PSN for the $\boldsymbol{\lambda}_{\mathtt{s}}$-calculus

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Abstract. The goal of this technical paper is to show preservation of strong normalisation for the $\lambda_{\mathtt{s}}$-calculus, a sub-calculus of the prismoid of resources [KR11]. For that, we use a modular technique given in [Kes09], which uses the labelling approach.

The modular technique requires Full Composition (proved in [KR11] independently of PSN) and the (IE) property requiring that the normalisation of a term with an implicit substitution implies normalisation of the same term with an explicit substitution:

$$(IE)\ u \in SN_{\lambda_{\mathtt{s}}}\ \&\ t\{x/u\}\overline{v_n} \in SN_{\lambda_{\mathtt{s}}} \text{ imply } t[x/u]\overline{v_n} \in SN_{\lambda_{\mathtt{s}}}$$

To achieve this difficult goal, we proceed by doing the following steps:

1. Add to the grammar of $\lambda\mathtt{s}$ (the $\mathtt{s}$-terms) labelled substitutions to mark strongly normalising $\mathtt{s}$-terms used as parameters of substitutions. For instance $t[\![x/u]\!]$ indicates that u is an $\mathtt{s}$-term (without labels) s.t $u \in SN_{\lambda_{\mathtt{s}}}$.
2. Enrich the reduction system $\rightarrow_{\lambda_{\mathtt{s}}}$ with another system $\rightarrow_{\underline{\mathtt{s}}}$ (thus obtaining a new system $\rightarrow_{\lambda_{\underline{\mathtt{s}}}}$) used only to propagate terminating labelled substitutions.

These two first points are developed in Section 1. After that, we can simplify the (IE) property into two more elementary steps:

3. Show that $u \in SN_{\lambda_{\mathtt{s}}}\ \&\ t\{x/u\}\overline{v_n} \in SN_{\lambda_{\mathtt{s}}}$ imply $t[\![x/u]\!]\overline{v_n} \in \mathcal{SN}_{\lambda_{\underline{\mathtt{s}}}}$.
4. Show that $t[\![x/u]\!]\overline{v_n} \in \mathcal{SN}_{\lambda_{\underline{\mathtt{s}}}}$ implies $t[x/u]\overline{v_n} \in \mathcal{SN}_{\lambda_{\mathtt{s}}}$.

These two last points are developed in Section 2. The modular approach implying PSN from the (IE) property is finally shown in Section 3.

1 The Labelling Technique

In this section we introduce the set of labelled terms and their associated reduction systems. The key idea is that bodies of labelled substitutions are strongly normalising terms, this invariant being kept during the reduction. The main rewriting system $\rightarrow_{\lambda_{\underline{\mathtt{s}}}}$ associated to labelled terms is split in two relations $\rightarrow_{\lambda\underline{\mathtt{s}}^i}$ and $\rightarrow_{\lambda\underline{\mathtt{s}}^e}$. The idea is that $\rightarrow_{\lambda\underline{\mathtt{s}}^i}$ steps will be weakly projected (eventually empty steps) into $\lambda_{\mathtt{s}}$ whereas $\rightarrow_{\lambda\underline{\mathtt{s}}^e}$ will be strongly projected (at least one step) into $\lambda_{\mathtt{s}}$. Formally if $\mathtt{xc}(\_)$ is a function mapping labelled terms to $\mathtt{s}$-terms, and t, t' are labelled terms then $t \rightarrow_{\lambda\underline{\mathtt{s}}^i} t'$ implies $\mathtt{xc}(t) \rightarrow^*_{\lambda_{\mathtt{s}}} \mathtt{xc}(t')$ and $t \rightarrow_{\lambda\underline{\mathtt{s}}^e} t'$ implies $\mathtt{xc}(t) \rightarrow^+_{\lambda_{\mathtt{s}}} \mathtt{xc}(t')$

The key lemma of this section states that $\rightarrow_{\lambda\underline{\mathtt{s}}^i}$ is terminating.

1.1 The Labelled Terms

Given a set of variables $\mathbb{S}$, the $\mathbb{S}$-labelled terms or $\mathbb{S}$-terms (or simply labelled terms if $\mathbb{S}$ is clear from the context), are given by:

$$T_{\mathbb{S}} ::= x \mid T_{\mathbb{S}}\ T_{\mathbb{S}} \mid \lambda x.T_{\mathbb{S}} \mid T_{\mathbb{S}}[x/T_{\mathbb{S}}] \mid T_{\mathbb{S}}[\![x/v]\!](v \in T \cap SN_{\lambda_{\mathtt{s}}}\ \&\ \mathtt{fv}(v) \subseteq \mathbb{S})$$

Labelled substitutions can only contain $\lambda_{\mathtt{s}}$-terms so in particular they cannot contain other labelled substitutions inside them. Remark that for instance, if $\mathbb{S} = \{y\}$ the $\mathbb{S}$-term $x[\![x/y]\!][y/u]$ is α-equivalent to $x[\![x/z]\!][z/u]$, leading to a term not in the grammar anymore. To be stable under α-conversion, we will thus only consider $\mathbb{S}$-terms where $x \notin \mathbb{S}$ in the terms $u[x/v]$, $u[\![x/v]\!]$, and $\lambda x.v$.

