INTERNSHIP PROPOSAL: ARRANGEMENTS OF DP-RIBBONS

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Location. The internship will take place at the "Institut de Mathématiques de Jussieu-Paris Rive Gauche" in Jussieu.

Scientific context and objectives. A DP-ribbon is a topological cylinder (a sphere with two boundaries) with a distinguished core circle with a distinguished side and an arrangement of DP-ribbons is a finite family of at least two DP-ribbons pairwise attached as shown in Fig. 1.

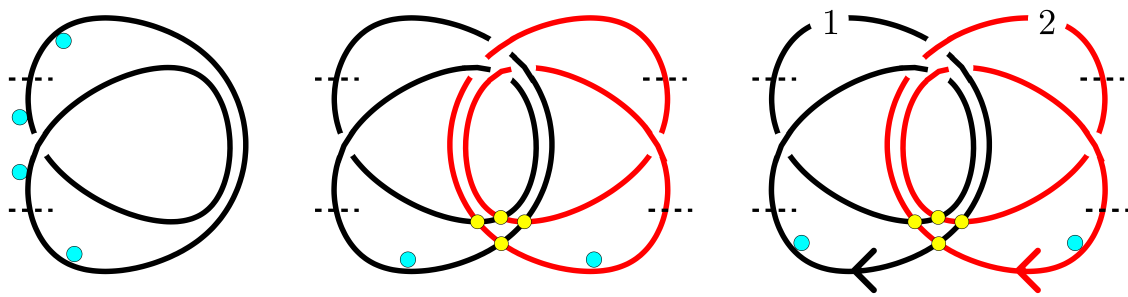

FIGURE 1. A DP-ribbon embedded in three-space (only the core circle is drawn, the distinguished side is indicated by small sky blue disks, and half-twists of the ribbon are indicated by horizontal dashed line segments), an arrangement of two DP-ribbons, and an indexed arrangement of two oriented DP-ribbons

Note that the underlying surface of an arrangement of two DP-ribbons is a sphere with one crosscap and five boundaries. The genus of an arrangement of DP-ribbons is the genus of its underlying surface. The interest of this class of arrangements lies in the following three statements.

Theorem 1 ([9]). *The arrangements of DP-ribbons of genus* 1 *are exactly, modulo the adjunction of topological disks along their boundaries, the so-called arrangements of double pseudolines, i.e., the dual arrangements of finite families of pairwise disjoint convex bodies of (real two-dimensional) projective planes.*

Theorem 2 ([9]). *An arrangement of DP-ribbons is of genus 1 if and only if its subarrangements of size 3, 4 and 5 are of genus 1.*

Theorem 3 ([9]). *There is a natural one-to-one and onto correspondence between indexed arrangements of n oriented DP-ribbons and the n-tuples of shuffles of the $n-1$ circular sequences $\overline{jj}jj$, $j = 2, 3, \ldots, n$. In particular the number b_n of indexed arrangements of n oriented DP-ribbons is*

$$\left\{4^{n-2}\binom{4n-5}{3,4,4,\ldots,4}\right\}^n$$

and the number a_n of arrangements of n DP-ribbons is bounded from below by

$$b_n/(2^n n!).$$

The class of arrangements of double pseudolines is an extension of the well-studied class of arrangements of pseudolines [7]. It plays a central role in the algorithmic of two-dimensional visibility graphs [14, 1], in the algorithmic of pseudotriangulations [8, 13, 15], in two-dimensional line transversal theory [5, 17, 10], and (more recently) in the classical $(1, k)$-separation problem of Tverberg [16, 12].

The driving goal of the internship is to check the following conjecture.

Conjecture 1. *An arrangement of five (hence any number of) DP-ribbons is of genus 1 if and only if its subarrangements of size 3 and 4 are of genus 1.*

We ask both for a non computer-assisted proof and a computer-assisted proof. So far we only know that an arrangement of five DP-ribbons whose subarrangements of size 4 are of genus 1 is of genus 1 or its subarrangements of size 4 belong to a well-defined family of few dozens of arrangements [9, Theorem 46]. A computer-assisted proof is therefore doable using modest computing ressources. Using classical enumeration algorithms for multiset permutations [18, 11] among other things, preliminary investigations[1] lead to the following values for the numbers $a_4^*(g)$ of arrangements of DP-ribbons of size 4 and genus g whose subarrangements of size 3 are of genus 1 and the numbers $b_4^*(g)$ of indexed arrangements of oriented DP-ribbons of size 4 and genus g whose subarrangements of size 3 are of genus 1:

g	1	2	3	4	5	6	7	≥ 8
$a_4^*(g)$	6 570	0	455	0	18	0	1	0
$b_4^*(g)$	2 415 112	0	135 664	0	4 560	0	16	0
$\lceil b_4^*(g)/2^4 4! \rceil$	6290	0	354	0	12	0	1	0

The code for the computer-assisted proof will aggregate the general-purpose platform to manipulate DP-ribbons of genus 1 developped in [6] and could potentially be reused to check conjectures related to the lines of research cited above (line transversal theory and so forth). Depending on time and the expectations of the intern, a multi-dimensional version of arrangements of double pseudolines (modeled on the notion of pseudohyperplane arrangements [2, 3, 4]) could be investigated.

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[1]Supported by the TEOMATRO grant ANR-10-BLAN 0207 through the founding of a postdoc position during the academic year 2012-2013.

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