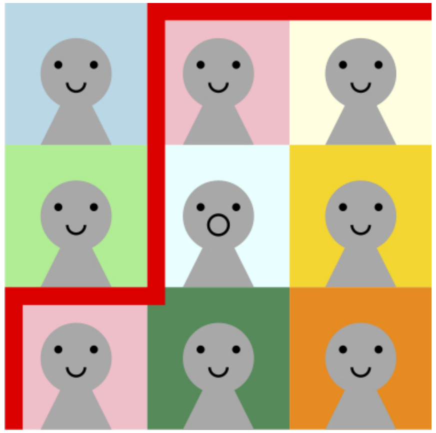
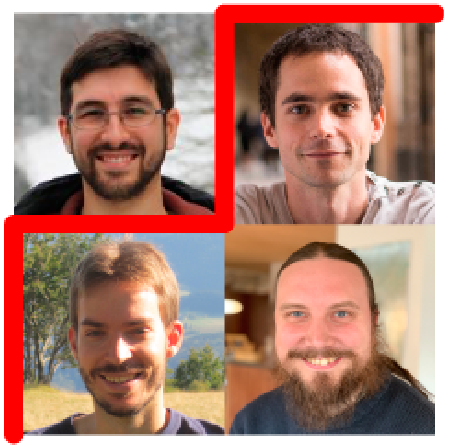
On type cones of g-vector fans





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V. PILAUD
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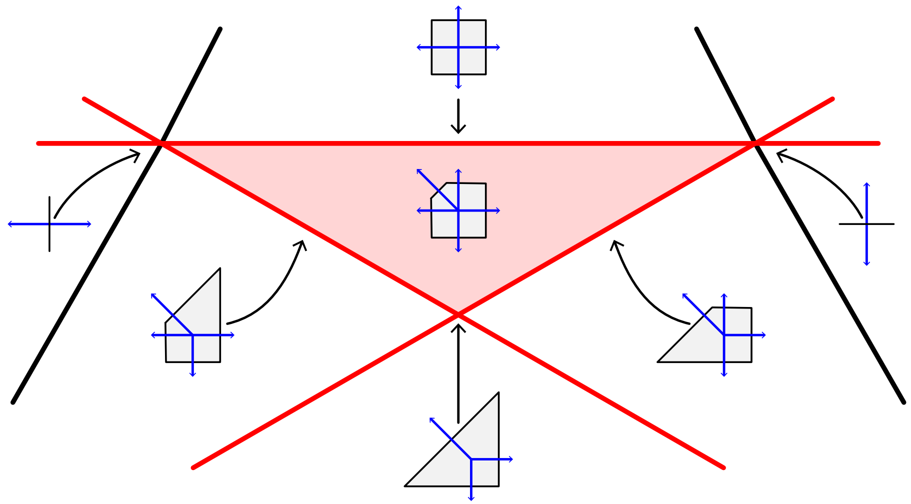
(Univ. Paris-Saclay → Univ. Versailles)



slides: http://www.lix.polytechnique.fr/~pilaud/FPSAC20.pdf

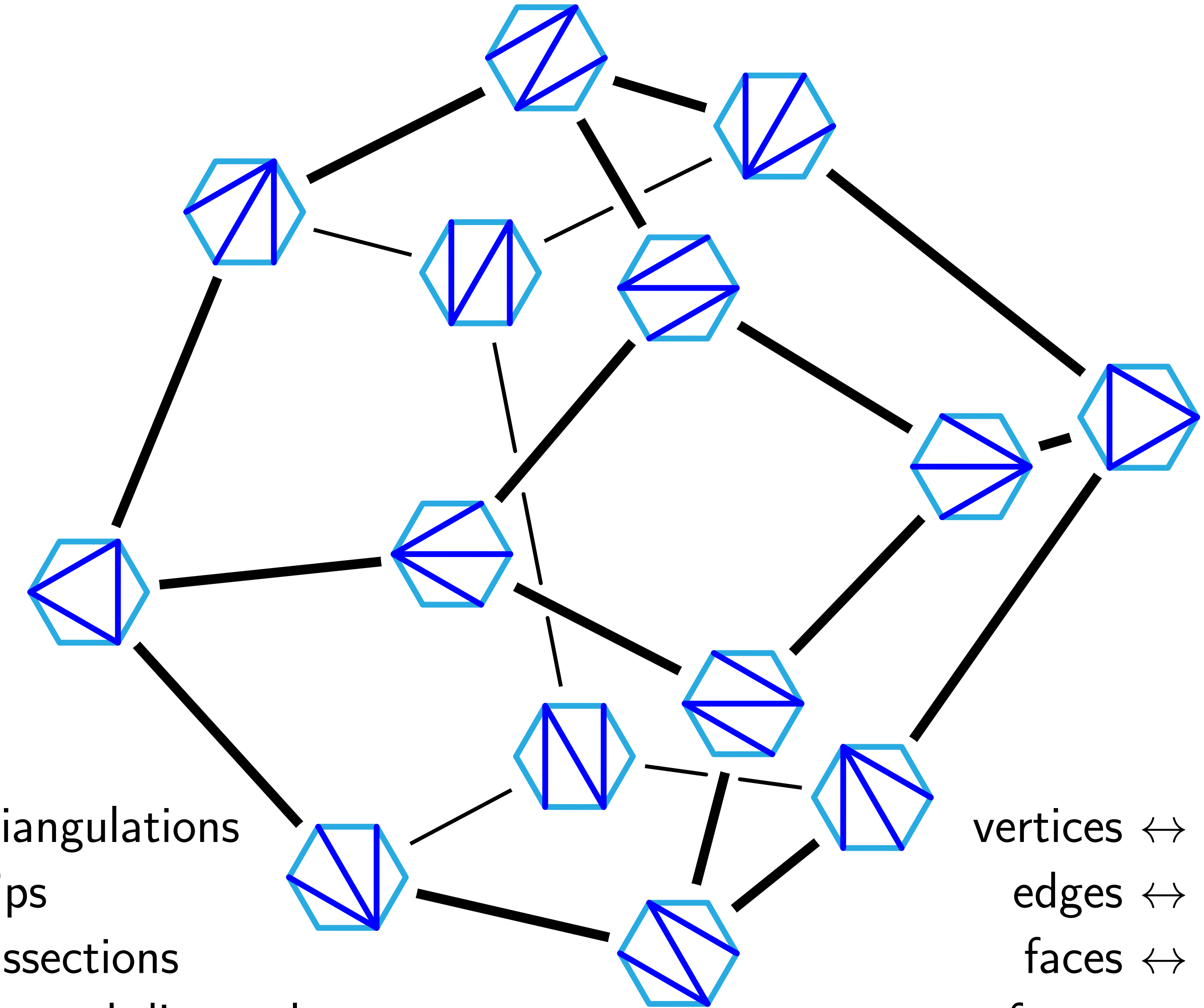
preprint: https://arxiv.org/pdf/1906.06861.pdf

This talk is being recorded

KINEMATIC ASSOCIAHEDRON

ASSOCIAHEDRON

associahedron = polytope whose graph is the flip graph on triangulations of a polygon



vertices ↔ triangulations
edges ↔ flips
faces ↔ dissections
facets ↔ internal diagonals

vertices ↔ binary trees
edges ↔ rotations
faces ↔ Schröder trees
facets ↔ corollas

Tamari ('51) Stasheff ('63)

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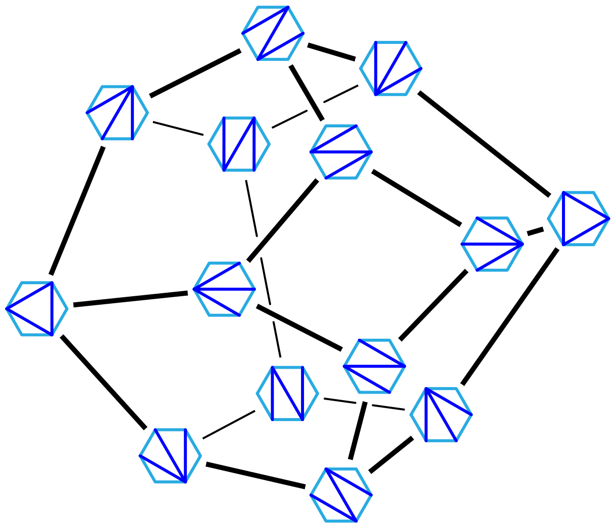
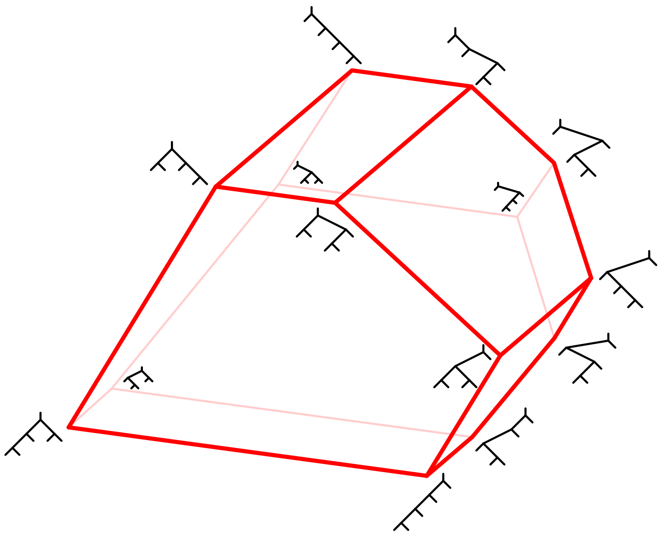
Three families of constructions (with non-equivalent normal fans):

SECONDARY FAN



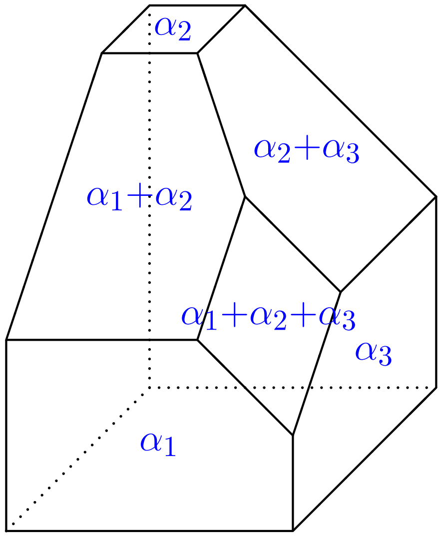
Gelfand–Kapranov–Zelevinsky ('94)
Billera–Filliman–Sturmfels ('90)

G-VECTOR FAN



Shnider–Sternberg ('93)
Loday ('04)
Hohlweg–Lange ('07)
Hohlweg–Lange–Thomas ('12)
Hohlweg–P.–Stella ('18)

D-VECTOR FAN



(Pictures by CFZ)

Chapoton–Fomin–Zelevinsky ('02)
Ceballos–Santos–Ziegler ('11)

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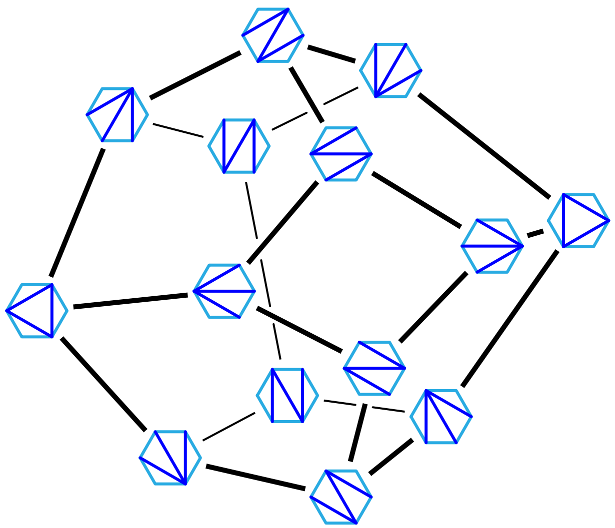

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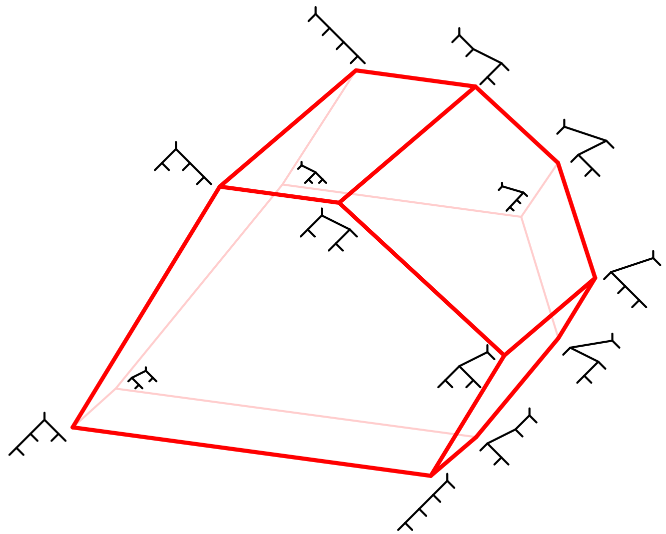

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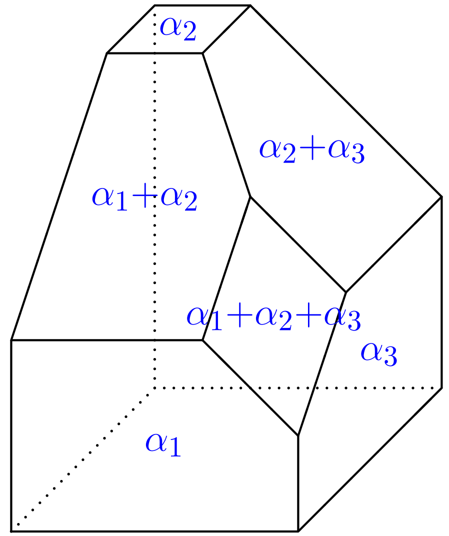