Bodies of labelled substitutions are normalising by definition and do not loose this property thanks to the semantics of the rules propagating labelled substitutions. Indeed, these rules guarantee that labelled substitutions can traverse/commute normal substitutions but not the converse. The $\mathbb{S}$-terms will thus be trivially stable by the reduction defined by the following set of equations and rules:

Equations :

$$\begin{array}{llll}
t[x/u][\![y/v]\!] & \equiv_{\underline{\mathtt{SS_C}}} & t[\![y/v]\!][x/u] & y \notin \mathtt{fv}(u)\ \&\ x \notin \mathtt{fv}(v) \\
t[\![x/u]\!][\![y/v]\!] & \equiv_{\underline{\mathtt{SS_C}}} & t[\![y/v]\!][\![x/u]\!] & y \notin \mathtt{fv}(u)\ \&\ x \notin \mathtt{fv}(v)
\end{array}$$

Rules :

$$\begin{array}{llll}
t[\![x/u]\!] & \rightarrow_{\underline{\mathtt{SGc}}} & t & x \notin \mathtt{fv}(t) \\
x[\![x/u]\!] & \rightarrow_{\underline{\mathtt{V}}} & u & \\
(\lambda y.t)[\![x/u]\!] & \rightarrow_{\underline{\mathtt{SL}}} & \lambda y.t[\![x/u]\!] & \\
(t\ v)[\![x/u]\!] & \rightarrow_{\underline{\mathtt{SA_L}}} & t[\![x/u]\!]\ v & x \notin \mathtt{fv}(v) \\
(t\ v)[\![x/u]\!] & \rightarrow_{\underline{\mathtt{SA_R}}} & t\ v[\![x/u]\!] & x \notin \mathtt{fv}(t) \\
t[y/v][\![x/u]\!] & \rightarrow_{\underline{\mathtt{SS}}} & t[y/v[\![x/u]\!]] & x \in \mathtt{fv}(v) \setminus \mathtt{fv}(t) \\
t[\![x/u]\!] & \rightarrow_{\underline{\mathtt{SDup}}} & t_{x \rightsquigarrow y}[\![x/u]\!][\![y/u]\!] & |t|_x > 1\ \&\ y \text{ fresh}
\end{array}$$

The $\Rightarrow_{\underline{\mathtt{s}}}$ (resp $\rightarrow_{\underline{\mathtt{s}}}$) reduction relation is generated by the previous reduction rules (resp. modulo $\equiv_{\underline{\mathtt{SS_C}}}$) conversion. They can be simulated by reduction on their unlabelled corresponding $\mathtt{s}$-terms. We also consider the relation $\rightarrow_{\lambda_{\underline{\mathtt{s}}}} = \rightarrow_{\lambda_{\mathtt{s}}} \cup \rightarrow_{\underline{\mathtt{s}}}$ on labelled terms.

1.2 Internal and External Reductions

We now split $\rightarrow_{\lambda_{\underline{\mathtt{s}}}}$ in two disjoint relations $\Rightarrow_{\lambda_{\underline{\mathtt{s}}^i}}$ and $\Rightarrow_{\lambda_{\underline{\mathtt{s}}^e}}$ which will be projected into $\lambda_{\mathtt{s}}$-reduction sequences differently.

Definition 1. *The internal reduction relation $\rightarrow_{\lambda_{\underline{\mathtt{s}}^i}}$ is taken as the following reduction relation $\Rightarrow_{\lambda_{\underline{\mathtt{s}}^i}}$ on $\equiv_{\mathtt{SS_C},\underline{\mathtt{SS_C}}}$-equivalence classes:*

- *If $u \Rightarrow_{\lambda_{\mathtt{s}}} u'$, then $t[\![x/u]\!] \Rightarrow_{\lambda_{\underline{\mathtt{s}}^i}} t[\![x/u']\!]$.*
- *If $t \Rightarrow_{\underline{\mathtt{s}}} t'$, then $t \Rightarrow_{\lambda_{\underline{\mathtt{s}}^i}} t'$.*
- *If $t \Rightarrow_{\lambda_{\underline{\mathtt{s}}^i}} t'$, then $t\ u \Rightarrow_{\lambda_{\underline{\mathtt{s}}^i}} t'\ u, u\ t \Rightarrow_{\lambda_{\underline{\mathtt{s}}^i}} u\ t'$, $\lambda x.t \Rightarrow_{\lambda_{\underline{\mathtt{s}}^i}} \lambda x.t', t[x/u] \Rightarrow_{\lambda_{\underline{\mathtt{s}}^i}} t'[x/u]$, $u[x/t] \Rightarrow_{\lambda_{\underline{\mathtt{s}}^i}} u[x/t']$, $t[\![x/u]\!] \Rightarrow_{\lambda_{\underline{\mathtt{s}}^i}} t'[\![x/u]\!]$.*

The external reduction relation $\rightarrow_{\lambda\underline{\mathtt{s}}^e}$ is taken as the following reduction relation $\Rightarrow_{\lambda\underline{\mathtt{s}}^e}$ on $\equiv_{\mathtt{SS_C},\underline{\mathtt{SS_C}}}$-equivalence classes:

- *If $t \Rightarrow_{\lambda_{\mathtt{s}}} t'$ occurs outside a labelled substitution, then $t \Rightarrow_{\lambda\underline{\mathtt{s}}^e} t'$.*
- *If $t \Rightarrow_{\lambda\underline{\mathtt{s}}^e} t'$, then $tu \Rightarrow_{\lambda\underline{\mathtt{s}}^e} t'u, ut \Rightarrow_{\lambda\underline{\mathtt{s}}^e} ut', \lambda x.t \Rightarrow_{\lambda\underline{\mathtt{s}}^e} \lambda x.t', t[x/u] \Rightarrow_{\lambda\underline{\mathtt{s}}^e} t'[x/u], u[x/t] \Rightarrow_{\lambda\underline{\mathtt{s}}^e} u[x/t'], t[\![x/u]\!] \Rightarrow_{\lambda\underline{\mathtt{s}}^e} t'[\![x/u]\!]$.*

Lemma 1. $\rightarrow_{\lambda_{\underline{\mathtt{s}}}} = \rightarrow_{\lambda\underline{\mathtt{s}}^i} \cup \rightarrow_{\lambda\underline{\mathtt{s}}^e}$

Proof. As in [Kes09].

1.3 Termination of $\rightarrow_{\lambda\underline{\mathtt{s}}^i}$

As $\rightarrow_{\lambda\underline{\mathtt{s}}^i}$ will only be weakly projected into $\lambda_{\mathtt{s}}$, we need to guarantee that there are no infinite $\rightarrow_{\lambda\underline{\mathtt{s}}^i}$ reductions starting from a labelled term. This will be useful in Section 2 to relate termination of $\lambda_{\mathtt{s}}$ to that of $\rightarrow_{\lambda_{\underline{\mathtt{s}}}}$.

We show termination of $\rightarrow_{\lambda\underline{\mathtt{s}}^i}$ using several measures that will be combined using a lexicographic order.

The first one counts the number of free occurrences of variables, giving to them more weight if they appear in the body of labelled substitutions.

Definition 2.

$$
\begin{array}{llll}
\mathtt{af}_x(z) &= 0 & \mathtt{af}_x(\lambda y.t) &= \mathtt{af}_x(t) \\
\mathtt{af}_x(x) &= 1 & \mathtt{af}_x(t[\![y/u]\!]) &= \mathtt{af}_x(t) + \mathtt{af}_y(t).\mathtt{af}_x(u) \\
\mathtt{af}_x(tu) &= \mathtt{af}_x(t) + \mathtt{af}_x(u) & \mathtt{af}_x(t[y/u]) &= \mathtt{af}_x(t) + \mathtt{af}_x(u)
\end{array}
$$

Remark that $\mathtt{af}_x(t) = 0$ if $x \notin \mathtt{fv}(t)$ and thus $\mathtt{af}_x(t[\![y/u]\!]) = \mathtt{af}_x(t)$ if $x \notin \mathtt{fv}(u)$. We also have $\mathtt{af}_x(t) = \mathtt{af}_y(t\{x/y\})$ for any y fresh.

Lemma 2. $\mathtt{af}_x(u[\![x]\!]) = \mathtt{af}_x(u[\![y]\!]) + \mathtt{af}_y(u[\![y]\!])$ *with y fresh.*

Proof. By induction on $|u|_x$.

We now define another function which counts the number of variables and give more weight to those appearing inside bodies of labelled substitution.

Definition 3. *Let* $\exp(k) = 2^{2^k}$.

$$
\begin{array}{llll}
\mathtt{dep}(x) &= 1 & \mathtt{dep}(t[x/u]) &= \mathtt{dep}(t) + \mathtt{dep}(u) \\
\mathtt{dep}(\lambda y.t) &= \mathtt{dep}(t) & \mathtt{dep}(tu) &= \mathtt{dep}(t) + \mathtt{dep}(u) \\
\mathtt{dep}(t[\![x/u]\!]) &= \mathtt{dep}(t) + \exp(\mathtt{af}_x(t)).\mathtt{dep}(u) & &
\end{array}
$$

Let $\phi(t) = 1 + \eta_{\lambda_{\mathtt{s}}}(t) + \mathtt{maxk}_{\lambda_{\mathtt{s}}}(t)$ where $\mathtt{maxk}_{\lambda_{\mathtt{s}}}(t) = \mathtt{max}\{\mathtt{k}(t') | t \rightarrow^*_{\lambda_{\mathtt{s}}} t'\}$ with the following definition for the function k:

$$
\begin{array}{llll}
\mathtt{k}(x) &= 1 & \mathtt{k}(t[x/u]) &= \mathtt{k}(t).(\mathtt{k}(u) + 1) \\
\mathtt{k}(\lambda x.t) &= \mathtt{k}(t) + 1 & \mathtt{k}(t[\![x/u]\!]) &= \mathtt{k}(t).\phi(u) \\
\mathtt{k}(tu) &= \mathtt{k}(t) + \mathtt{k}(u) + 1 & &
\end{array}
$$

Remark that $\mathtt{k}$ and ϕ are not mutually recursive because in the case of the labelled substitution of $\mathtt{k}$, there are no labelled substitutions in the subterm u so when $\phi(u)$ calls one more time $\mathtt{k}(\_)$, the case of the labelled substitution cannot be reached.

