Combinatoire des polytopes
Examen du 21/02/2020

Les notes de cours, les TDs (et leurs corrections), et vos notes personnelles sont autorisées. Les appareils électroniques sont interdits (en particulier les téléphones portables). Il est demandé de répondre sur des feuilles simples.

Les exercices de cet énoncé sont indépendants et peuvent être traités dans n'importe quel ordre. Attention à bien noter les numéros d'exercice et de questions devant vos réponses.

La précision des réponses, la qualité de la rédaction, et les efforts de présentation seront pris en compte dans la notation.

Exercice 1 (p-sequences)**.** For a polytope P, let $p_k(P)$ be its number of k-gonal 2-faces for each $k \geq 3$.

(1) Show that for a simple 3-polytope P, we have

$$\sum_{k \geq 3} (6-k) \cdot p_k(P) = 12.$$

(2) Show that every simple 3-polytope contains at least four faces each of which has at most five edges.

(3) Let $\mathcal{C} \subset \mathbb{R}^3$ be the convex hull of the set of points $(p_3(P), p_4(P), p_5(P))$ for all simple 3-polytopes P. Show that $\mathcal{C}$ is a polyhedron and give its descriptions as intersection of halfspaces, and as polytope and recession cone.

Exercice 2 (Permutahedron)**.** For $n \geq 1$, the *permutahedron* $\mathsf{Perm}(n)$ is defined as the convex hull of the points $(\sigma(1), \dots, \sigma(n))$ for all permutations $\sigma \in \mathfrak{S}_n$.

(1) Draw the permutahedra $\mathsf{Perm}(1)$, $\mathsf{Perm}(2)$ and $\mathsf{Perm}(3)$.

(2) What is the intrinsic dimension of $\mathsf{Perm}(n)$? Justify.

(3) What is the number of vertices of $\mathsf{Perm}(n)$? Justify.

(4) For $\varnothing \neq I \subsetneq [n]$, show that the inequality $\sum_{i \in I} x_i \geq |I|(|I|+1)/2$ defines a facet F_I of $\mathsf{Perm}(n)$ whose combinatorial type is that of the Cartesian product $\mathsf{Perm}(|I|) \times \mathsf{Perm}(n-|I|)$.

An *ordered partition* of $[n]$ is a partition $[n] = I_1 \sqcup \dots \sqcup I_k$ where the parts are ordered (but the order among the elements inside each part is irrelevant). We write such a partition as $I_1|I_2|\dots|I_k$. For instance, the ordered partitions $12|35|4$ and $4|12|35$ are distinct since they have the same parts but in different order, while the ordered partitions $12|35|4$ and $21|53|4$ are the same.

(5) Show that, for an ordered partition $\pi = I_1|I_2|\dots|I_k$, the intersection of the facets F_{I_1}, $F_{I_1 \cup I_2}$, $\dots$, $F_{I_1 \cup \dots \cup I_{k-1}}$ defines a $(n-k)$-dimensional face F_π of $\mathsf{Perm}(n)$. Describe the combinatorics of F_π.

(6) Conversely, given a non-zero vector $c = (c_1, \dots, c_n)$, describe (in terms of the coordinates of c) the ordered partition π such that F_π is the face of $\mathsf{Perm}(n)$ minimizing c.

(7) Describe the face lattice of $\mathsf{Perm}(n)$.

(8) Let $k, k_1, \dots, k_p$ and $n, n_1, \dots, n_p$ be integers such that $k = k_1 + \dots + k_p$ and $n = k_1 n_1 + \dots + k_p n_p$. What is the number of faces of $\mathsf{Perm}(n)$ with combinatorial type $\mathsf{Perm}(n_1)^{k_1} \times \dots \times \mathsf{Perm}(n_p)^{k_p}$?