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Unidentified Construction of the Associahedron = KINEMATIC ASSOCIAHEDRON

Arkani-Hamed–Bai–He–Yan ('18)   Bazier-Matte–Douville–Mousavand–Thomas–Yıldırım ('18^{+})

KINEMATIC ASSOCIAHEDRON

kinematic associahedron = n-dimensional associahedron constructed in the $n(n+3)/2$-dimensional kinematic space as a section of the positive orthant with an n-dimensional affine subspace

Arkani-Hamed–Bai–He–Yan ('18)

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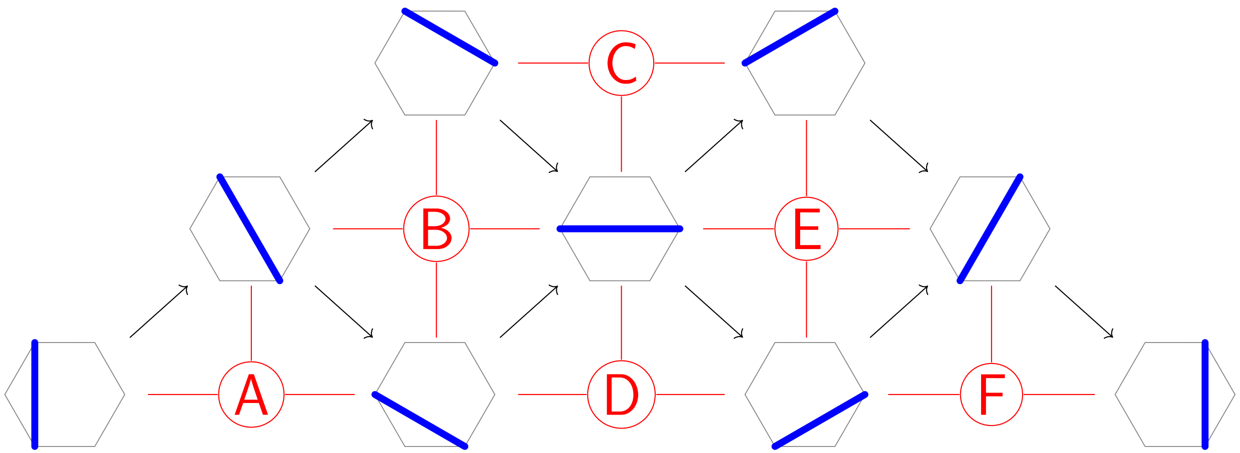
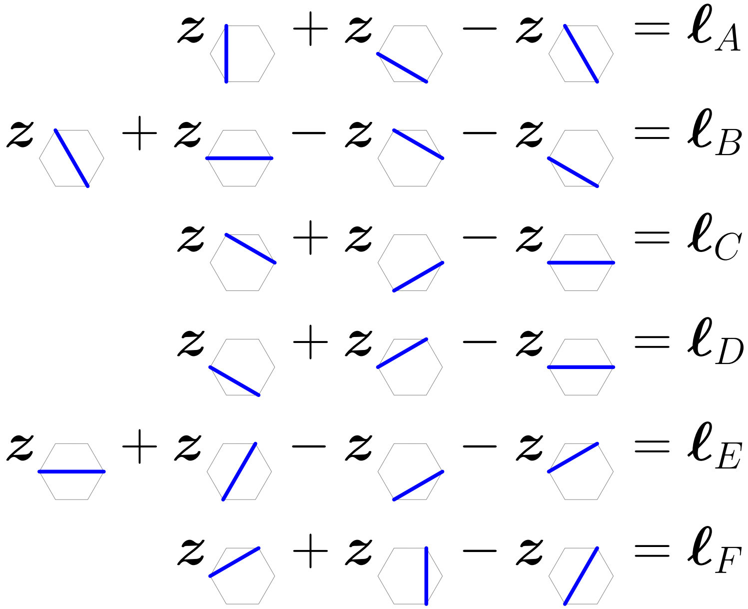
Arkani-Hamed–Bai–He–Yan ('18)

fix parameters $\boldsymbol{\ell}_A, \dots, \boldsymbol{\ell}_F > 0$

$\boldsymbol{z} \geq 0$ indexed by internal diagonals of $(n+3)$-gon





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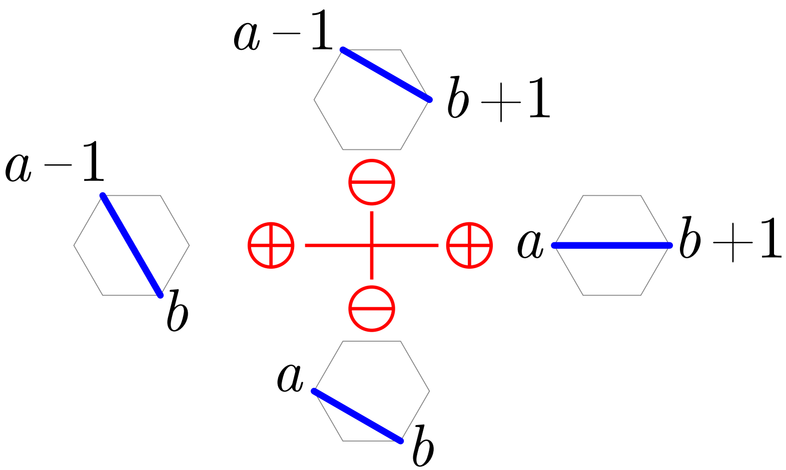
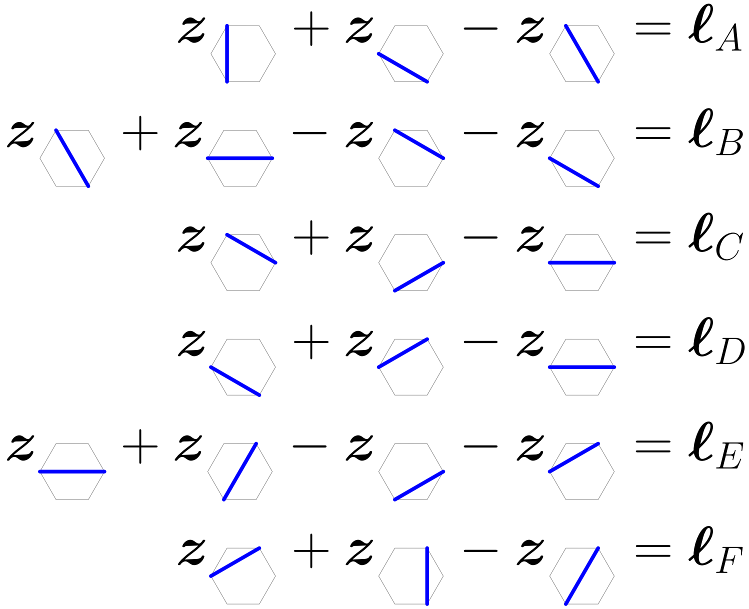




Let $X(n) = \{(a,b) \mid 0 \leq a < b \leq n+2\}$ and $Y(n) = \{(a,b) \mid 1 \leq a < b \leq n+1\}$.
For any $\boldsymbol{\ell} \in \mathbb{R}_{>0}^{Y(n)}$, the polytope

$$\left\{ \boldsymbol{z} \in \mathbb{R}^{X(n)} \;\middle|\; \begin{array}{l} \boldsymbol{z} \geq 0, \quad \boldsymbol{z}_{(0,n+2)} = 0 \quad \text{and} \quad \boldsymbol{z}_{(a,a+1)} = 0 \text{ for all } 0 \leq a \leq n+1 \\ \boldsymbol{z}_{(a-1,b)} + \boldsymbol{z}_{(a,b+1)} - \boldsymbol{z}_{(a,b)} - \boldsymbol{z}_{(a-1,b+1)} = \boldsymbol{\ell}_{(a,b)} \text{ for all } (a,b) \in Y(n) \end{array} \right\}$$

is an associahedron.

Arkani-Hamed–Bai–He–Yan ('18)

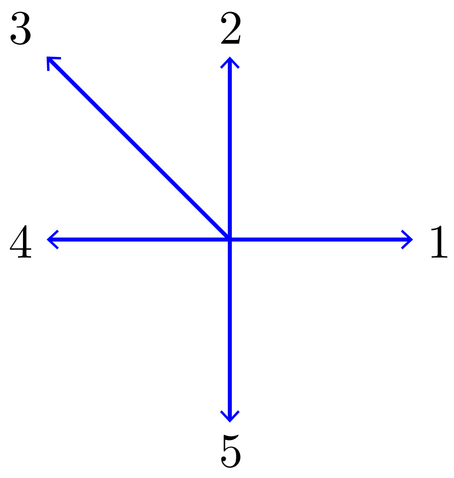
TYPE CONES

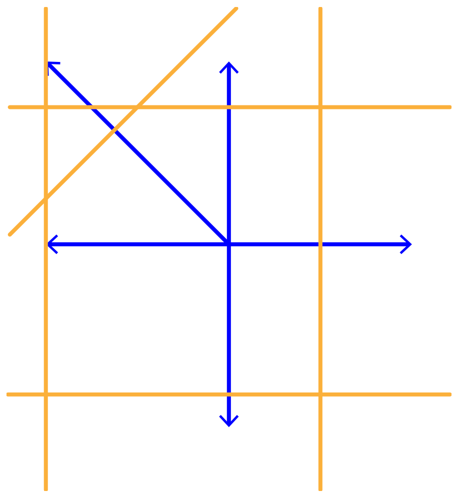
CHOOSING RIGHT-HAND-SIDES

$\mathcal{F}$ = complete simplicial fan in $\mathbb{R}^n$ with N rays

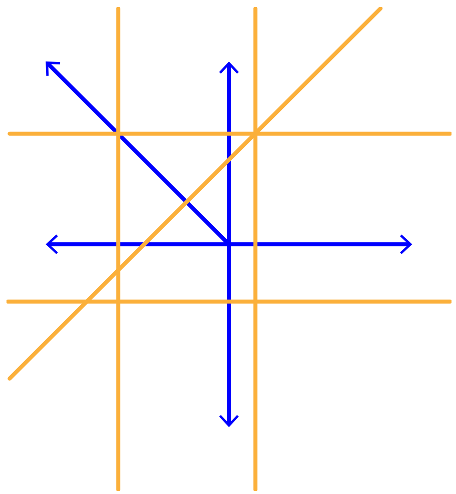
$\boldsymbol{G}$ = $(N \times n)$-matrix whose rows are representatives of the rays of $\mathcal{F}$

for a height vector $\boldsymbol{h} \in \mathbb{R}^N_{>0}$, consider the polytope $\mathbb{P}_{\boldsymbol{h}} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h}\}$

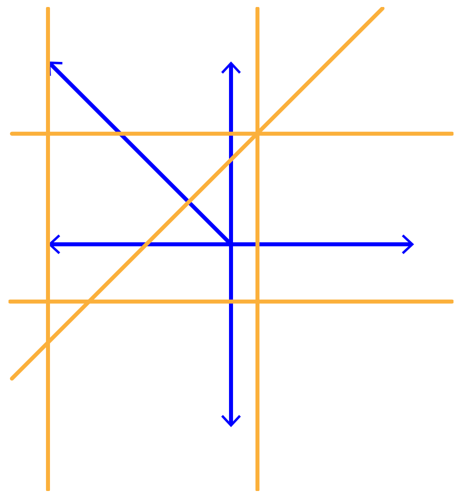




A



B



C

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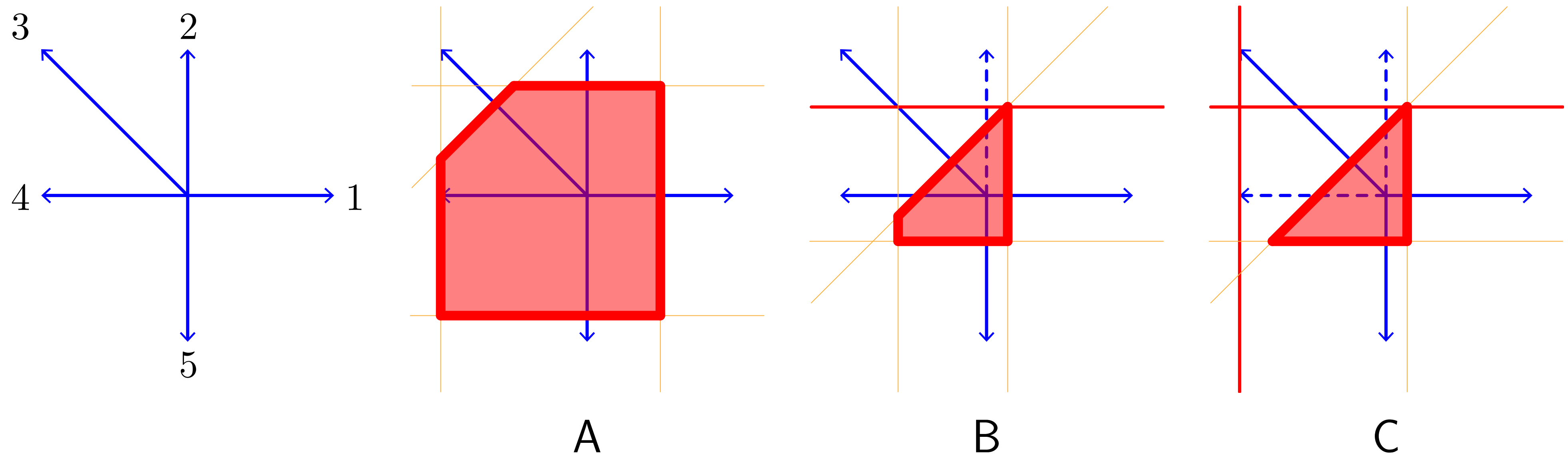