We have the following properties on the previous functions:

- $\phi(v) \geq 2$
- $v \rightarrow_{\lambda_{\mathtt{s}}} v'$ implies $\eta_{\lambda_{\mathtt{s}}}(v) > \eta_{\lambda_{\mathtt{s}}}(v')$ and $\mathtt{maxk}_{\lambda_{\mathtt{s}}}(v) \geq \mathtt{maxk}_{\lambda_{\mathtt{s}}}(v')$ so that $\phi(v) > \phi(v')$.

Furthermore, we need extra properties concerning the non-deterministic replacement:

Lemma 3. *Let x, z be distinct variables, and y a fresh variable s.t. $y \notin \mathbb{S}$.*

1. $\mathtt{dep}(t) = \mathtt{dep}(t_{x \rightsquigarrow y})$
2. $\mathtt{af}_z(t_{x \rightsquigarrow y}) = \mathtt{af}_z(t)$
3. $\mathtt{af}_x(t) = \mathtt{af}_x(t_{x \rightsquigarrow y}) + \mathtt{af}_y(t_{x \rightsquigarrow y})$

Proof. The first point is true since $\mathtt{dep}(\_)$ does not directly take into account variables, the only way there could be a difference would be a call to a call to $\mathtt{af}_x(\_)$ or $\mathtt{af}_y(\_)$ which is impossible by α-conversion.

The other points are straightforward.

We show that $\mathtt{af}_x(t)$ is stable under reduction, and that $\mathtt{dep}(\_)$ and $\mathtt{k}(\_)$ decrease in such a way that we can prove that $\rightarrow_{\lambda \underline{\mathtt{S}}^i}$ terminates:

Lemma 4. *Let t, u be $\mathbb{S}$-terms and let $z \notin \mathbb{S}$.*

1. $t \equiv_{\mathtt{P}_{\underline{\mathtt{C}}}} t'$ *implies* $\mathtt{af}_z(t) = \mathtt{af}_z(t')$, $\mathtt{dep}(t) = \mathtt{dep}(t')$, *and* $\mathtt{k}(t) = \mathtt{k}(t')$.
2. $t \rightarrow_{\underline{\mathtt{SL}}, \underline{\mathtt{SA}_\mathtt{L}}, \underline{\mathtt{SA}_\mathtt{R}}, \underline{\mathtt{SS}}} t'$ *implies* $\mathtt{af}_z(t) = \mathtt{af}_z(t')$, $\mathtt{dep}(t) = \mathtt{dep}(t')$, *and* $\mathtt{k}(t) > \mathtt{k}(t')$.
3. $t \rightarrow_{\underline{\mathtt{V}}, \underline{\mathtt{SGc}}, \underline{\mathtt{SDup}}} t'$ *implies* $\mathtt{af}_z(t) = \mathtt{af}_z(t')$ *and* $\mathtt{dep}(t) > \mathtt{dep}(t')$.

Proof. We only show the interesting cases, as the other ones are straightforward:

- If $t = t_1[\![x/u]\!][\![y/v]\!] \equiv_{\underline{\mathtt{SS}_\mathtt{C}}} t_1[\![y/v]\!][\![x/u]\!] = t'$, with $y \notin \mathtt{fv}(u)$ & $x \notin \mathtt{fv}(v)$, then
  - $$\begin{array}{lr} \mathtt{af}_z(t_1[\![x/u]\!][\![y/v]\!]) & = \\ \mathtt{af}_z(t_1[\![x/u]\!]) + \mathtt{af}_y(t_1[\![x/u]\!]).\mathtt{af}_y(v) & = \\ \mathtt{af}_z(t_1) + \mathtt{af}_x(t_1).\mathtt{af}_z(u) + \mathtt{af}_y(t_1[\![x/u]\!]).\mathtt{af}_z(v) & = \\ \mathtt{af}_z(t_1) + \mathtt{af}_x(t_1).\mathtt{af}_z(u) + (\mathtt{af}_y(t_1) + \mathtt{af}_x(t_1).\mathtt{af}_y(u)).\mathtt{af}_z(v) & = \\ \mathtt{af}_z(t_1) + \mathtt{af}_x(t_1).\mathtt{af}_z(u) + \mathtt{af}_y(t_1).\mathtt{af}_z(v) & = \\ \mathtt{af}_z(t_1) + \mathtt{af}_y(t_1).\mathtt{af}_z(v) + (\mathtt{af}_x(t_1) + \mathtt{af}_y(t_1).\mathtt{af}_x(v)).\mathtt{af}_z(u) & = \\ \mathtt{af}_z(t_1[\![y/v]\!]) + \mathtt{af}_x(t_1[\![y/v]\!]).\mathtt{af}_z(u) & = \\ \mathtt{af}_z(t_1[\![y/v]\!][\![x/u]\!]) & \end{array}$$

