Combinatoire des polytopes
TD D – Oriented Matroids and zonotopes

1 Normal fans

Consider a polytope $P \in \mathbb{R}^d$. The *normal cone* of a face F of P is the cone

$$N(F) := \big\{ \overline{c} \in (\mathbb{R}^d)^* \;\big|\; \langle \overline{c} \mid \overline{x} \rangle \geq \langle \overline{c} \mid \overline{x}' \rangle \text{ for all } \overline{x} \in F \text{ and } \overline{x}' \in P \big\}$$

of all linear functionals that are maximized on a face containing F. The *normal fan* of P is the collection $\mathcal{F}(P) := \{N(F) \mid F \text{ face of } P\}$ of all normal cones of the faces of P.

Exercice 1 (Cartesian products and normal fans)**.** Describe the normal fan of the Cartesian product $P \times Q := \{(\overline{p}, \overline{q}) \mid \overline{p} \in P \text{ and } \overline{q} \in Q\}$ in terms of the normal fans of the polytopes P and Q.

Exercice 2 (Projections and normal fans)**.** Consider an *affine map* $\pi : \mathbb{R}^p \to \mathbb{R}^d$ defined by $\pi(\overline{x}) = A\overline{x} + \overline{b}$ for some $(d \times p)$-matrix A and vector $b \in \mathbb{R}^d$, and denote its dual map by $\pi^* : (\mathbb{R}^d)^* \to (\mathbb{R}^p)^*$. Let P be a polytope in $\mathbb{R}^p$ and $Q = \pi(P)$ be its image in $\mathbb{R}^d$ under the map π. Show that

(1) for any face F of Q, the preimage $\pi^{-1}(F) \cap P$ is a face of P (conversely, is the image of a face of P always a face of Q?),
(2) π^{-1} is an order preserving map from the face lattice of Q to the face lattice of P,
(3) a linear functional $\overline{c} \in (\mathbb{R}^d)^*$ defines F if and only if the linear functional $\pi^*(\overline{c}) \in (\mathbb{R}^p)^*$ defines $\pi^{-1}(F)$,
(4) the normal fan of Q is isomorphic via π^* to the section of the normal fan of P by the vector space $\pi^*\big((\mathbb{R}^d)^*\big)$.

Exercice 3 (Minkowski sum and normal fans)**.** Show that the normal fan of the *Minkowski sum* $P + Q := \{\overline{p} + \overline{q} \mid \overline{p} \in P \text{ and } \overline{q} \in Q\}$ is the common refinement of the normal fans of the two polytopes P and Q, meaning that the cones of $\mathcal{F}(P+Q)$ are the intersections of a cones of $\mathcal{F}(P)$ by cones of $\mathcal{F}(Q)$.

2 Zonotopes

Exercice 4 (Two equivalent definitions)**.** Let V be a $(d \times p)$-matrix with columns vectors $\overline{v}_1, \dots, \overline{v}_p \in \mathbb{R}^d$. Show that the following two polytopes coincide:

- the projection of the p-dimensional cube $\square_p$ by the affine map $\pi_V : \mathbb{R}^p \to \mathbb{R}^d$ defined by $\pi_V(\overline{x}) = V\overline{x}$,
- the Minkowski sum of the polytopes $[-\overline{v}_1, \overline{v}_1], \dots, [-\overline{v}_p, \overline{v}_p]$.

This polytope is the *zonotope* $Z(V)$.

Exercice 5 (Two examples)**.** Describe the zonotope $Z(V)$ and its faces in the following two situations:

- when $\overline{v}_1, \dots, \overline{v}_p$ are linearly independent,
- when $\overline{v}_1, \dots, \overline{v}_p$ leave in a plane.

Exercice 6 (Central symmetry and zonotopes)**.** A polytope P is *centrally symmetric* if $P - \overline{b} = -P + \overline{b}$ where $\overline{b}$ is the barycenter of P. Show that:

(1) a projection of a centrally symmetric polytope is centrally symmetric,
(2) any centrally symmetric polytope is the projection of a cross-polytope,
(3) the following conditions are equivalent for a polytope P:
  (i) P is a zonotope (projection of a cube),
  (ii) all faces of P are zonotopes,
  (iii) all 2-dimensional faces of P are zonotopes,
  (iv) all faces of P are centrally symmetric,
  (v) all 2-dimensional faces of P are centrally symmetric,
  (vi) any edge of P is a Minkowski summand of P (there exists a polytope P' such that $P = P' + e$).

Exercice 7 (Zonotopes and hyperplane arrangements)**.** Show that the normal fan of the zonotope $Z(V)$ is the fan defined by the arrangement $\mathcal{A}(V)$ of hyperplanes $H_i := \big\{\overline{x} \in (\mathbb{R}^d)^* \;\big|\; \langle\, \overline{x} \mid \overline{v}_i \,\rangle = 0\big\}$ for $i \in [p]$.

For the vector configuration $V = \{\overline{v}_1, \dots, \overline{v}_p\} \subseteq \mathbb{R}^d$, recall that the relative positions of its vectors (called its *oriented matroid*) can be recorded by several combinatorial collections, in particular:

- its *signed vectors* are the sign vectors of its linear dependences

$$\mathcal{V}(V) := \big\{\operatorname{sign}(\overline{d}) \;\big|\; \overline{d} \in \mathbb{R}^p \text{ such that } V\overline{d} = \overline{0}\big\} \subseteq \{+,-,0\}^p$$

- its *signed circuits* are its support minimal vectors,
- its *signed covectors* are the sign vectors of its linear evaluations

$$\mathcal{V}^*(V) := \{\operatorname{sign}(\overline{c}V) \mid \overline{c} \in (\mathbb{R}^n)^*\} \subseteq \{+,-,0\}^p.$$

- its *signed cocircuits* are its support minimal covectors,

Exercice 8 (Faces of the zonotope versus covectors of the matroid)**.** Show that there following three families are in bijection:

- the non-empty faces of the zonotope $Z(V)$,
- the faces of the hyperplane arrangement $\mathcal{A}(V)$,
- the sign covectors of the vector configuration V.

Deduce that the following three families are in bijection:

- the facets of the zonotope $Z(V)$,
- the rays of the hyperplane arrangement $\mathcal{A}(V)$,
- the sign cocircuits of the vector configuration V.

3 Oriented matroids from graphs

Exercice 9 (Graphical matroid)**.** Consider a directed graph $G = (V, E)$ and its *incidence configuration* $I(G) := \{\overline{e}_w - \overline{e}_v \mid (v,w) \in E\} \subset \mathbb{R}^V$. Describe the circuits and cocircuits of the vector configuration $I(G)$.

Exercice 10 (Graphical zonotope)**.** Consider a graph $G = (V, E)$ and its *graphical zonotope*

$$Z(G) := \sum_{(v,w) \in E} [\overline{e}_u, \overline{e}_v].$$

Describe its normal fan and its face structure.