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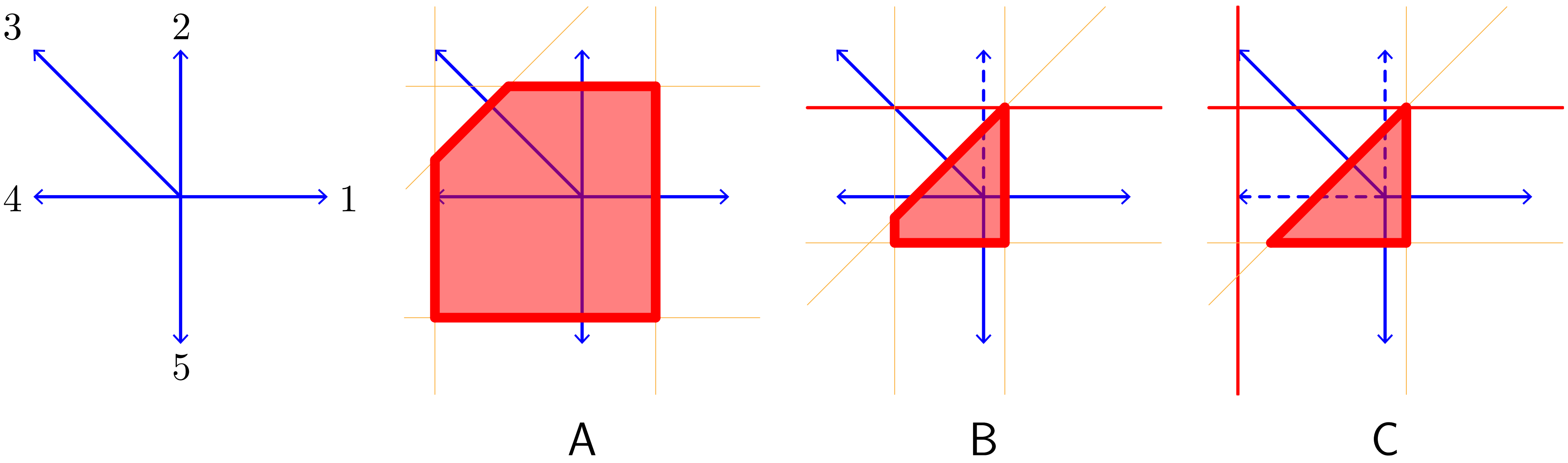

When is $\mathcal{F}$ the normal fan of $\mathbb{P}_{\boldsymbol{h}}$?

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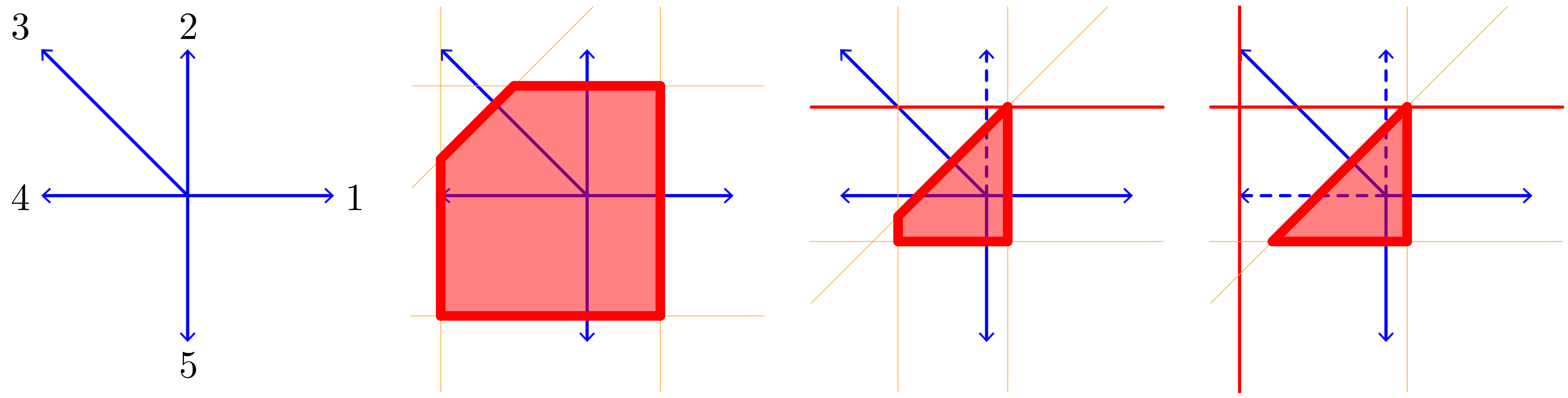

A B C

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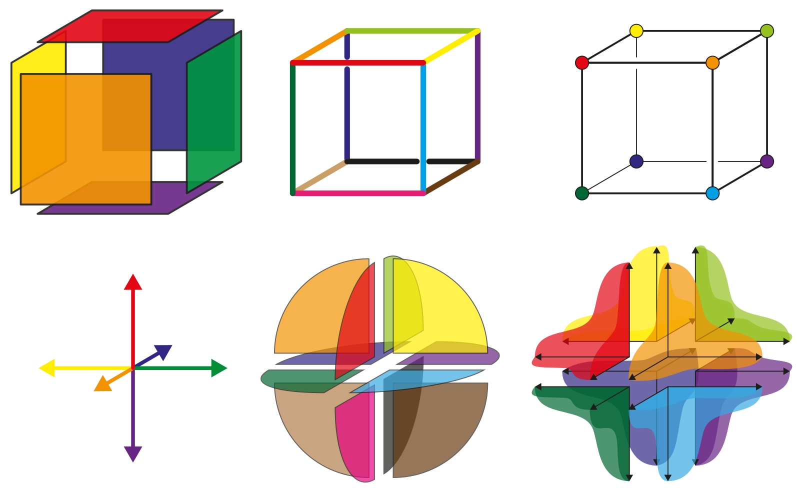

face $\mathbb{F}$ of polytope $\mathbb{P}$

normal cone of $\mathbb{F}$ = positive span of the outer normal vectors of the facets containing $\mathbb{F}$

normal fan of $\mathbb{P}$ = $\{$ normal cone of $\mathbb{F}$ $\mid$ $\mathbb{F}$ face of $\mathbb{P}$ $\}$

WALL-CROSSING INEQUALITIES

$\mathcal{F}$ = complete simplicial fan in $\mathbb{R}^n$ with N rays

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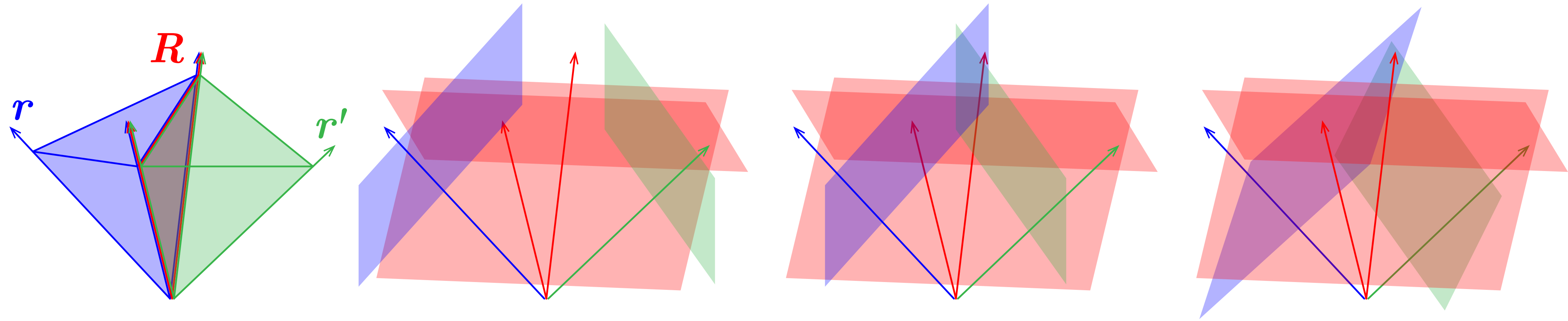

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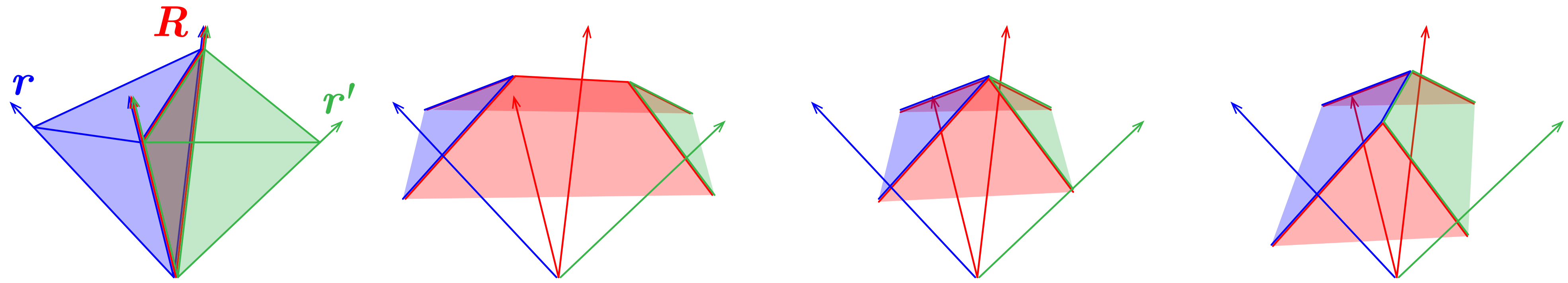

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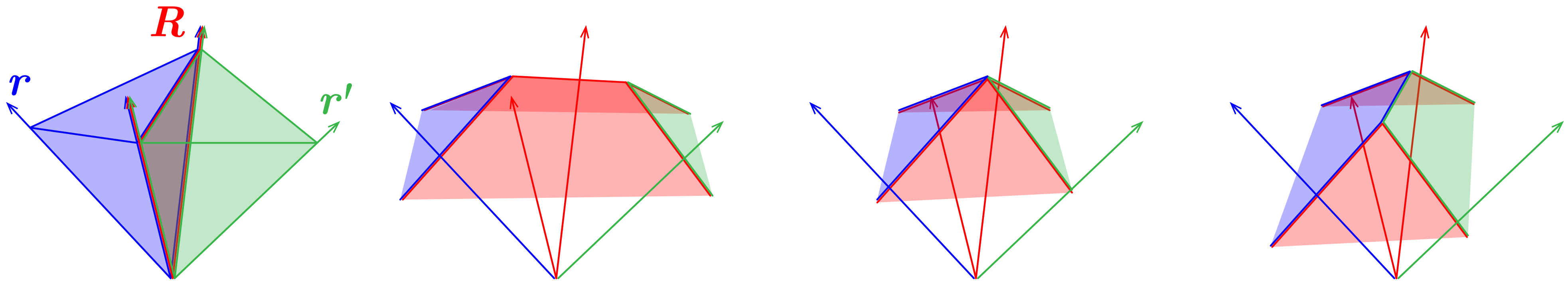

wall-crossing inequality for a wall $\boldsymbol{R}$ = $\displaystyle\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \{\boldsymbol{r}, \boldsymbol{r'}\}} \alpha_{\boldsymbol{R},\boldsymbol{s}}\, h_{\boldsymbol{s}} > 0$ where

- $\boldsymbol{r}, \boldsymbol{r'}$ = rays such that $\boldsymbol{R} \cup \{\boldsymbol{r}\}$ and $\boldsymbol{R} \cup \{\boldsymbol{r'}\}$ are chambers of $\mathcal{F}$
- $\alpha_{\boldsymbol{R},\boldsymbol{s}}$ = coeff. of unique linear dependence $\displaystyle\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \{\boldsymbol{r}, \boldsymbol{r'}\}} \alpha_{\boldsymbol{R},\boldsymbol{s}}\, \boldsymbol{s} = 0$ with $\alpha_{\boldsymbol{R},\boldsymbol{r}} + \alpha_{\boldsymbol{R},\boldsymbol{r'}} = 2$

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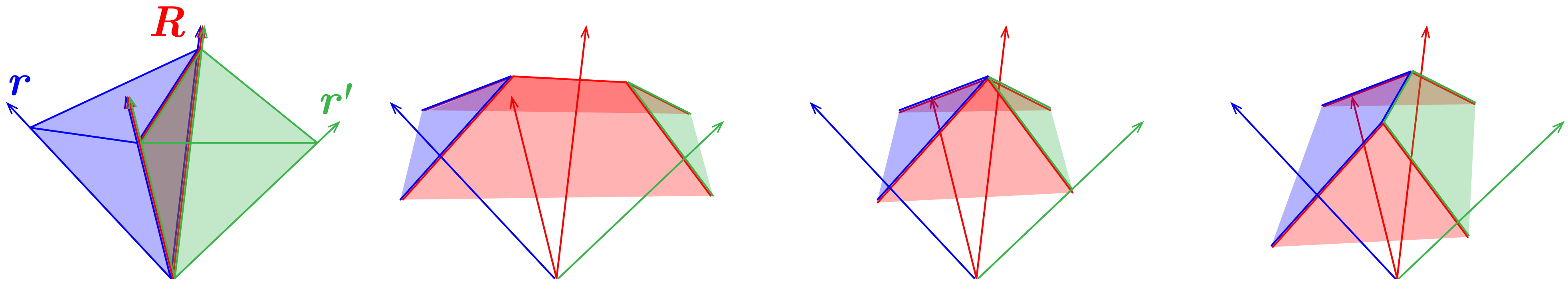