- $$\begin{array}{ll} \mathtt{dep}(t_1[\![x/u]\!][\![y/v]\!]) & = \\ \mathtt{dep}(t_1[\![x/u]\!]) + \exp(\mathtt{af}_y(t_1[\![x/u]\!])).\mathtt{dep}(v)) & = \\ \mathtt{dep}(t_1) + \exp(\mathtt{af}_x(t_1)).\mathtt{dep}(u) + \exp(\mathtt{af}_y(t_1[\![x/u]\!])).\mathtt{dep}(v) & = \\ \mathtt{dep}(t_1) + \exp(\mathtt{af}_x(t_1)).\mathtt{dep}(u) + \exp(\mathtt{af}_y(t_1)).\mathtt{dep}(v) & = \\ \mathtt{dep}(t_1[\![y/v]\!]) + \exp(\mathtt{af}_x(t_1[\![y/v]\!])).\mathtt{dep}(u) & = \\ \mathtt{dep}(t_1[\![y/v]\!][\![x/u]\!]) & \end{array}$$
- $\mathtt{k}(t) = \mathtt{k}(t_1).\phi(u).\phi(v) = \mathtt{k}(t')$

– If $t_1[y/t_2][\![x/v]\!] \rightarrow_{\underline{\mathtt{SS}}} t_1[y/t_2[\![x/v]\!]] = t'$ with $x \notin \mathtt{fv}(t_1)$ & $x \in \mathtt{fv}(t_2)$

- $$\begin{array}{ll} \mathtt{af}_z(t_1[y/t_2][\![x/v]\!]) & = \\ \mathtt{af}_z(t_1[y/t_2]) + \mathtt{af}_x(t_1[y/t_2]).\mathtt{af}_z(v) & = \\ \mathtt{af}_z(t_1[y/t_2]) + \mathtt{af}_x(t_2).\mathtt{af}_z(v) & = \\ \mathtt{af}_z(t_1) + \mathtt{af}_z(t_2) + \mathtt{af}_x(t_2).\mathtt{af}_z(v) & = \mathtt{af}_z(t') \end{array}$$
- $$\mathtt{dep}(t) = \mathtt{dep}(t_1) + \mathtt{dep}(t_2) + \exp(\mathtt{af}_x(t_2)).\mathtt{dep}(v) = \mathtt{dep}(t')$$
- $$\begin{array}{ll} \mathtt{k}(t_1[y/t_2][\![x/v]\!]) & = \\ \mathtt{k}(t_1[y/t_2]).\phi(v) & = \\ \mathtt{k}(t_1).(\mathtt{k}(t_2)+1).\phi(v) & = \\ \mathtt{k}(t_1).(\mathtt{k}(t_2).\phi(v) + \phi(v)) & > \\ \mathtt{k}(t_1).(\mathtt{k}(t_2[\![x/v]\!]) + 1) & = \mathtt{k}(t') \end{array}$$

– If $t = x[\![x/v]\!] \rightarrow_{\underline{\mathtt{V}}} v = t'$, then

- $$\begin{array}{ll} \mathtt{af}_z(x[\![x/v]\!]) & = \\ \mathtt{af}_z(x) + \mathtt{af}_x(x).\mathtt{af}_z(v) & = \\ \mathtt{af}_z(v) & \end{array}$$
- $$\begin{array}{ll} \mathtt{dep}(x[\![x/v]\!]) & = \\ \mathtt{dep}(x) + \exp(\mathtt{af}_x(x)).\mathtt{dep}(v) & = \\ 1 + 4.\mathtt{dep}(v) & > \\ \mathtt{dep}(v) & \end{array}$$

– $t[\![x/u]\!] \rightarrow_{\underline{\mathtt{SDup}}} t_{x \rightsquigarrow y}[\![x/u]\!][\![y/u]\!]$ with $|t|_x > 1$

- $$\begin{array}{ll} \mathtt{af}_z(t[\![x/u]\!]) & = \\ \mathtt{af}_z(t) + \mathtt{af}_x(t).\mathtt{af}_z(u) & =_{L.\ 3} \\ \mathtt{af}_z(t) + (\mathtt{af}_x(t_{x \rightsquigarrow y}) + \mathtt{af}_y(t_{x \rightsquigarrow y})).\mathtt{af}_z(u) & =_{L.\ 3} \\ \mathtt{af}_z(t_{x \rightsquigarrow y}) + \mathtt{af}_x(t_{x \rightsquigarrow y}).\mathtt{af}_z(u) + \mathtt{af}_y(t_{x \rightsquigarrow y}).\mathtt{af}_z(u) & = \\ \mathtt{af}_z(t_{x \rightsquigarrow y}[\![x/u]\!]) + \mathtt{af}_y(t_{x \rightsquigarrow y}).\mathtt{af}_z(u) & =_{(y \notin \mathtt{fv}(u))} \\ \mathtt{af}_z(t_{x \rightsquigarrow y}[\![x/u]\!]) + \mathtt{af}_y(t_{x \rightsquigarrow y}[\![x/u]\!]).\mathtt{af}_z(u) & = \mathtt{af}_z(t') \end{array}$$

