The fixpoint is computed with the abstract elements obtained in each Kleene iteration.



Conclusion and future work

Conclusion

- We develop a new numerical abstract domain based on support function.
- Our abstract domain depends on a set of finite directions.
- The fixpoint is computed in a polynomial time.

Perspectives

- Implements this domain on APRON.
- According to the program to analyse, defines a relevant set of directions.