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$\mathcal{F}$ is the normal fan of $\mathbb{P}_{\boldsymbol{h}}$ $\iff$ $\boldsymbol{h}$ satisfies all wall-crossing inequalities of $\mathcal{F}$

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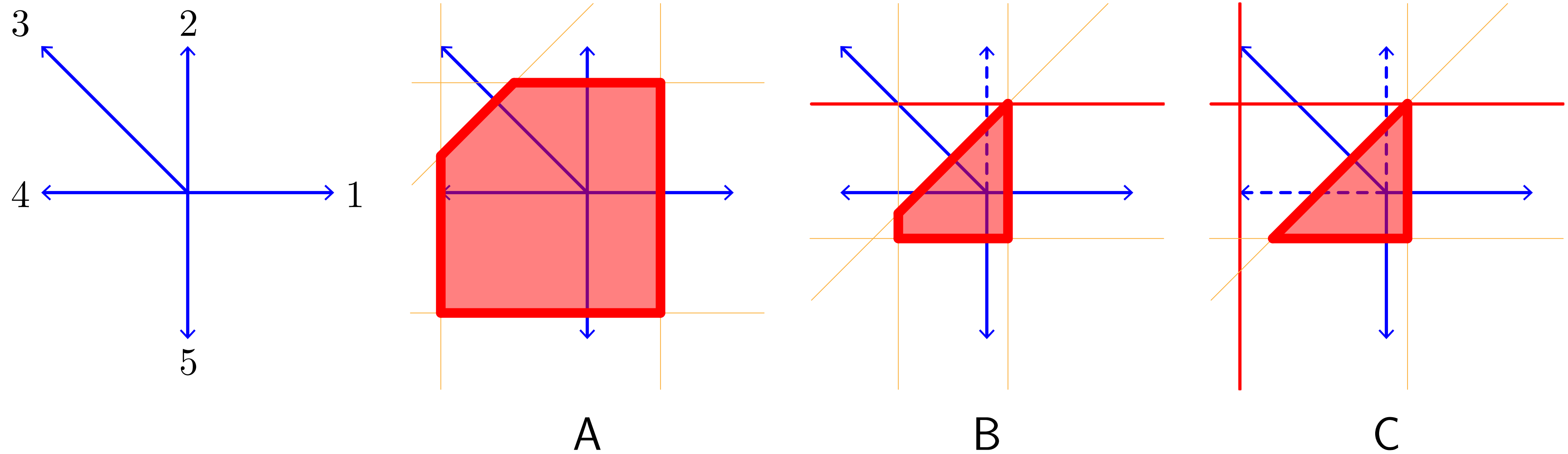
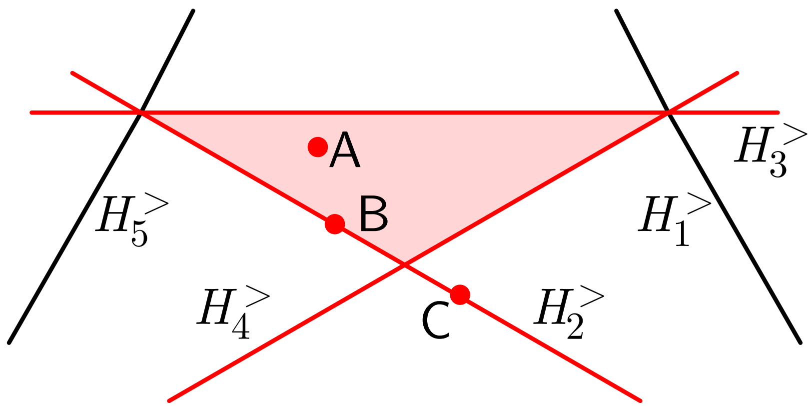


wall-crossing inequalities:

$$\begin{aligned}
\text{wall } 1 : &\quad h_2 + h_5 > 0 \\
\text{wall } 2 : &\quad h_1 + h_3 > h_2 \\
\text{wall } 3 : &\quad h_2 + h_4 > h_3 \\
\text{wall } 4 : &\quad h_3 + h_5 > h_4 \\
\text{wall } 5 : &\quad h_1 + h_4 > 0
\end{aligned}$$



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type cone $\mathbb{TC}(\mathcal{F})$ = realization space of $\mathcal{F}$ — McMullen ('73)

$= \{\boldsymbol{h} \in \mathbb{R}^N \mid \mathcal{F}$ is the normal fan of $\mathbb{P}_{\boldsymbol{h}}\}$

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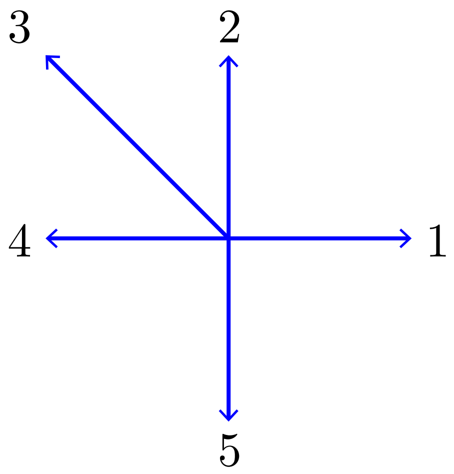

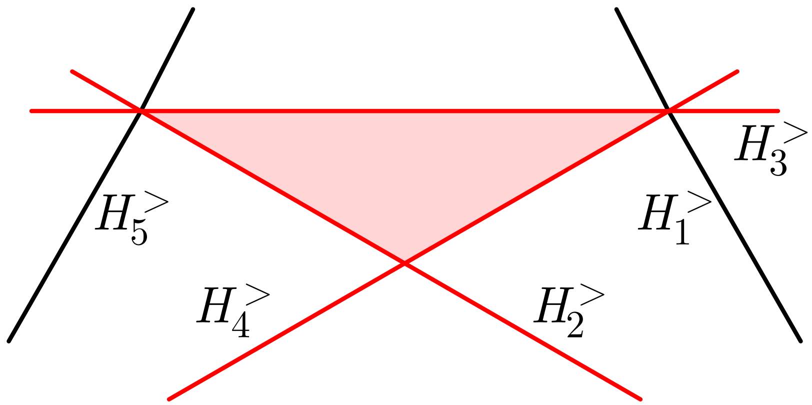

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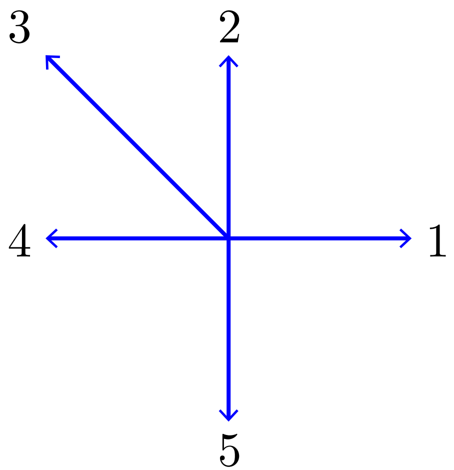

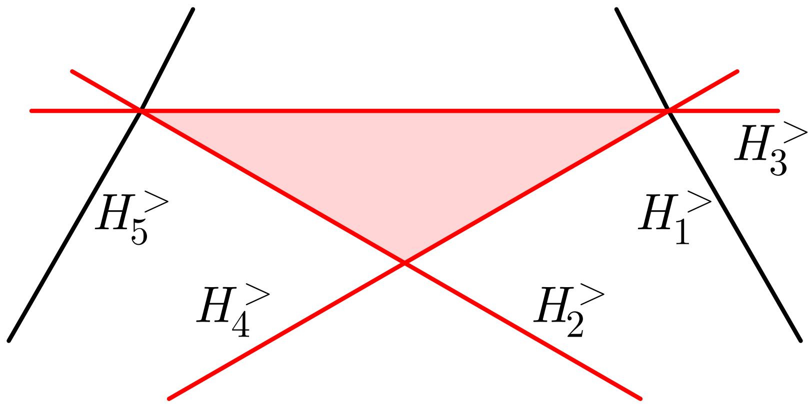

some properties of $\mathbb{TC}(\mathcal{F})$:

- $\mathbb{TC}(\mathcal{F})$ is an open cone (dilations preserve normal fans)
- $\mathbb{TC}(\mathcal{F})$ has lineality space $\boldsymbol{G}\,\mathbb{R}^n$ (translations preserve normal fans)
- dimension of $\mathbb{TC}(\mathcal{F})/\boldsymbol{G}\,\mathbb{R}^n = N - n$

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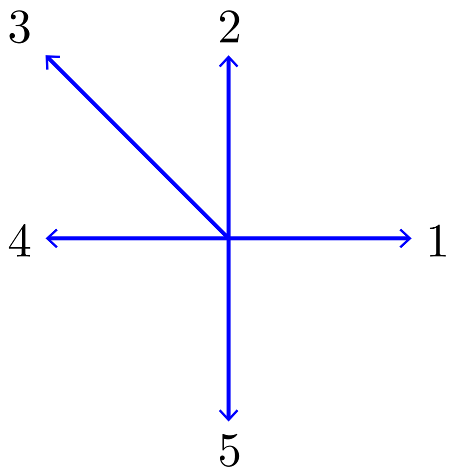
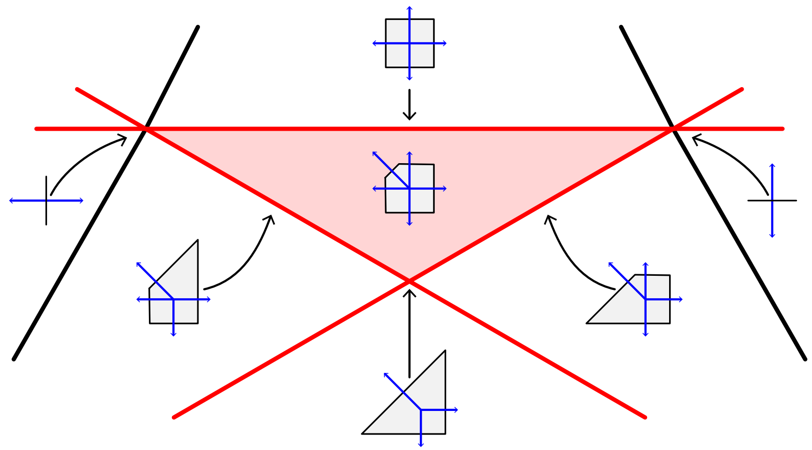
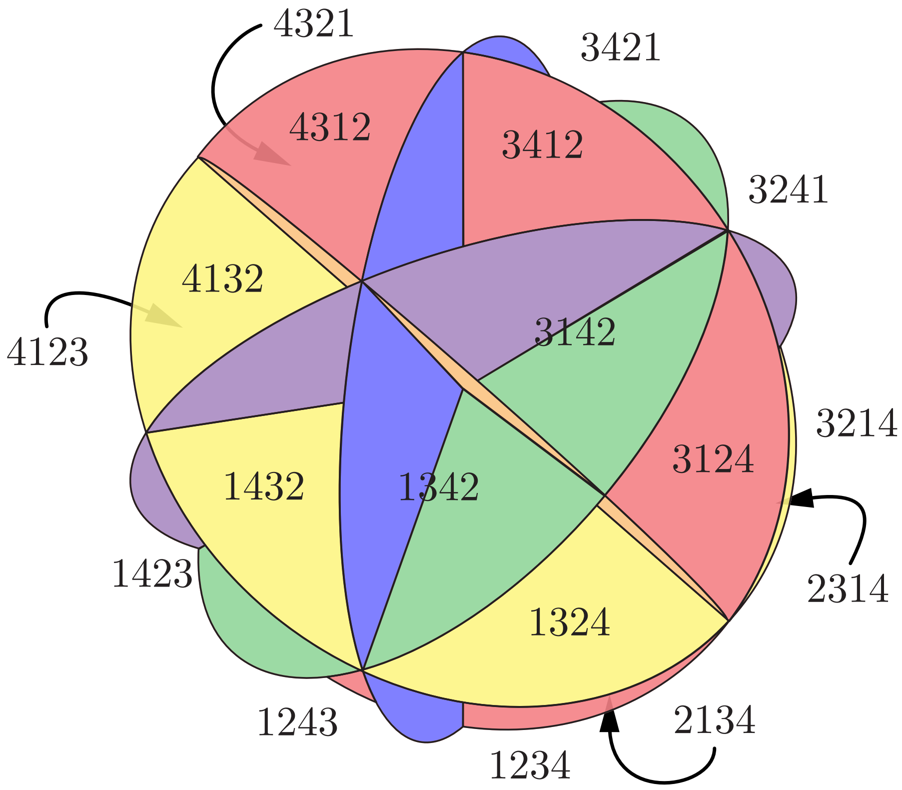