- $$\begin{array}{ll}
\mathtt{dep}(t[\![x/u]\!]) & = \\
\mathtt{dep}(t) + \exp(\mathtt{af}_x(t)).\mathtt{dep}(u) & =_{L.\ 3} \\
\mathtt{dep}(t) + \exp(\mathtt{af}_x(t_{x \leadsto y}) + \mathtt{af}_y(t_{x \leadsto y})).\mathtt{dep}(u) & =_{L.\ 3} \\
\mathtt{dep}(t_{x \leadsto y}) + \exp(\mathtt{af}_x(t_{x \leadsto y}) + \mathtt{af}_y(t_{x \leadsto y})).\mathtt{dep}(u) & > \\
\mathtt{dep}(t_{x \leadsto y}) + \exp(\mathtt{af}_x(t_{x \leadsto y})).\mathtt{dep}(u) + \exp(\mathtt{af}_y(t_{x \leadsto y})).\mathtt{dep}(u) & = \\
\mathtt{dep}(t_{x \leadsto y}[\![x/u]\!]) + \exp(\mathtt{af}_y(t_{x \leadsto y})).\mathtt{dep}(u) & =_{(y \notin \mathtt{fv}(u))} \\
\mathtt{dep}(t_{x \leadsto y}[\![x/u]\!]) + \exp(\mathtt{af}_y(t_{x \leadsto y}[\![x/u]\!])).\mathtt{dep}(u) & = \mathtt{dep}(t')
\end{array}$$

– All inductive cases easily hold because the measures take into account all the subterms.

Lemma 5. *The reduction relation $\to_{\underline{\mathtt{s}}}$ is terminating.*

Proof. Since $t \to_{\underline{\mathtt{s}}} t'$ implies $\langle \mathtt{dep}(t), \mathtt{k}(t) \rangle >_{lex} \langle \mathtt{dep}(t'), \mathtt{k}(t') \rangle$ by Lemma 4 and $>_{lex}$ is a well-founded relation, then $\to_{\underline{\mathtt{s}}}$ terminates.

We can now conclude this section:

Lemma 6. *The reduction relation $\to_{\lambda \underline{\mathtt{s}}^i}$ is terminating.*

Proof. The proof which uses Lemma 4 is exactly the same as in [Kes09].

2 The IE Property

We can now prove the first elementary step of the (IE) property, connecting the normalisation of an implicit substitution in the reduction system $\lambda_{\mathtt{s}}$ to the normalisation of a labelled substitution in $\lambda_{\underline{\mathtt{s}}}$:

$$u \in SN_{\lambda_{\mathtt{s}}} \ \& \ t\{x/u\}\overline{v_n} \in SN_{\lambda_{\mathtt{s}}} \text{ imply } t[\![x/u]\!]\overline{v_n} \in \mathcal{S}N_{\lambda_{\underline{\mathtt{s}}}}$$

This requires, in addition to the termination of the reduction system $\to_{\lambda \underline{\mathtt{s}}^i}$, to project the steps of the labelled reductions in $\to_{\lambda_{\mathtt{s}}}$-steps.

2.1 Projection

We define a function $\mathtt{xc}$ which maps labelled terms to $\mathtt{s}$-terms.

$$\begin{array}{llll}
\mathtt{xc}(x) & = x & \mathtt{xc}(t[x/u]) & = \mathtt{xc}(t)[x/\mathtt{xc}(u)] \\
\mathtt{xc}(tu) & = \mathtt{xc}(t)\ \mathtt{xc}(u) & \mathtt{xc}(t[\![x/v]\!]) & = \mathtt{xc}(t)\{x/v\} \\
\mathtt{xc}(\lambda y.t) & = \lambda y.\mathtt{xc}(t) & &
\end{array}$$

Lemma 7. *Let t be a labelled term. If $t \to_{\underline{\mathtt{s}}} t'$, then $\mathtt{xc}(t) = \mathtt{xc}(t')$*

Proof. The interesting case is $t = t_1[y/t_2][\![x/v]\!] \to_{\mathtt{SS}} t_1[y/t_2[\![x/v]\!]] = t'$ with $x \in \mathtt{fv}(t_2) \setminus \mathtt{fv}(t_1)$:

$$\begin{array}{ll}
\mathtt{xc}(t) & = \\
\mathtt{xc}(t_1)[y/\mathtt{xc}(t_2)]\{x/v\} & = \\
\mathtt{xc}(t_1)\{x/v\}[y/\mathtt{xc}(t_2)\{x/v\}] & = \\
\mathtt{xc}(t_1)[y/\mathtt{xc}(t_2)\{x/v\}] & = \mathtt{xc}(t')
\end{array}$$

Lemma 8 (Projecting $\to_{\lambda_{\mathtt{s}}}$). *Let t, t' be labelled terms. Then,*

1. $t \equiv_{\mathtt{SS_C}} t'$ *or* $t \equiv_{\underline{\mathtt{SS_C}}} t'$ *implies* $\mathtt{xc}(t) = \mathtt{xc}(t')$
2. $t \Rightarrow_{\lambda_{\underline{\mathtt{s}}}^{i}} t'$ *implies* $\mathtt{xc}(t) \to^{*}_{\lambda_{\mathtt{s}}} \mathtt{xc}(t')$
3. $t \Rightarrow_{\lambda_{\underline{\mathtt{s}}}^{e}} t'$ *implies* $\mathtt{xc}(t) \to^{+}_{\lambda_{\mathtt{s}}} \mathtt{xc}(t')$

Proof. As $\mathtt{xc}(\_)$ does not alter application, lambda and substitution, the proof is the same that the one in [Kes09].

