Combinatoire des polytopes
Examen du 21/02/2020

Les notes de cours, les TDs (et leurs corrections), et vos notes personnelles sont autorisées. Les appareils électroniques sont interdits (en particulier les téléphones portables). Il est demandé de répondre sur des feuilles simples.

Les exercices de cet énoncé sont indépendants et peuvent être traités dans n'importe quel ordre. Attention à bien noter les numéros d'exercice et de questions devant vos réponses.

La précision des réponses, la qualité de la rédaction, et les efforts de présentation seront pris en compte dans la notation.

Exercice 1 (p-sequences)**.** For a polytope P, let $p_k(P)$ be its number of k-gonal 2-faces for each $k \geq 3$.

(1) Show that for a simple 3-polytope P, we have

$$\sum_{k\geq 3}(6-k)\cdot p_k(P) = 12.$$

(2) Show that every simple 3-polytope contains at least four faces each of which has at most five edges.

(3) Let $\mathcal{C} \subset \mathbb{R}^3$ be the convex hull of the set of points $(p_3(P), p_4(P), p_5(P))$ for all simple 3-polytopes P. Show that $\mathcal{C}$ is a polyhedron and give its descriptions as intersection of halfspaces, and as polytope and recession cone.

Solution.

(1) From Euler's equation we have $f_0 - f_1 + f_2 = 2$. Moreover, $f_2 = \sum_{k\geq 3} p_k$ and $2\,f_1 = \sum_{k\geq 3} k\,p_k$. And, by the simplicity of P, we also have $3\,f_0 = 2\,f_1$. The combination of these equations gives the desired result.

(2) From the formula, we have that $3(p_3+p_4+p_5) \geq 3\,p_3 + 2\,p_4 + p_5 = 12 + \sum_{k\geq 7}(k-6)p_k \geq 12$. We recover that $p_3+p_4+p_5 \geq 4$.

(3) Refining the previous answer, we know that $3\,p_3 + 2\,p_4 + p_5 \geq 12$ and that $p_i \geq 0$. We thus know that $\mathcal{C}$ is contained in the polyhedron

$$\mathcal{P} = \{(p_3,p_4,p_5) \mid 3\,p_3 + 2\,p_4 + p_5 \geq 12,\ p_3 \geq 0,\ p_4 \geq 0 \text{ and } p_5 \geq 0\}\,.$$

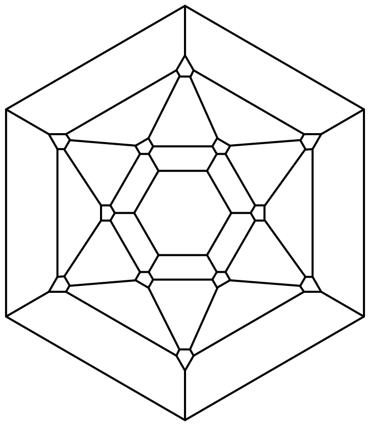
This is an unbounded polyhedron with 4 facets and 3 vertices: $(4,0,0)$, $(0,6,0)$, and $(0,0,12)$. It has 3 unbounded rays, in the directions $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Hence

$$\mathcal{P} = \mathrm{conv}(\{(4,0,0),(0,6,0),(0,0,12)\}) + \mathrm{cone}(\{(1,0,0),(0,1,0),(0,0,1)\}).$$

We will show that $\mathcal{C} = \mathcal{P}$.

- To show that $\mathrm{conv}(\{(4,0,0),(0,6,0),(0,0,12)\}) \subset \mathcal{C}$ we observe that the vertices correspond to the tetrahedron, the cube and the dodecahedron.
- To show that $\mathrm{cone}(\{(1,0,0),(0,1,0),(0,0,1)\})$ is in the recession cone of $\mathcal{C}$, it suffices to show that there are arbitrarily large numbers N such that $(N,0,0)$, $(0,N,0)$, and $(0,0,N)$ belong to $\mathcal{C}$. Taking a prism over an n-gon, $n \geq 6$, we obtain a simple 3-polytope with $2n$ vertices and two n-gons and n squares, and hence the vector $(0,n,0) \in \mathcal{C}$ for $n \geq 6$. Truncating its $2n$ vertices, we recover a simple 3 polytope with $6n$ vertices and $2n$-gons, $2n$ triangles and n 8-gons. Therefore, $(2n,0,0) \in \mathcal{C}$ for $n \geq 6$. Now, take a twisted prism over an n-gon, which

has two n-gons and $2n$ triangles. Make a connected sum with a prism over an n-gon on both n-gonal faces. We get a 3-polytope with $2n$ vertices of degree 3 and $2n$ vertices of degree 5. Its has $2n$ triangular faces, $2n$ square faces, and two n-gons. Now truncate the $2n$ vertices of degree 5. The squares and triangles become hexagons, and we add $2n$ pentagons. The resulting polytope is a simple 3-polytope with $12n$ vertices, two n-gons, $4n$ hexagons and $2n$ pentagons (see the drawing for $n = 6$. Hence $(2n, 0, 0) \in \mathcal{C}$ for $n \geq 6$.