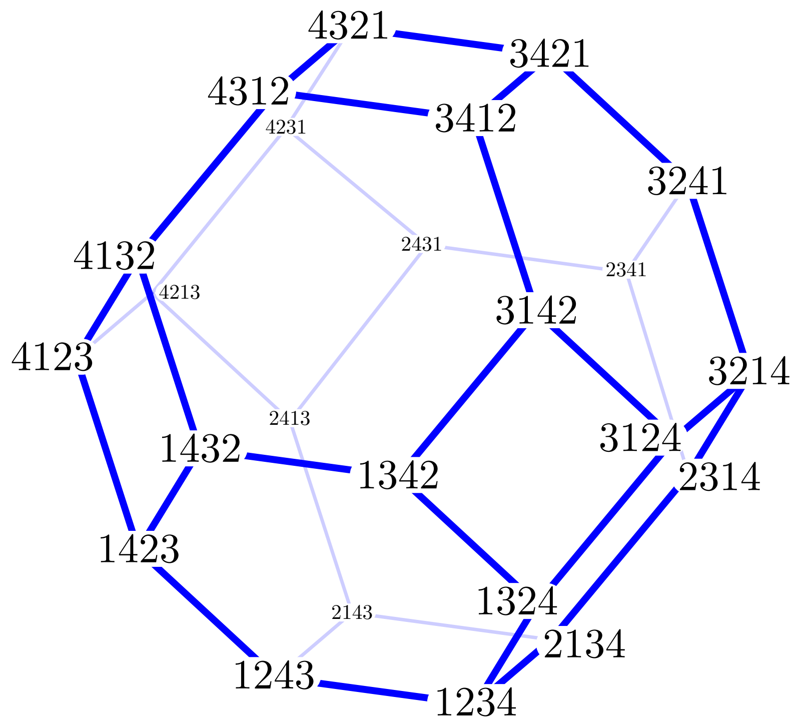


some properties of $\mathbb{TC}(\mathcal{F})$:

- closure of $\mathbb{TC}(\mathcal{F})$ = polytopes whose normal fan coarsens $\mathcal{F}$ = deformation cone
- Minkowski sums $\longleftrightarrow$ positive linear combinations

EXM: SUBMODULAR FUNCTIONS





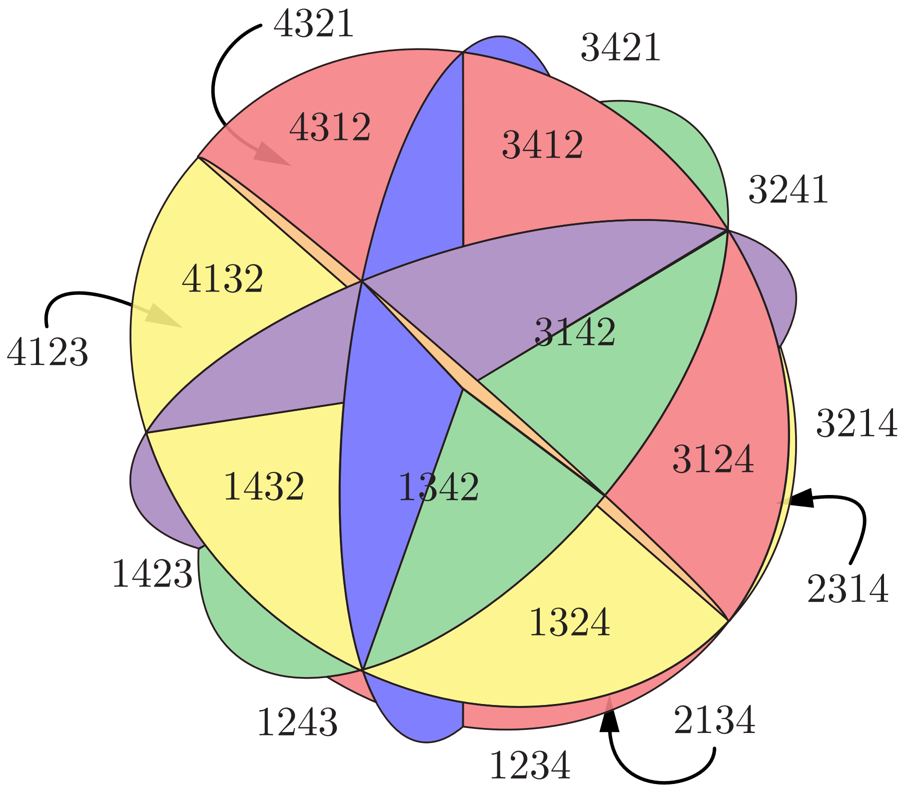
braid fan $=$

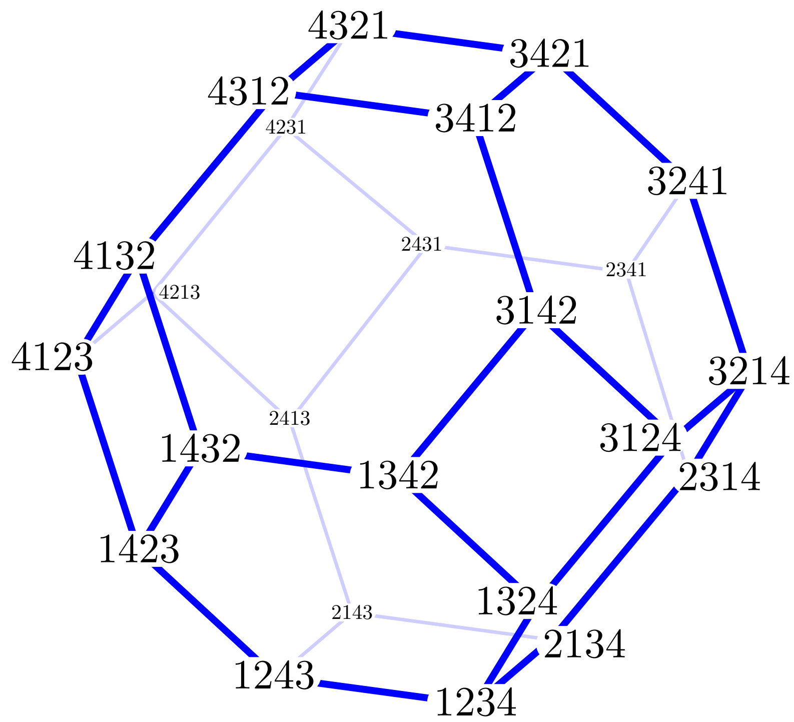
$$\mathbb{C}(\sigma) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \right\}$$

permutahedron $=$

$$\operatorname{conv} \left\{ [\sigma^{-1}(i)]_{i \in [n]} \;\middle|\; \sigma \in \mathfrak{S}_n \right\}$$

EXM: SUBMODULAR FUNCTIONS





braid fan =

$$\mathbb{C}(\sigma) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_{\sigma(1)} \le \dots \le x_{\sigma(n)} \right\}$$

permutahedron =

$$\operatorname{conv}\left\{ [\sigma^{-1}(i)]_{i \in [n]} \;\middle|\; \sigma \in \mathfrak{S}_n \right\}$$

closed type cone of braid fan = {deformed permutahedra} = {submodular functions}

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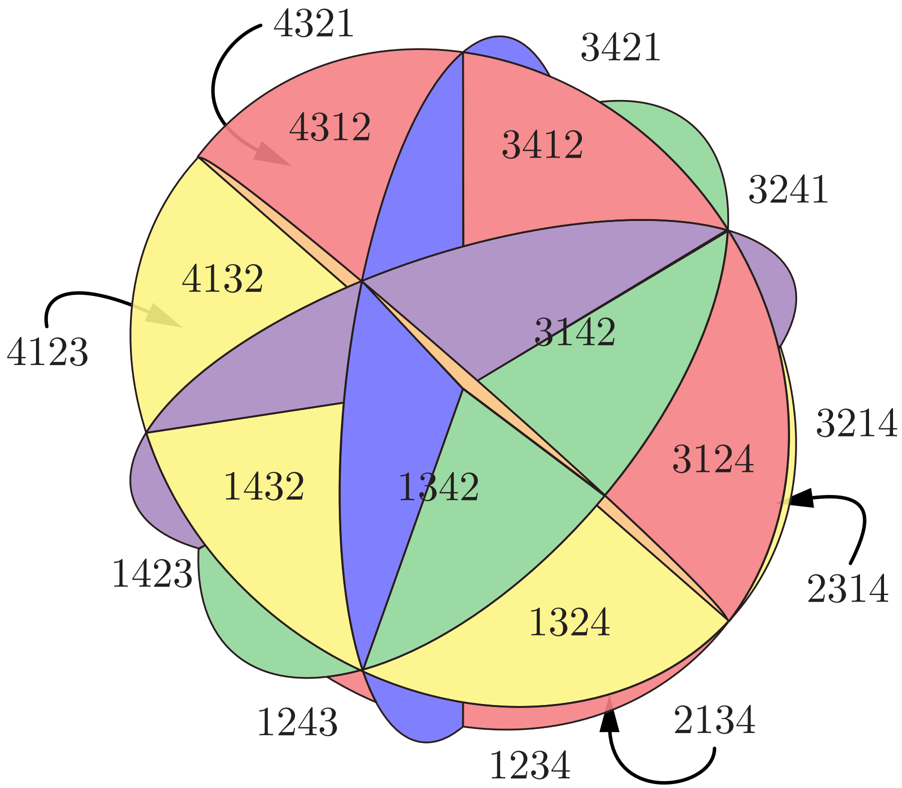

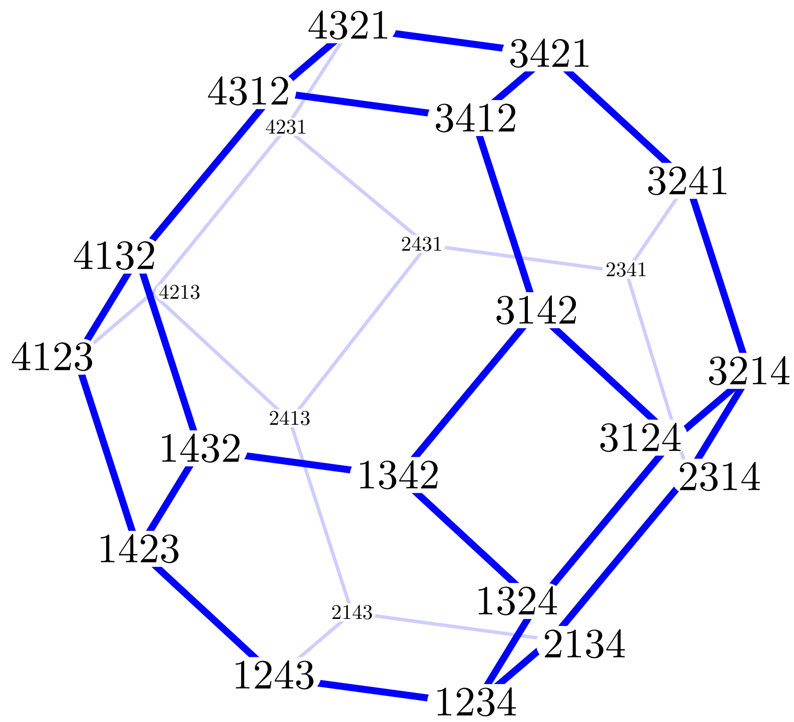

braid fan $=$

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permutahedron $=$

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deformed permutahedron $=$ polytope whose normal fan coarsens the braid fan

$$\mathbb{D}\mathrm{efo}(\boldsymbol{z}) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \langle\, \mathbb{1} \mid \boldsymbol{x} \,\rangle = \boldsymbol{z}_{[n]} \text{ and } \langle\, \mathbb{1}_R \mid \boldsymbol{x} \,\rangle \ge \boldsymbol{z}_R \text{ for all } R \subseteq [n] \right\}$$

for some vector $\boldsymbol{z} \in \mathbb{R}^{2^{[n]}}$ such that $\boldsymbol{z}_R + \boldsymbol{z}_S \le \boldsymbol{z}_{R \cup S} + \boldsymbol{z}_{R \cap S}$ and $\boldsymbol{z}_\varnothing = 0$

Postnikov ('09) Postnikov–Reiner–Williams ('08)

SIMPLICIAL TYPE CONE

$\mathcal{F}$ = complete simplicial fan in $\mathbb{R}^n$ with N rays

$\boldsymbol{G} = (N \times n)$-matrix whose rows are representatives of the rays of $\mathcal{F}$

$\boldsymbol{K} = (N - n) \times N$-matrix that spans the left kernel of $\boldsymbol{G}$ (ie. $\boldsymbol{K}\boldsymbol{G} = \mathbf{0}$)

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Classical affine transformation on polytopes:

$$\begin{array}{ccc} \mathbb{P}_{\boldsymbol{h}} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{Gx} \le \boldsymbol{h}\} & \longrightarrow & \mathbb{Q}_{\boldsymbol{h}} = \left\{\boldsymbol{z} \in \mathbb{R}^N \;\middle|\; \boldsymbol{z} \ge 0 \text{ and } \boldsymbol{Kz} = \boldsymbol{Kh}\right\} \\ \boldsymbol{x} & \longmapsto & \boldsymbol{z} = \boldsymbol{h} - \boldsymbol{Gx} \end{array}$$

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All polytopal realizations of $\mathcal{F}$ are affinely equivalent to

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for any $\boldsymbol{h}$ in the type cone $\mathbb{TC}(\mathcal{F})$.