Lemma 9. *Let t be a labelled term. If $\mathtt{xc}(t) \in \mathcal{SN}_{\lambda_{\mathtt{s}}}$ then $t \in \mathcal{SN}_{\lambda_{\underline{\mathtt{s}}}}$*

Proof. The proof is the same one in [Kes09] and uses Lemma 8, Lemma 1, and Lemma 6.

Corollary 1. *Let $t, u, \overline{v_n}$ be $\mathtt{s}$-terms. If $u \in \mathcal{SN}_{\lambda_{\mathtt{s}}}$ and $t\{x/u\}\overline{v_n} \in \mathcal{SN}_{\lambda_{\mathtt{s}}}$ then $t[\![x/u]\!]\overline{v_n} \in \mathcal{SN}_{\lambda_{\underline{\mathtt{s}}}}$*

Proof. Take $\mathbb{S} = \mathtt{fv}(u)$. The hypothesis $u \in \mathcal{SN}_{\lambda_{\mathtt{s}}}$ allows us to construct the $\mathbb{S}$-labelled term $t[\![x/u]\!]\overline{v_n}$. Moreover $\mathtt{xc}(t) = t$ so that $\mathtt{xc}(t[\![x/u]\!]\overline{v_n}) = t\{x/u\}\overline{v_n}$ and we thus conclude by Lemma 9.

We can now prove the last elementary step, connecting the normalisation of a labelled substitution in the reduction system $\lambda_{\underline{\mathtt{s}}}$ to the normalisation of an explicit substitution in $\lambda_{\mathtt{s}}$:

$$t[\![x/u]\!]\overline{v_n} \in \mathcal{SN}_{\lambda_{\underline{\mathtt{s}}}} \text{ implies } t[x/u]\overline{v_n} \in \mathcal{SN}_{\lambda_{\mathtt{s}}}$$

This requires to perform an unlabelling, transforming labelled substitutions into regular explicit substitutions.

2.2 Unlabelling

We define a function $\mathtt{U}$ which maps $\mathbb{S}$-terms to $\mathtt{ex}$-terms.

Definition 4.

$$\begin{array}{ll}
\mathtt{U}(x) & = x \\
\mathtt{U}(tu) & = \mathtt{U}(t)\mathtt{U}(u) \\
\mathtt{U}(\lambda x.t) & = \lambda x.\mathtt{U}(t) \\
\mathtt{U}(t[x/u]) & = \mathtt{U}(t)[x/\mathtt{U}(u)] \\
\mathtt{U}(t[\![x/u]\!]) & = \mathtt{U}(t)[x/\mathtt{U}(u)]
\end{array}$$

Remark that $\mathtt{fv}(\mathtt{U}(t)) = \mathtt{fv}(t)$.

Lemma 10. *Let t_1 be labelled term s.t. $\mathtt{U}(t_1) \to_{\lambda_{\mathtt{s}}} u_2$. Then there exists a labelled term u_1 s.t. $t_1 \to_{\lambda_{\underline{\mathtt{s}}}} u_1$ and $\mathtt{U}(u_1) = u_2$.*

Proof. By induction on $\to_{\lambda_{\mathtt{s}}}$.

– The new interesting case w.r.t [Kes09] is when $t_1 = t[\![x/u]\!]$ and:

$$\mathtt{U}(t_1) = \mathtt{U}(t)[x/\mathtt{U}(u)] \to_{\mathtt{SDup}} \mathtt{U}(t)_{x \rightsquigarrow y}[x/\mathtt{U}(u)][y/\mathtt{U}(u)] = u_2$$

Notice that we can apply $\to_{\mathtt{SDup}}$ thanks to the preservation of free variables of $\mathtt{U}()$. We can conclude with $t_1 \to_{\underline{\mathtt{SDup}}} t_{x \rightsquigarrow y}[\![x/u]\!][\![y/u]\!] = u_1$.

– The case where $\mathtt{U}(t_1) = t[x/u][y/v] \to t[x/u[y/v]]$ with $y \in \mathtt{fv}(u) \setminus \mathtt{fv}(t)$ is the case which justifies the need of the set $\mathbb{S}$. Indeed there are several ways to label the $\mathtt{s}$-term $\mathtt{U}(t_1)$. For instance the labelling $t[\![x/u]\!][\![y/v]\!]$ (with $y \in \mathtt{fv}(u)$) cannot reduce on $t[\![x/u[\![y/v]\!]]\!]$. However thanks to the fact that $y \notin \mathbb{S}$ we have a contradiction.