Exercice 2 (Permutahedron)**.** For $n \geq 1$, the *permutahedron* $\mathsf{Perm}(n)$ is defined as the convex hull of the points $(\sigma(1), \dots, \sigma(n))$ for all permutations $\sigma \in \mathfrak{S}_n$.

(1) Draw the permutahedra $\mathsf{Perm}(1)$, $\mathsf{Perm}(2)$ and $\mathsf{Perm}(3)$.

(2) What is the intrinsic dimension of $\mathsf{Perm}(n)$? Justify.

(3) What is the number of vertices of $\mathsf{Perm}(n)$? Justify.

(4) For $\varnothing \neq I \subsetneq [n]$, show that the inequality $\sum_{i \in I} x_i \geq |I|(|I|+1)/2$ defines a facet F_I of $\mathsf{Perm}(n)$ whose combinatorial type is that of the Cartesian product $\mathsf{Perm}(|I|) \times \mathsf{Perm}(n - |I|)$.

An *ordered partition* of $[n]$ is a partition $[n] = I_1 \sqcup \dots \sqcup I_k$ where the parts are ordered (but the order among the elements inside each part is irrelevant). We write such a partition as $I_1|I_2|\dots|I_k$. For instance, the ordered partitions $12|35|4$ and $4|12|35$ are distinct since they have the same parts but in different order, while the ordered partitions $12|35|4$ and $21|53|4$ are the same.

(5) Show that, for an ordered partition $\pi = I_1|I_2|\dots|I_k$, the intersection of the facets F_{I_1}, $F_{I_1 \cup I_2}$, $\dots$, $F_{I_1 \cup \dots \cup I_{k-1}}$ defines a $(n-k)$-dimensional face F_π of $\mathsf{Perm}(n)$. Describe the combinatorics of F_π.

(6) Conversely, given a non-zero vector $c = (c_1, \dots, c_n)$, describe (in terms of the coordinates of c) the ordered partition π such that F_π is the face of $\mathsf{Perm}(n)$ minimizing c.

(7) Describe the face lattice of $\mathsf{Perm}(n)$.

(8) Let $k, k_1, \dots, k_p$ and $n, n_1, \dots, n_p$ be integers such that $k = k_1 + \dots + k_p$ and $n = k_1 n_1 + \dots + k_p n_p$. What is the number of faces of $\mathsf{Perm}(n)$ with combinatorial type $\mathsf{Perm}(n_1)^{k_1} \times \dots \times \mathsf{Perm}(n_p)^{k_p}$?

Solution.

(1) Classical pictures. See Ziegler.

(2) The dimension of $\mathsf{Perm}(n)$ is $n - 1$. Indeed, all permutations are contained in the affine hyperplane $\sum_{i \in [n]} x_i = n(n+1)/2$ and the identity permutation together with the $n-1$ simple transpositions $(i\ i+1)$ give n affinely independent points of $\mathsf{Perm}(n)$.

(3) By symmetry, all permutations appear on the convex hull (one can also argue that they are all in the affine hyperplane $\sum_{i \in [n]} x_i = n(n+1)/2$ and at the same distance from the point $((n+1)/2, \dots, (n+1)/2))$. Therefore the number of vertices of $\mathsf{Perm}(n)$ is $n!$.

(4) The inequality is clearly satisfied (since the entries at the positions given by I are at least the $|I|$ first integers), with equality if and only if $\sigma^{-1}(I) = [|I|]$. In other words, all permutations in this face are obtained from any one by permuting its coordinates in I and independently its coordinates in $[n] \smallsetminus I$. This shows that this face is isomorphic to the Cartesian product $\mathsf{Perm}(|I|) \times \mathsf{Perm}(n - |I|)$, and is thus of dimension $|I| - 1 + n - |I| - 1 = n - 2$, so that it is a facet of $\mathsf{Perm}(n)$.