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Assume that the type cone $\mathbb{TC}(\mathcal{F})$ is simplicial.
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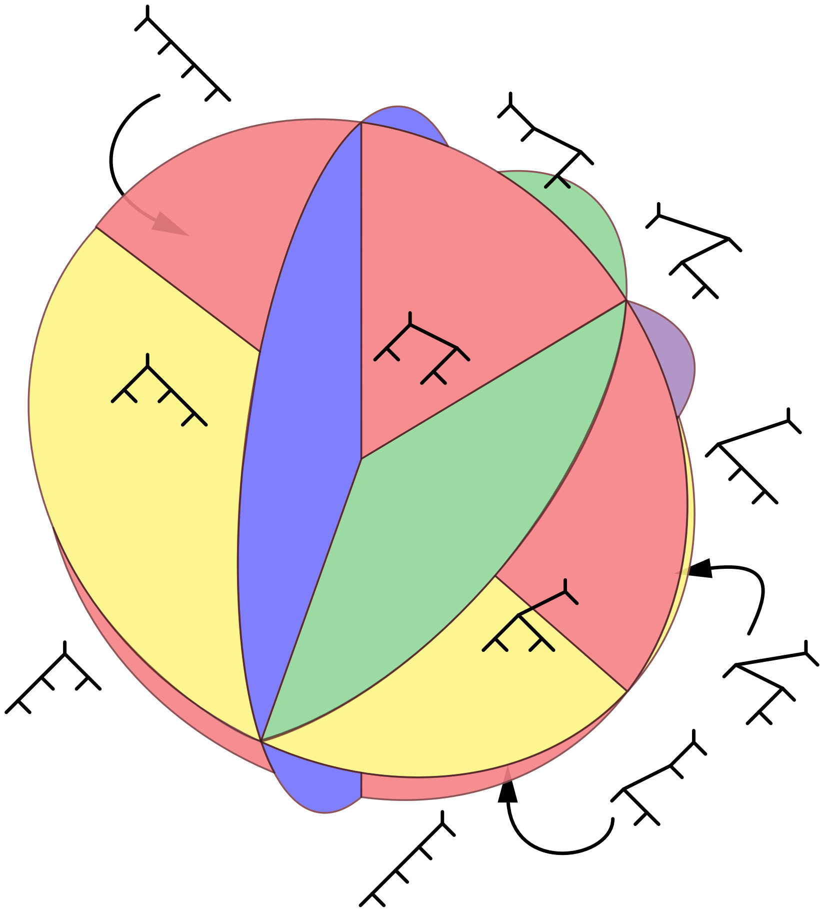
$$\mathbb{R}_{\boldsymbol{\ell}} = \left\{\boldsymbol{z} \in \mathbb{R}^N \;\middle|\; \boldsymbol{z} \ge 0 \text{ and } \boldsymbol{Kz} = \boldsymbol{\ell}\right\}$$

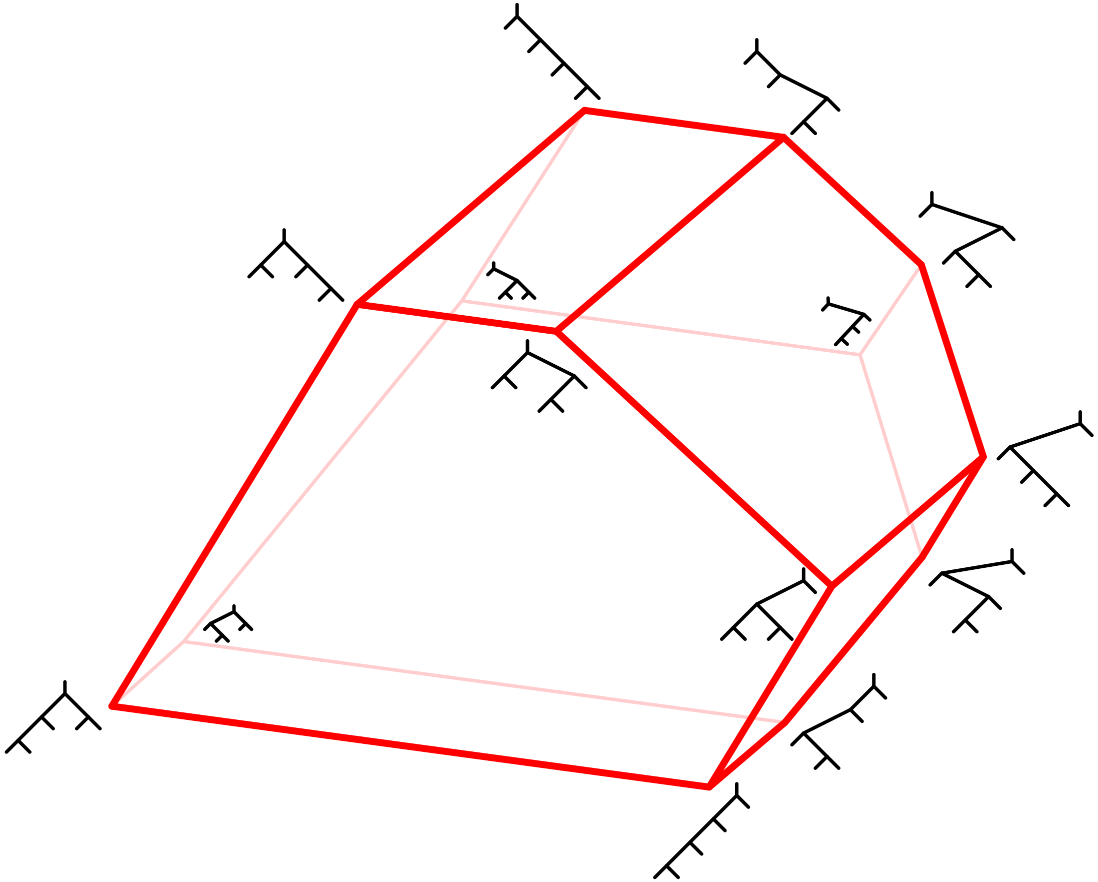
for any positive vector $\boldsymbol{\ell} \in \mathbb{R}^{N-n}_{>0}$.

Padrol–Palu–P.–Plamondon ('19$^+$)

TYPE CONES OF G-VECTOR FANS

SYLVESTER FAN AND CLASSICAL ASSOCIAHEDRON





sylvester fan $=$

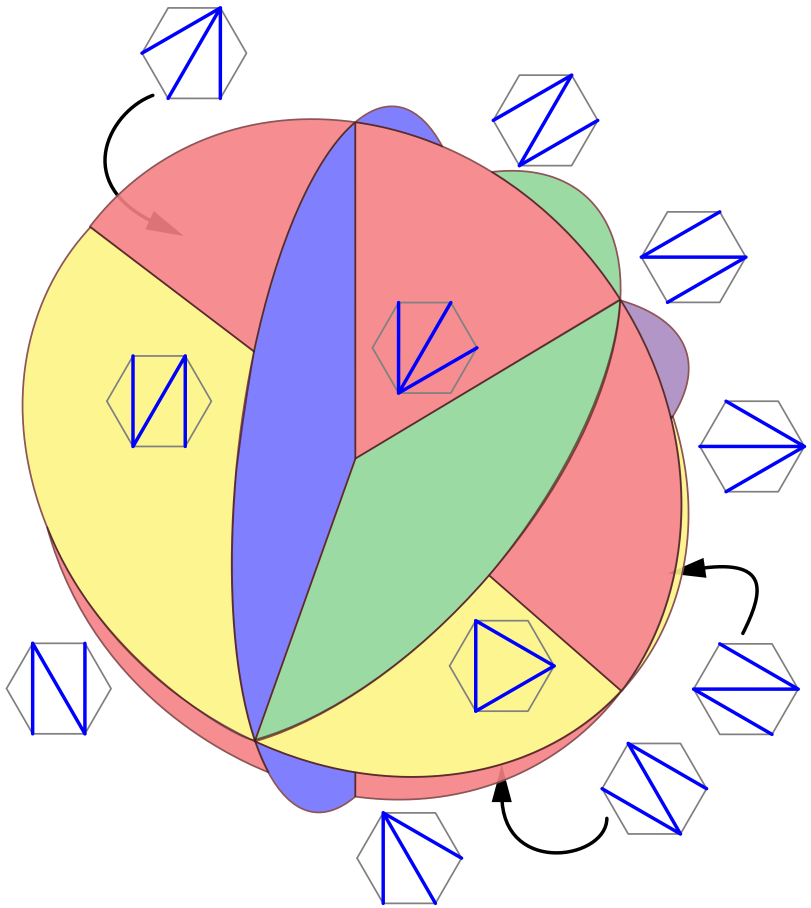
$\mathbb{C}(T) = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \le x_j$ if $i \to j$ in T$\}$

associahedron $=$

$\mathrm{conv}\left\{ [\ell(T,i) \cdot r(T,i)]_{i \in [n]} \;\middle|\; \mathrm{T}\text{ binary tree} \right\}$

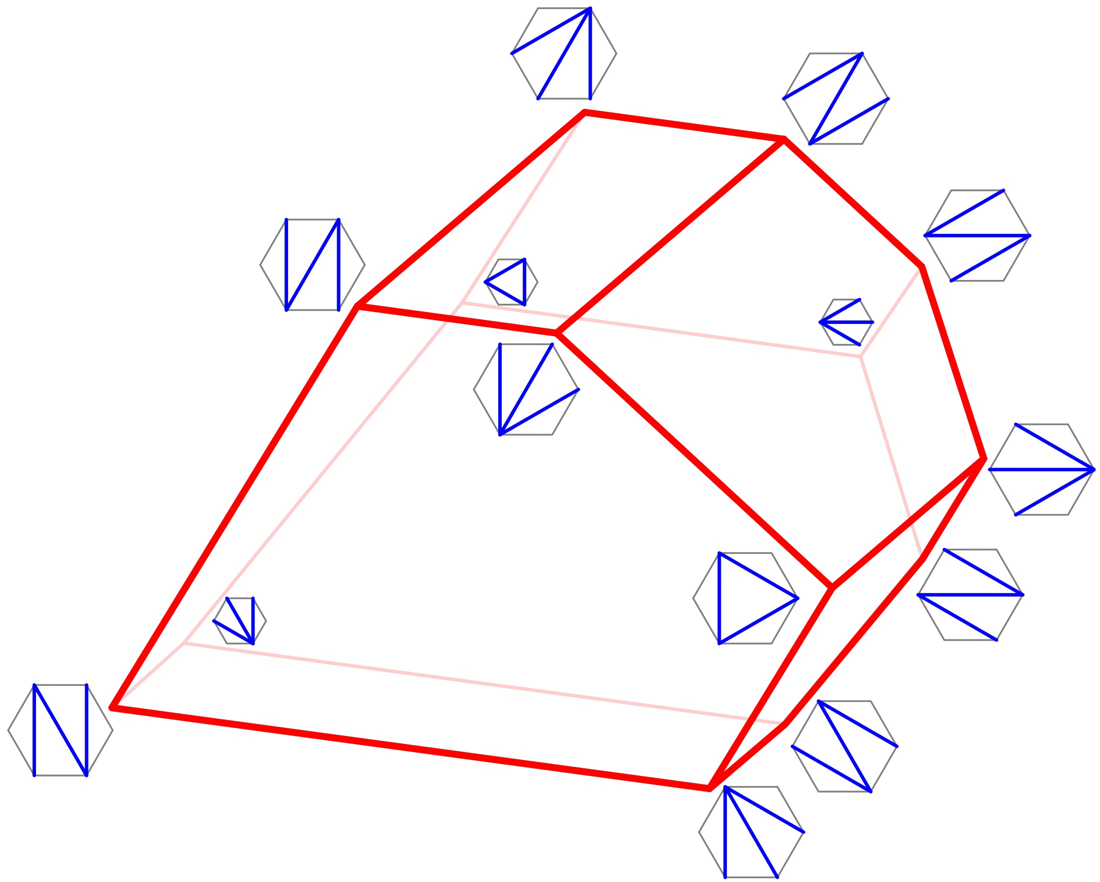
Shnider–Sternberg ('93)
Loday ('04)

SYLVESTER FAN AND CLASSICAL ASSOCIAHEDRON



sylvester fan

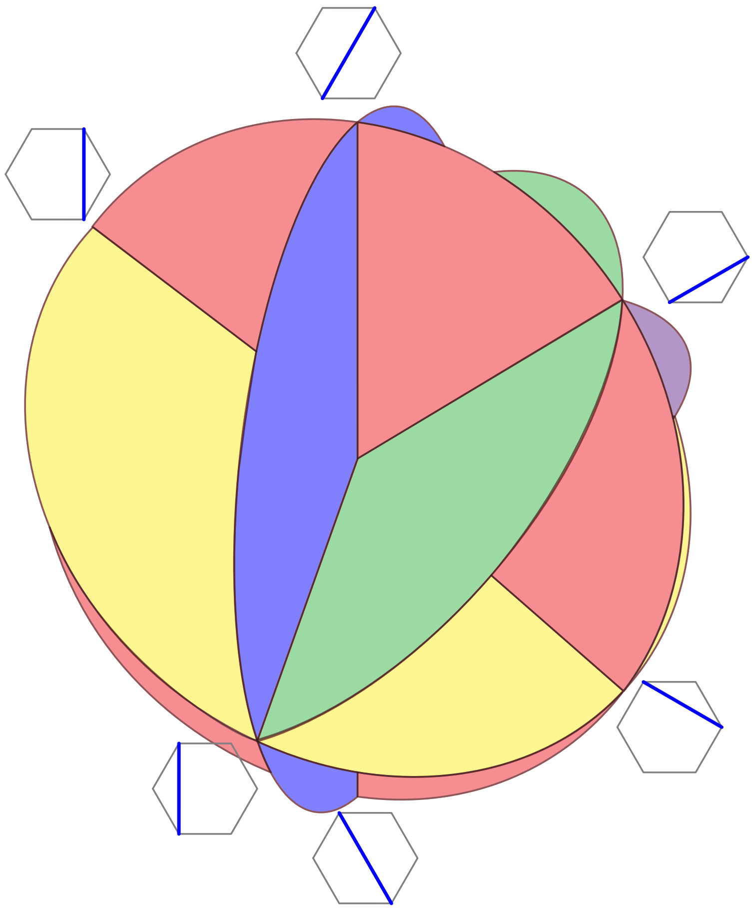
chambers $\longleftrightarrow$ triangulations