Lemma 11. *Let $t \in T_{\mathbb{S}}$. If $t \in \mathcal{SN}_{\lambda_{\underline{\mathtt{s}}}}$, then $\mathtt{U}(t) \in SN_{\lambda_{\mathtt{s}}}$*

Proof. To show that $\mathtt{U}(t) \in SN_{\lambda_{\mathtt{s}}}$, we have to show that all reducts are in $SN_{\lambda_{\mathtt{s}}}$ i.e. $\forall t' \mathtt{U}(t) \equiv t_1 \Rightarrow_{\lambda_{\mathtt{s}}} t_2 \equiv t' \Rightarrow t' \in SN_{\lambda_{\mathtt{s}}}$. This is done by induction on $\eta_{\lambda_{\underline{\mathtt{s}}}}(t)$ using Lemma 10.

Taking $\mathbb{S} = \mathtt{fv}(u)$ and transforming the $\mathtt{s}$-term $s[x/u]\overline{u_n}$ into the $\mathbb{S}$-term $s[\![x/u]\!]\overline{u_n}$ we have the following special case:

Corollary 2. *If $s[\![x/u]\!]\overline{u_n} \in \mathcal{SN}_{\lambda_{\underline{\mathtt{s}}}}$, then we get $s[x/u]\overline{u_n} \in SN_{\lambda_{\mathtt{s}}}$.*

We can finally conclude this section:

Lemma 12 ((IE) Property). *If $u \in SN_{\lambda_{\mathtt{s}}}$ and $s\{x/u\}\overline{u_n} \in SN_{\lambda_{\mathtt{s}}}$ then $s[x/u]\overline{u_n} \in SN_{\lambda_{\mathtt{s}}}$.*

Proof. By Corollaries 1 and 2.

3 The PSN Proof

The modular approach of [Kes09] allows to deduce from the (IE) property that the following inductive definition gives exactly the set of strongly normalising $\mathtt{s}$-terms:

Definition 5. *The inductive set $\mathcal{ISN}$ is defined as follows:*

$$\frac{t_1, ..., t_n \in \mathcal{ISN} \quad n \geq 0}{x t_1 .. t_n \in \mathcal{ISN}}\ (\mathtt{var}) \qquad \frac{u[x/v] t_1 ... t_n \in \mathcal{ISN} \quad n \geq 0}{(\lambda x.u) v t_1 ... t_n \in \mathcal{ISN}}\ (\mathtt{app})$$

$$\frac{u\{x/v\} t_1 ... t_n \in \mathcal{ISN} \quad v \in \mathcal{ISN} \quad n \geq 0}{u[x/v] t_1 ... t_n \in \mathcal{ISN}}\ (\mathtt{subs}) \qquad \frac{u \in \mathcal{ISN}}{\lambda x.u \in \mathcal{ISN}}\ (\mathtt{abs})$$

Lemma 13. $\mathcal{ISN} = \mathcal{SN}_{\lambda_{\mathtt{s}}}$

Proof. It is guaranteed by the (IE) property [Kes09].

We can also use the inductive definition for strongly normalising λ-terms:

Definition 6 (Inductive definition of $\mathcal{S}N_\beta$[vR96]).

$$\frac{u \in \mathcal{S}N_\beta}{\lambda x.u \in \mathcal{S}N_\beta}\ (\mathtt{abs}_\beta) \qquad\qquad \frac{t_1, ..., t_n \in \mathcal{S}N_\beta \quad n \geq 0}{xt_1..t_n \in \mathcal{S}N_\beta}\ (\mathtt{var}_\beta)$$

$$\frac{u\{x/v\}t_1...t_n \in \mathcal{S}N_\beta \quad n \geq 0 \quad v \in \mathcal{S}N_\beta}{(\lambda x.u)vt_1...t_n \in \mathcal{S}N_\beta}\ (\mathtt{app}_\beta)$$

Theorem 1 (PSN for λ-terms). *If $t \in \mathcal{S}N_\beta$, then $t \in SN_{\lambda_\mathsf{s}}$.*

Proof. As the inductive set $\mathcal{I}SN$ By induction on the definition of $\mathcal{S}N_\beta$, using Definition 5 thanks to Lemma 13.

- If $t = xt_1...t_n$ with $t_i \in \mathcal{S}N_\beta$, then $t_i \in SN_{\lambda_\mathsf{s}}$ by the i.h. so that the ($\mathtt{var}$) rule allows to conclude.
- The case $t = \lambda x.u$ is similar.
- If $t = (\lambda x.u)vt_1...t_n$ with $u\{x/v\}t_1...t_n \in \mathcal{S}N_\beta$ and $v \in \mathcal{S}N_\beta$, then both terms are in SN_{λ_s} by the i.h. so that the ($\mathtt{subs}$) gives $u[x/v]t_1...t_n \in SN_{\lambda_\mathsf{s}}$ and the ($\mathtt{app}$) rule gives $(\lambda x.u)vt_1...t_n \in SN_{\lambda_\mathsf{s}}$.

References

Kes09. Delia Kesner. A theory of explicit substitutions with safe and full composition. *Logical Methods in Computer Science*, pages 1–29, 2009.

KR11. Delia Kesner and Fabien Renaud. A prismoid framework for languages with resources. *Theoretical Computer Science*, In Press, Accepted Manuscript, 2011.

vR96. Femke van Raamsdonk. *Confluence and Normalization for Higher-Order Rewriting*. PhD thesis, Amsterdam University, Netherlands, 1996.