(5) The facets $F_{I_1}, \dots, F_{I_1 \cup \dots \cup I_{k-1}}$ define a face F_π. This face F_π contains precisely the permutations whose first $|I_1|$ elements are at the positions of I_1, the next $|I_2|$ elements are at the positions of I_2, etc. In other words, it has the combinatorics of the Cartesian product $\mathsf{Perm}(|I_1|) \times \cdots \times \mathsf{Perm}(|I_k|)$. In particular, it is $(n-k)$-dimensional.

(6) Let π be the partition of $[n]$ such that $c_i = c_j$ if i and j lie in the same part of π and $c_i < c_j$ if the part of π containing i is after the part of π containing j. Then c is minimized by F_π.

(7) According to the previous questions, the faces of $\mathsf{Perm}(n)$ are in bijection with the ordered partitions of $[n]$. The inclusion of faces translates to the refinement of ordered partitions (meaning glueing parts together). The face lattice of $\mathsf{Perm}(n)$ is thus isomorphic to the refinement poset of ordered partitions of $[n]$.

(8) To choose a face with combinatorial type $\mathsf{Perm}(n_1)^{k_1} \times \cdots \times \mathsf{Perm}(n_p)^{k_p}$, we have to choose an ordered partition of $[n]$ with k_i parts of size n_i for all $i \in [p]$. We first choose where we place the bars (we have $k!/\prod_{i=1}^p k_i!$ choices), then what numbers we put in the parts (we have $n!/\prod_{i=1}^p n_i!^{k_i}$ choices). This gives

$$\frac{k!\, n!}{\prod_{i=1}^p (k_i!\, n_i!^{k_i})}$$

Exercice 3 (Minkowski summands)**.** The Minkowski sum $P + Q$ of two polytopes $P, Q \in \mathbb{R}^d$ is the set $P + Q = \{p + q \mid p \in P, q \in Q\}$. We say that Q is a *Minkowski summand* of P (written $Q \preceq P$) if there is a polytope R such that $P = Q + R$.

(1) Characterize the condition $Q \preceq P$ when Q and P are 1-dimensional.

(2) Prove that if $P \preceq Q$ and $Q \preceq P$, then $P = Q + t$ for some $t \in \mathbb{R}^d$.

(3) For $u \in \mathbb{R}^d \smallsetminus 0$, let P^u be the face of P maximized in direction u. Show that if $Q \preceq P$ then $Q^u \preceq P^u$.

(4) Characterize the Minkowski summands of a polygon $P \subset \mathbb{R}^2$. To this end, we label its vertices by $p_1, \dots, p_n$ clockwise and we consider its edge directions $v_i = p_i - p_{i-1}$ for $1 \le i \le n$ (with the convention $p_0 = p_n$).

- Prove that any polygon with the exact same edge directions must be a translate of P.
- Characterize the values $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n_{\ge 0}$ such that there is a polygon $Q \subset \mathbb{R}^2$ with edge directions $\lambda_i \cdot v_i$ (we set $\lambda_i = 0$ if no multiple of v_i appears as an edge direction of Q).
- Show that if $Q \preceq P$, then its edge directions are of the form $\lambda_i \cdot v_i$ for some $0 \le \lambda_i \le 1$.
- Show that $Q \preceq P$ if and only if its edge directions are of the form $\lambda_i \cdot v_i$ for some $0 \le \lambda_i \le 1$.

(5) Prove that $Q \preceq P$ if and only if

  (i) $\dim Q^u \le \dim P^u$ for all $u \in \mathbb{R}^d \smallsetminus 0$, and

  (ii) $Q^u \preceq P^u$ whenever $\dim P^u = 1$.

To prove the only if part

- Construct a map $p_i \mapsto q_i$ that associates a vertex $q_i \in Q$ to every vertex $p_i \in P$.
- Define $R = \operatorname{conv}\{r_i = p_i - q_i\}$.
- Show that $P = Q + R$ (by contradiction).

(6) For $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, let

$$\mathsf{Perm}(a) = \operatorname{conv}\left\{(a_{\sigma(1)}, \dots, a_{\sigma(n)}) \;\middle|\; \sigma \in \mathfrak{S}_n\right\} \subset \mathbb{R}^n.$$

Show that there is a $\lambda > 0$ such that $\lambda \cdot \mathsf{Perm}(a) \preceq \mathsf{Perm}(n)$, where $\mathsf{Perm}(n)$ is the permutahedron defined in the previous exercice.

Solution.