Exercice 3 (Minkowski summands)**.** The Minkowski sum $P + Q$ of two polytopes $P, Q \in \mathbb{R}^d$ is the set $P + Q = \{p + q \mid p \in P, q \in Q\}$. We say that Q is a *Minkowski summand* of P (written $Q \preceq P$) if there is a polytope R such that $P = Q + R$.

(1) Characterize the condition $Q \preceq P$ when Q and P are 1-dimensional.

(2) Prove that if $P \preceq Q$ and $Q \preceq P$, then $P = Q + t$ for some $t \in \mathbb{R}^d$.

(3) For $u \in \mathbb{R}^d \smallsetminus 0$, let P^u be the face of P maximized in direction u. Show that if $Q \preceq P$ then $Q^u \preceq P^u$.

(4) Characterize the Minkowski summands of a polygon $P \subset \mathbb{R}^2$. To this end, we label its vertices by $p_1, \dots, p_n$ clockwise and we consider its edge directions $v_i = p_i - p_{i-1}$ for $1 \le i \le n$ (with the convention $p_0 = p_n$).

- Prove that any polygon with the exact same edge directions must be a translate of P.
- Characterize the values $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n_{\ge 0}$ such that there is a polygon $Q \subset \mathbb{R}^2$ with edge directions $\lambda_i \cdot v_i$ (we set $\lambda_i = 0$ if no multiple of v_i appears as an edge direction of Q).
- Show that if $Q \preceq P$, then its edge directions are of the form $\lambda_i \cdot v_i$ for some $0 \le \lambda_i \le 1$.
- Show that $Q \preceq P$ if and only if its edge directions are of the form $\lambda_i \cdot v_i$ for some $0 \le \lambda_i \le 1$.

(5) Prove that $Q \preceq P$ if and only if

(i) $\dim Q^u \le \dim P^u$ for all $u \in \mathbb{R}^d \smallsetminus 0$, and

(ii) $Q^u \preceq P^u$ whenever $\dim P^u = 1$.

To prove the only if part

- Construct a map $p_i \mapsto q_i$ that associates a vertex $q_i \in Q$ to every vertex $p_i \in P$.
- Define $R = \operatorname{conv}\{r_i = p_i - q_i\}$.
- Show that $P = Q + R$ (by contradiction).

(6) For $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, let

$$\mathsf{Perm}(a) = \operatorname{conv}\left\{(a_{\sigma(1)}, \dots, a_{\sigma(n)}) \;\middle|\; \sigma \in \mathfrak{S}_n\right\} \subset \mathbb{R}^n.$$

Show that there is a $\lambda > 0$ such that $\lambda \cdot \mathsf{Perm}(a) \preceq \mathsf{Perm}(n)$, where $\mathsf{Perm}(n)$ is the permutahedron defined in the previous exercice.

Exercice 4 (One-point suspensions)**.** Let $V = (\binom{p_1}{1}, \dots, \binom{p_n}{1}) \in \mathbb{R}^{(d+1)\times n}$ be a vector configuration arising as the homogenization of the n vertices of a d-polytope P. Let $G = (g_1, \dots, g_n) \in \mathbb{R}^{(n-d-1)\times n}$ be its Gale dual vector configuration.

(1) Let $G' = (g'_0, g'_1, \dots, g'_n)$ be the vector configuration with $g'_0 = \frac{g_1}{2}$, $g'_1 = \frac{g_1}{2}$ and $g'_i = g_i$ for $2 \le i \le n$. Explain why G' is the Gale dual of (the vector configuration arising as the homogenization of the vertices of) a polytope P'. What is the dimension of P'?

(2) Describe the faces of P' (with respect to those of P).

(3) Describe the geometric operation that sends P to P'. It is called the *one-point suspension* of p_1 in P.

(4) Does every polytope combinatorially equivalent to P' arise from a one-point suspension of a polytope combinatorially equivalent to P?

(5) Show that P' has a vertex figure combinatorially equivalent to P.

(6) Argue why the realization space of P' is stably equivalent to the realization space of P. (Give only the main arguments, without writing a full formal proof.)