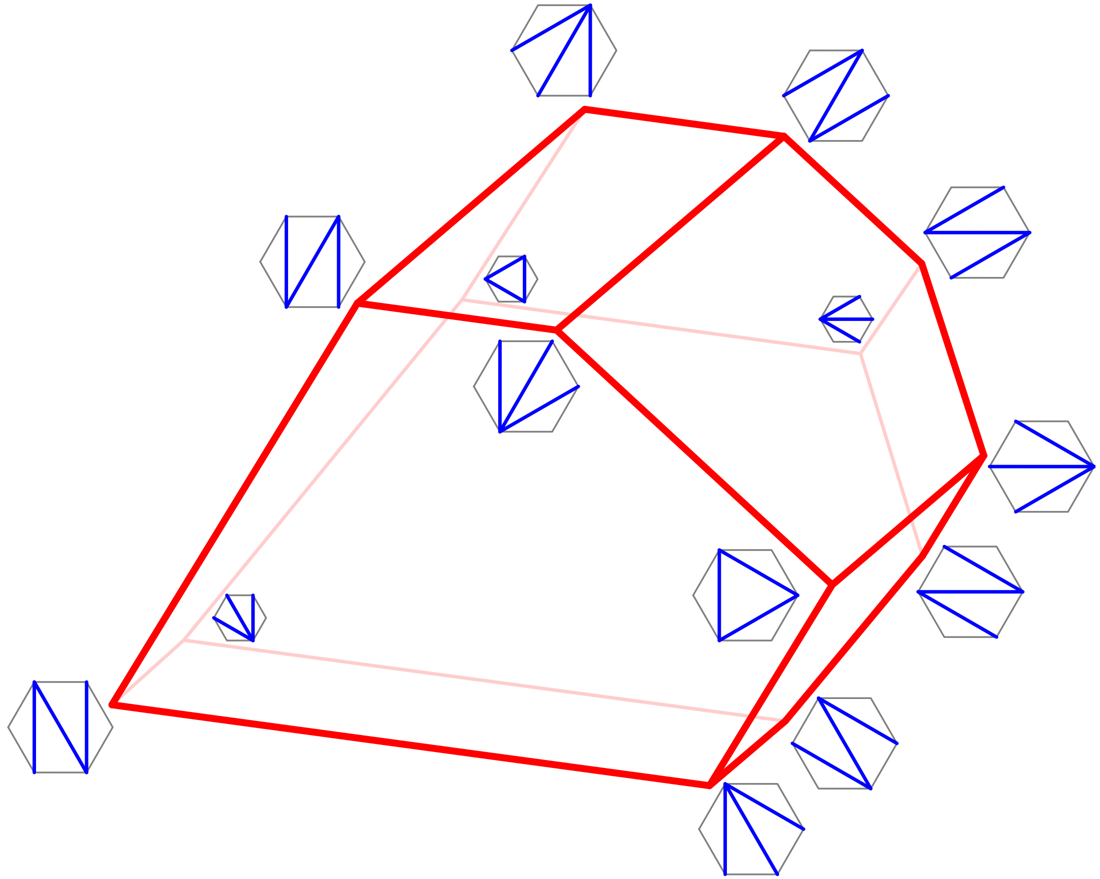


associahedron

vertices $\longleftrightarrow$ triangulations

SYLVESTER FAN AND CLASSICAL ASSOCIAHEDRON





sylvester fan

chambers $\longleftrightarrow$ triangulations

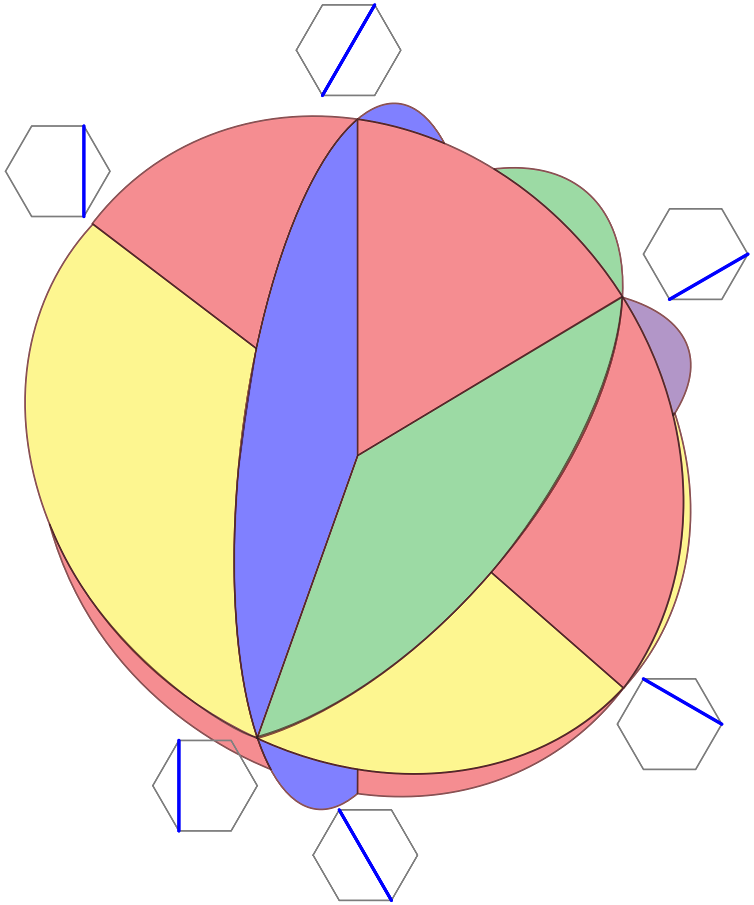
rays $\longleftrightarrow$ internal diagonals

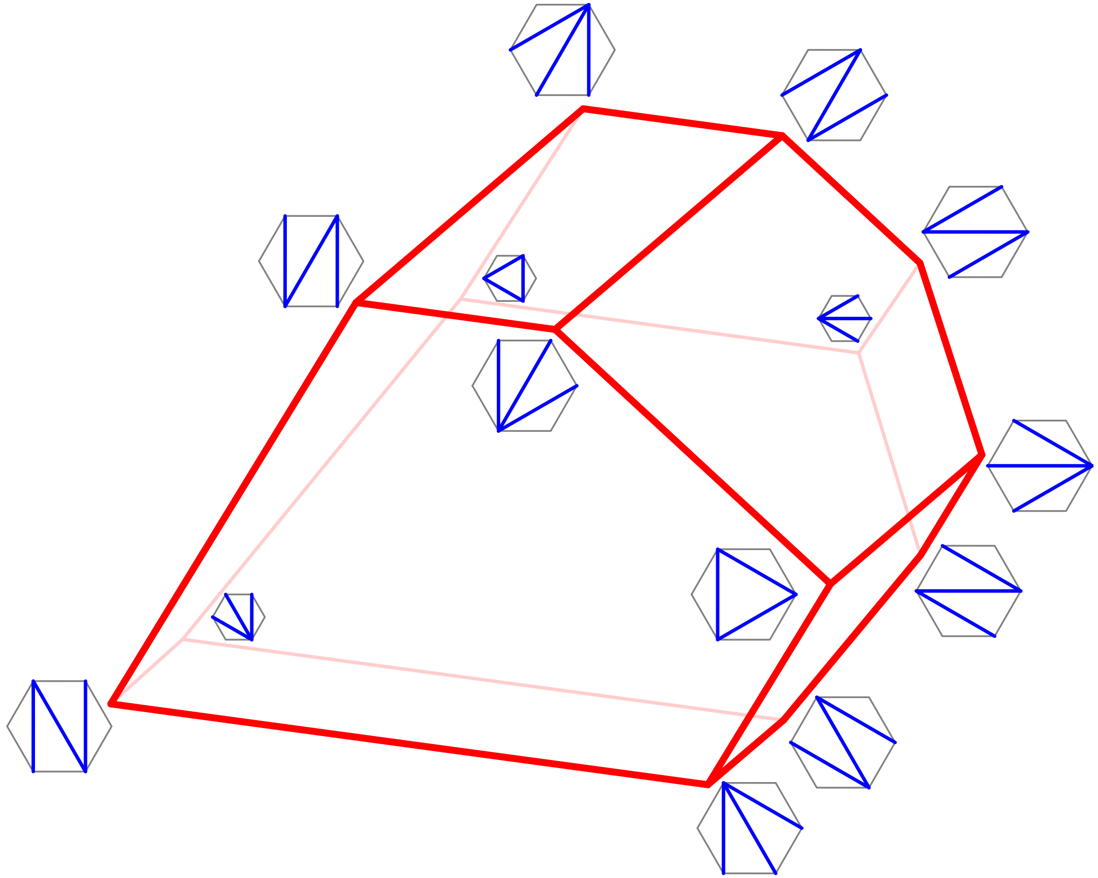
associahedron

vertices $\longleftrightarrow$ triangulations

facets $\longleftrightarrow$ internal diagonals

SYLVESTER FAN AND CLASSICAL ASSOCIAHEDRON





sylvester fan

chambers $\longleftrightarrow$ triangulations

rays $\longleftrightarrow$ internal diagonals

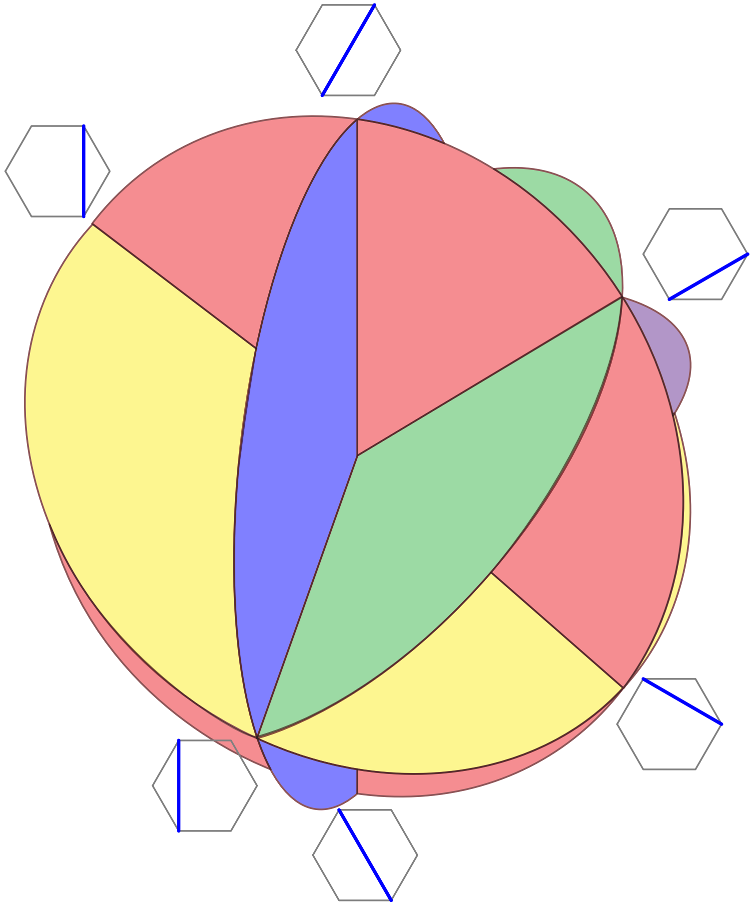
exch. rays $\longleftrightarrow$ pairs crossing diagonals

associahedron

vertices $\longleftrightarrow$ triangulations

facets $\longleftrightarrow$ internal diagonals

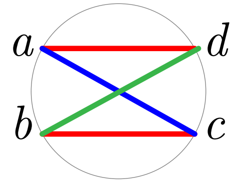
SYLVESTER FAN AND CLASSICAL ASSOCIAHEDRON



wall crossing inequalities $=$

for all $0 \le a < b < c < d \le n+2$,

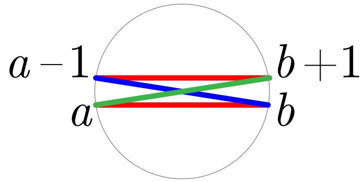
$\boldsymbol{z}_{(a,c)} + \boldsymbol{z}_{(b,d)} - \boldsymbol{z}_{(b,c)} - \boldsymbol{z}_{(a,d)} > 0$



facet defining inequalities $=$

for all $1 \le a < b \le n+1$,

$\boldsymbol{z}_{(a-1,b)} + \boldsymbol{z}_{(a,b+1)} - \boldsymbol{z}_{(a,b)} - \boldsymbol{z}_{(a-1,b+1)} > 0$



$\Longrightarrow$ simplicial type cone

($\#$ facets $= \binom{n+1}{2} = \frac{n(n+3)}{2} - n = N - n$)

Let $X(n) = \{(a,b) \mid 0 \le a < b \le n+2\}$ and $Y(n) = \{(a,b) \mid 1 \le a < b \le n-1\}$.
For any $\boldsymbol{\ell} \in \mathbb{R}^{Y(n)}_{>0}$, the polytope

$$\left\{ \boldsymbol{z} \in \mathbb{R}^{X(n)} \;\middle|\; \begin{array}{l} \boldsymbol{z} \ge 0, \quad \boldsymbol{z}_{(0,n+2)} = 0 \quad \text{and} \quad \boldsymbol{z}_{(a,a+1)} = 0 \text{ for all } 0 \le a \le n+1 \\ \boldsymbol{z}_{(a-1,b)} + \boldsymbol{z}_{(a,b+1)} - \boldsymbol{z}_{(a,b)} - \boldsymbol{z}_{(a-1,b+1)} = \boldsymbol{\ell}_{(a,b)} \text{ for all } (a,b) \in Y(n) \end{array} \right\}$$

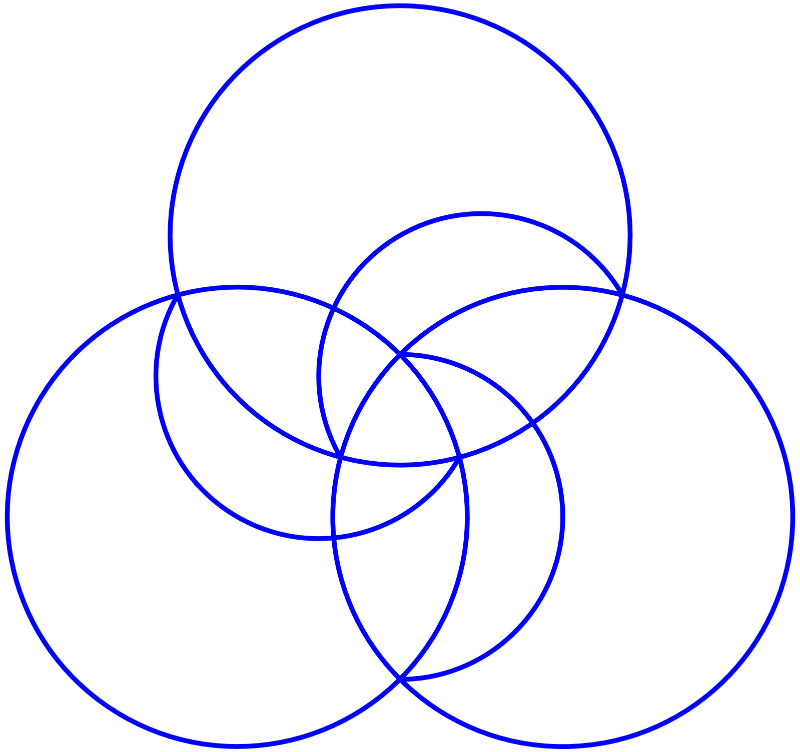
is an associahedron.