(1) If $P = [a, b]$, and $Q = [c, d]$, then $Q \preceq P$ if and only if $|d - c| \leq |b - a|$.

(2) If $P \preceq Q$ and $Q \preceq P$, then there are polytopes R, T such that $Q = P + R$ and $P = Q + T$. We conclude that $P = P + (R + T)$. This implies that $R + T = \{0\}$. Since the Minkowski sum does not reduce the dimension, we conclude that R and T are points in $\mathbb{R}^d$. That is, there is a $t \in \mathbb{R}^d$ such that $P = Q + \{t\}$.

(3) It follows from the fact that $(Q + R)^u = Q^u + R^u$.

(4)
- The vertices of P are of the form $p_k = p_0 + \sum_{1 \leq i \leq k} v_i$. Any other polygon with the same edge directions will have vertices $p'_k = p'_0 + \sum_{1 \leq i \leq k} v_i$, and hence will be a translate of P by $p'_0 - p_0$.
- We need $\sum_{1 \leq i \leq n} \lambda_i \cdot v_i = 0$. This condition is needed as this is the result of doing the full turn around the boundary of Q. It is sufficient, as the convex hull of the points $q_k = \sum_{1 \leq i \leq k} \lambda_i \cdot v_i$ gives the desired polygon.
- This follows from $Q^u \preceq P^u$ combined with the fact that the length characterizes Minkowski summands in $\mathbb{R}^1$.
- We need to provide R such that $Q + R = P$. We will take a polygon R with edge directions $(1 - \lambda_i)v_i$. Since $(Q + R)^u = Q^u + R^u$, the edge directions of $Q + R$ are $\lambda_i v_i + (1 - \lambda_i)v_i = v_i$. Hence, there is some $t \in \mathbb{R}^2$ such that $Q + R = P + t$. Setting $R' = R - t$ we get $Q + R' = P$.

(5) The "if" direction follows directly from the previous observation. For the only if, we follow the suggestion. To a vertex $p_i = P^u$ we associate $q_i = Q^u$.

We will show that $P = Q + R$. We know that $P \subseteq Q + R$. Assume that $P \neq Q + R$. That is, that there is some vertex x of $Q + R$ that does not belong to P. Then there is a separating hyperplane. That is, some $c \in \mathbb{R}^d$ such that if $\max_{p \in P} \langle\, c \mid p \,\rangle < \langle\, c \mid x \,\rangle$. Let $p_i \in P^c$. Then $x = q_i + r_j$ for some j. We have $\langle\, c \mid p_i \,\rangle < \langle\, c \mid q_i + r_j \,\rangle$. Equivalently, $\langle\, c \mid p_i - pj \,\rangle < \langle\, c \mid q_i - q_j \,\rangle$. Now, let $p_j = p_0, p_1, \dots, p_s = p_i$ be a path in the graph of P from p_j to p_i. Note that for $1 \leq k \leq s$ there is some $0 \leq \lambda_k \leq 1$ such that $q_k - q_{k-1} = \lambda_k(p_k - p_{k-1})$ by the second condition.

Then,

$$\langle\, c \mid q_i - q_j \,\rangle = \sum_{1 \leq k \leq s} \langle\, c \mid q_k - q_{k-1} \,\rangle = \sum_{1 \leq k \leq s} \lambda_k \langle\, c \mid p_k - p_{k-1} \,\rangle \leq \sum_{1 \leq k \leq s} \langle\, c \mid p_k - p_{k-1} \,\rangle = \langle\, c \mid p_i - p_j \,\rangle.$$

A contradiction.

(6) By construction, $\mathsf{Perm}(a)$ is of dimension at most $n - 1$, as the sum of coordinates of the vertices is constant. Moreover, see that $\sum_{i \in I} x_i \geq \min_{\sigma \in \mathfrak{S}_n}(\sum_{i \in I} x_{\sigma(i)})$ defines a face of G_I of $\mathsf{Perm}(a)$, which is isomorphic to $\mathsf{Perm}(a_I) \times \mathsf{Perm}(a_{[n] \setminus I})$, and hence must be of dimension at most $n - 2$. Furthermore, if $\mathsf{Perm}(n)^c$ is associated to the ordened partition $\pi = I_1 | I_2 | \dots | I_k$, then $\mathsf{Perm}(a)^c$ is isomorphic to $\mathsf{Perm}(a_{I_1}) \times \cdots \times \mathsf{Perm}(a_{I_k})$, and is hence of dimension at most $n - k$. Condition (i) is therefore fulfilled for every direction. And condition (ii) is automatically true for some $\lambda > 0$ small enough.

Exercice 4 (One-point suspensions). Let $V = \left(\binom{p_1}{1}, \dots, \binom{p_n}{1}\right) \in \mathbb{R}^{(d+1) \times n}$ be a vector configuration arising as the homogenization of the n vertices of a d-polytope P. Let $G = (g_1, \dots, g_n) \in \mathbb{R}^{(n-d-1) \times n}$ be its Gale dual vector configuration.

(1) Let $G' = (g'_0, g'_1, \dots, g'_n)$ be the vector configuration with $g'_0 = \frac{g_1}{2}$, $g'_1 = \frac{g_1}{2}$ and $g'_i = g_i$ for $2 \le i \le n$. Explain why G' is the Gale dual of (the vector configuration arising as the homogenization of the vertices of) a polytope P'. What is the dimension of P'?

(2) Describe the faces of P' (with respect to those of P).

(3) Describe the geometric operation that sends P to P'. It is called the *one-point suspension* of p_1 in P.

(4) Does every polytope combinatorially equivalent to P' arise from a one-point suspension of a polytope combinatorially equivalent to P?

(5) Show that P' has a vertex figure combinatorially equivalent to P.

(6) Argue why the realization space of P' is stably equivalent to the realization space of P. (Give only the main arguments, without writing a full formal proof.)

Solution.

(1) The condition is that $\sum_{g\in G'} g' = 0$. Since $\sum_{g\in G'} g' = \sum_{g\in G} g$ the result follows. The dimension is $(n+1) - (n-d-1) - 1 = d+1$.

(2) Let $(p'_0, p'_1, \dots, p'_n)$ be the vertices of P'. $\mathrm{conv}\{p'_i : i \in I\}$ forms a face of P' if and only if $\{g'_i : i \notin I\}$ form a positive dependence. Note that the positive dependences of G' are recovered by those of G by replacing each instance of g_1 by $g'_0 + g'_1$, $2g'_0$ or $2g'_1$. We conclude that for each face $\mathrm{conv}\{p_i : i \in I\}$ of P not containing p_1 ($1 \notin I$), the sets $\mathrm{conv}\{p'_i : i \in I\}$, $\mathrm{conv}\{p'_i : i \in I \cup \{0\}\}$ and $\mathrm{conv}\{p'_i : i \in I \cup \{1\}\}$ are faces of P'; and that for each face $\mathrm{conv}\{p_i : i \in I\}$ of P containing p_1 ($1 \in I$), the set $\mathrm{conv}\{p'_i : i \in I \cup \{0, 1\}\}$ is a face of P', and these are all faces of P'.

(3) One way to realize it is to embed P in the x_{d+1} plane of $\mathbb{R}^{d+1}$ and replace p_1 by $p_1 + e_{d+1}$ and $p_1 - e_{d+1}$.

(4) Not necessarily. If P is a pentagon, and P' is its one point suspension, then we recover a simplicial polytope with four coplanar points. There are realizations of this polytope without the coplanarity, by perturbation.

(5) The operation of vertex figure is dual to removing a vector in the Gale dual, and we recover G.

(6) Taking the vertex figure gives a projection from the realization space of P' to that of P. This projection of realization spaces can be explicitly constructed by selecting an affine basis of P and projecting P' radially from p'_0 to the hyperplane spanned by its neighbors correspnding to the affine basis. The fibers of this map are non-empty because we can do the previous operation $p_1 \pm e_{d+1}$ on any realization of P. To see that they are polynomially defined polyhedra, note that every vertex p'_i must be on the ray emanating from p'_0 through p_i. With an affine transformation, we can assume that p'_0 lies at the origin of $\mathbb{R}^{d+1}$, and we can parametrize the vertices of P' by the $\lambda_i > 0$ such that $p'_i = \lambda_i p_i$. The face lattice constraints can be written with determinantal inequalities which are linear on the (inverses of the) λ_i's and polynomial on the coordinates of the p_i's. Indeed, they is given by inequalities of the type

$$\begin{vmatrix} \lambda_{i_1} p_{i_1} & \cdots & \lambda_{i_{d+1}} p_{i_{d+1}} \\ 1 & \cdots & 1 \end{vmatrix} > 0,$$

which are equivalent to

$$\begin{vmatrix} p_{i_1} & \cdots & p_{i_{d+1}} \\ \frac{1}{\lambda_{i_1}} & \cdots & \frac{1}{\lambda_{i_{d+1}}} \end{vmatrix} > 0.$$