Arkani-Hamed–Bai–He–Yan ('18)

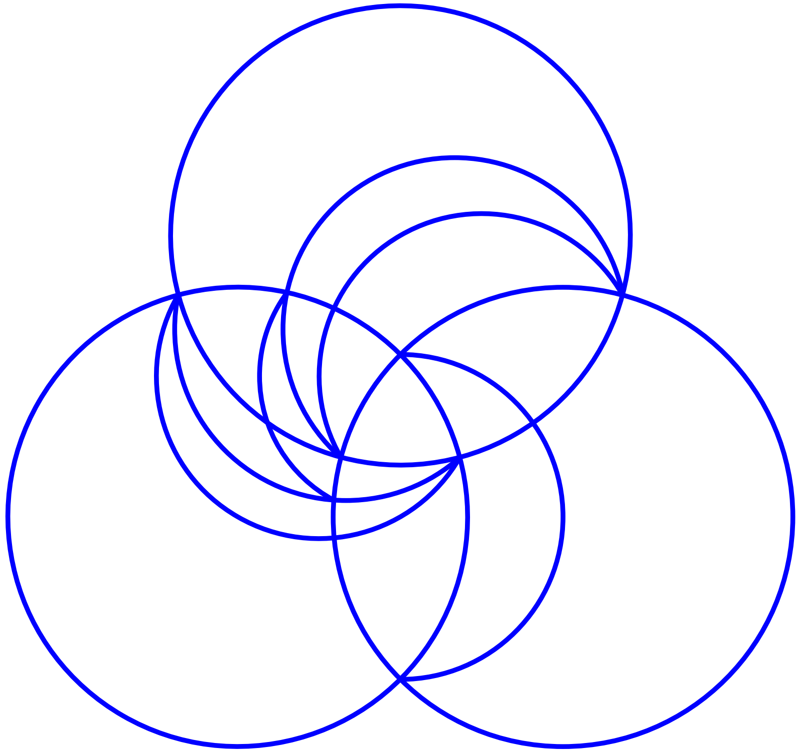
G-VECTOR FANS AND GENERALIZED ASSOCIAHEDRA

$B_\circ$ = finite type exchange matrix (acyclic or not, simply-laced or not)
$\mathcal{A}(B_\circ)$ = cluster algebra with principal coefficients and initial exchange matrix $B_\circ$
$\mathcal{F}(B_\circ)$ = $\boldsymbol{g}$-vector fan of $\mathcal{A}(B_\circ)$

Exm: stereographic projections of type A_3 and C_3 cyclic $\boldsymbol{g}$-vector fans



$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ -1 & 1 & 0 \end{bmatrix}$$

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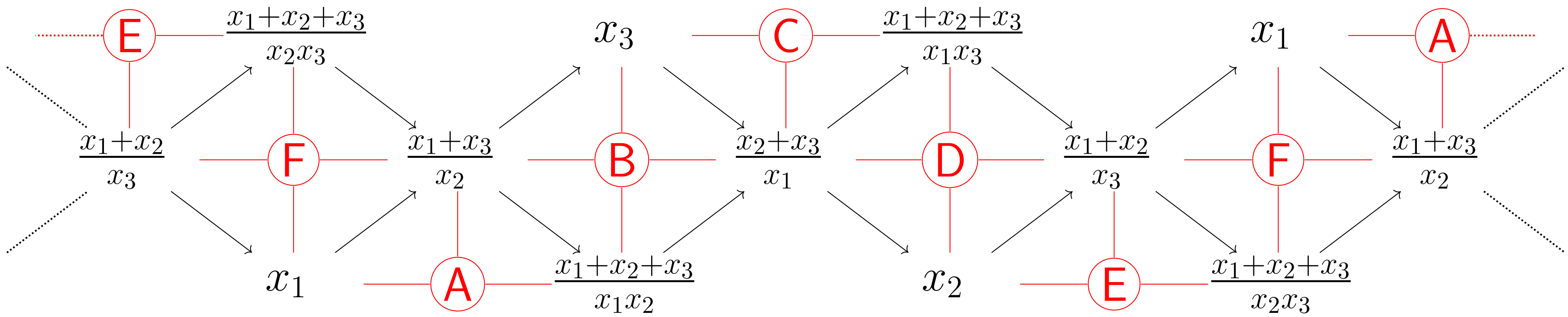

mesh mutation = mutation $(B, X) \to (B', X')$ with $X \smallsetminus \{x\} = X' \smallsetminus \{x'\}$
such that $b_{xy} \geq 0$ for all $y \in X$

initial mesh mutation = ends at an initial cluster variable x'

facets of type cone of $\mathcal{F}(B_\circ)$ = $\boldsymbol{g}$-vector dependences of non-initial mesh mutations

$\Longrightarrow$ simplicial type cone
(# non-initial mesh mutations = # cluster variables − # initial cluster variables)

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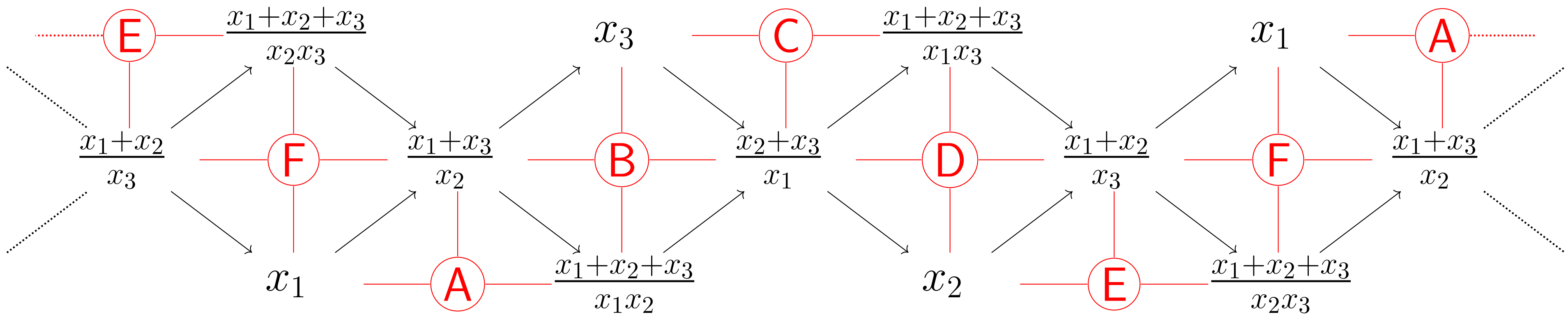

$\mathcal{V}(B_\circ) = \{$cluster variables$\}$.
$\mathcal{M}(B_\circ) = \{(x, x')$ exchangeable by a non-initial mesh mutation$\}$.
For any $\boldsymbol{\ell} \in \mathbb{R}_{>0}^{\mathcal{M}(B_\circ)}$, the polytope

$$\left\{ \boldsymbol{z} \in \mathbb{R}^{\mathcal{V}(B_\circ)} \;\middle|\; \boldsymbol{z} \ge 0 \;\text{ and }\; \boldsymbol{z}_x + \boldsymbol{z}_{x'} - \sum_y \alpha_{x,x'}(y)\boldsymbol{z}_y = \boldsymbol{\ell}_{x,x'} \text{ for all } (x,x') \in \mathcal{M}(B_\circ) \right\}$$

is a generalized associahedron.

$\uparrow$ $\alpha_{x,x'} = |b_{x,y}|$ if $y \in X$ and 0 otherwise

Bazier-Matte–Douville–Mousavand–Thomas–Yıldırım ('18$^+$)
Padrol–Palu–P.–Plamondon ('19$^+$)

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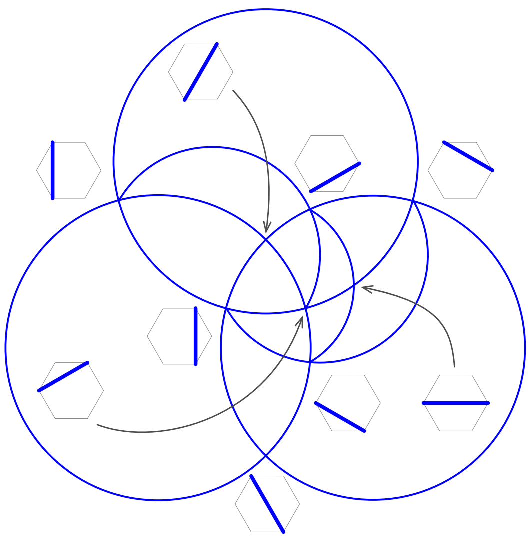
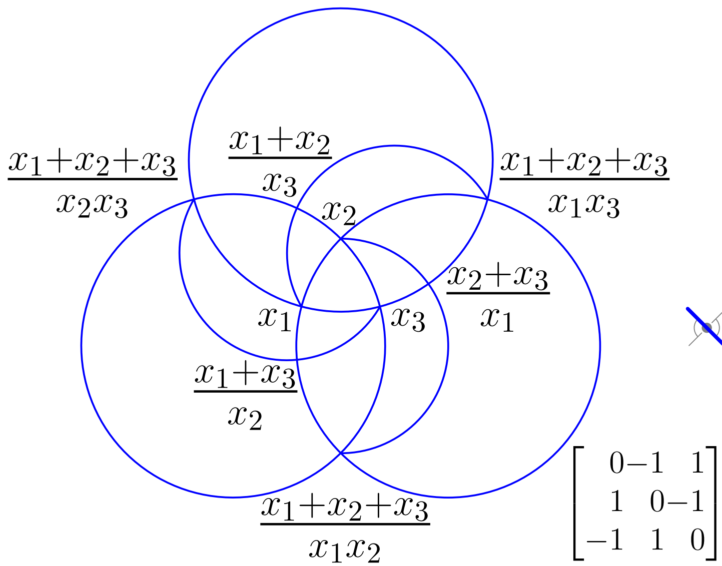
Padrol–Palu–P.–Plamondon ('19$^+$)

Fundamental exms: $\boldsymbol{g}$-vector fans of cluster-like complexes



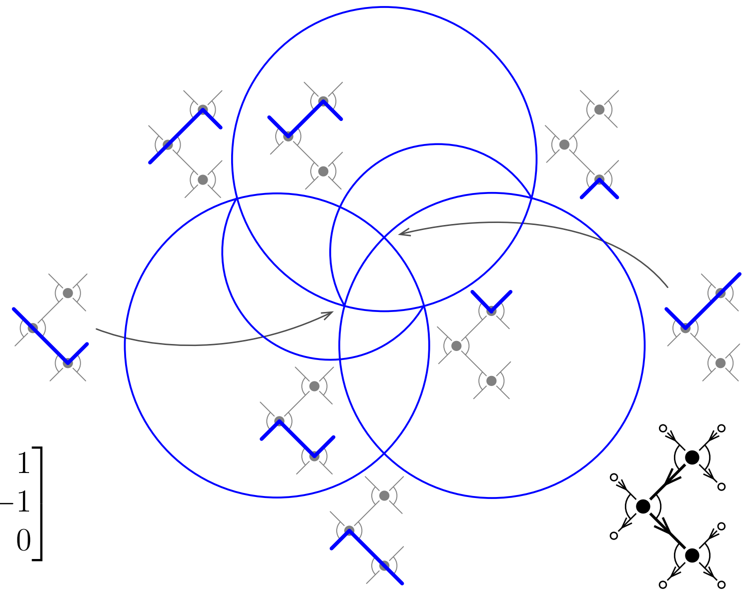
sylvester fans

Arkani-Hamed–Bai–He–Yan ('18)



finite type $\boldsymbol{g}$-vector fans
wrt any seed (acyclic or not)

BMDMTY ('18$^+$)



finite gentle fans
for brick and 2-acyclic quivers

Palu–P.–Plamondon ('18)