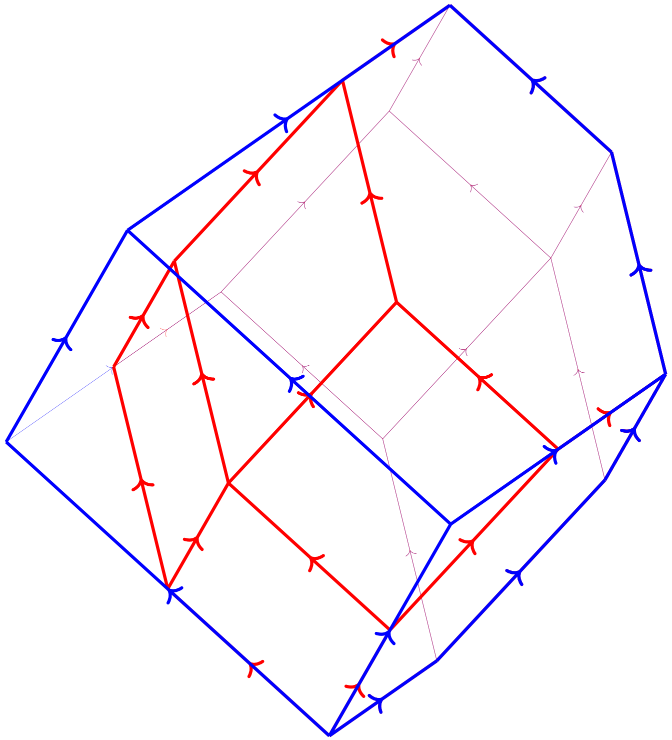
Acyclic reorientation lattices and their lattice quotients

V. PILAUD (CNRS & LIX, École Polytechnique)



Algebraic and Combinatorial Perspectives in the Mathematical Sciences
Friday March 25th, 2022

slides: http://www.lix.polytechnique.fr/~pilaud/documents/presentations/acyclicReorientationLatticesACPMS.pdf
preprint: http://www.arxiv.org/abs/2111.12387

PERMUTAHEDRA & ASSOCIAHEDRA

LATTICES: WEAK ORDER AND TAMARI LATTICE

lattice = partially ordered set L where any $X \subseteq L$ admits a meet $\bigwedge X$ and a join $\bigvee X$
lattice congruence = equivalence relation on L compatible with meets and joins

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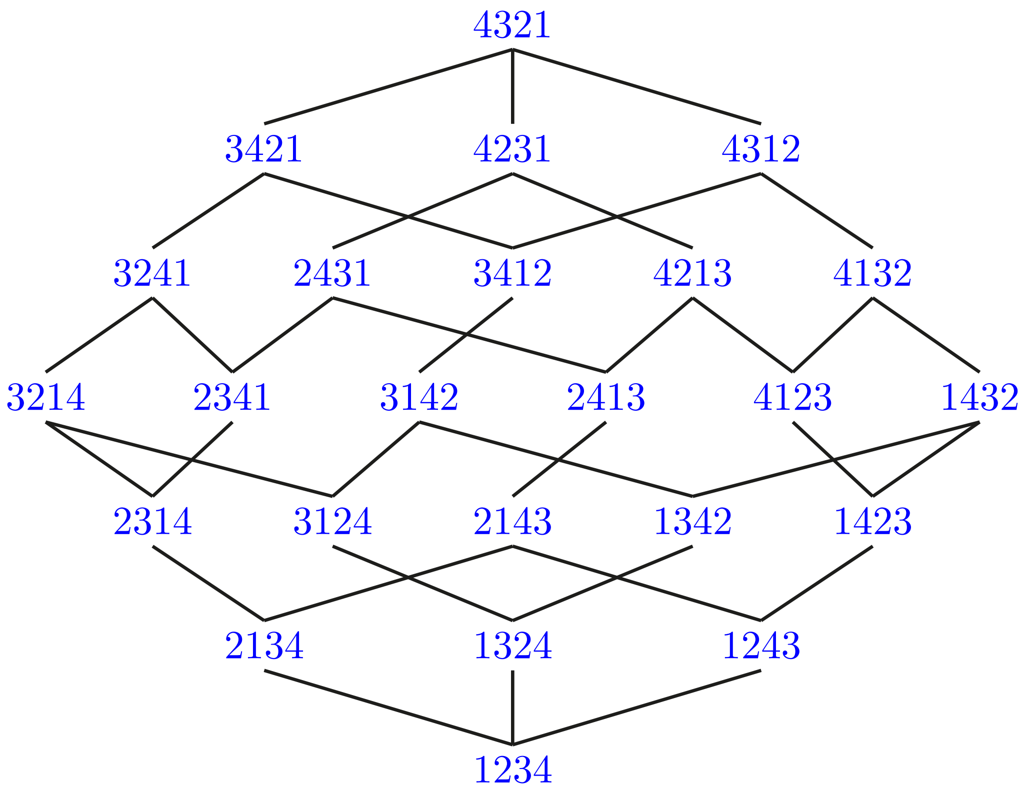

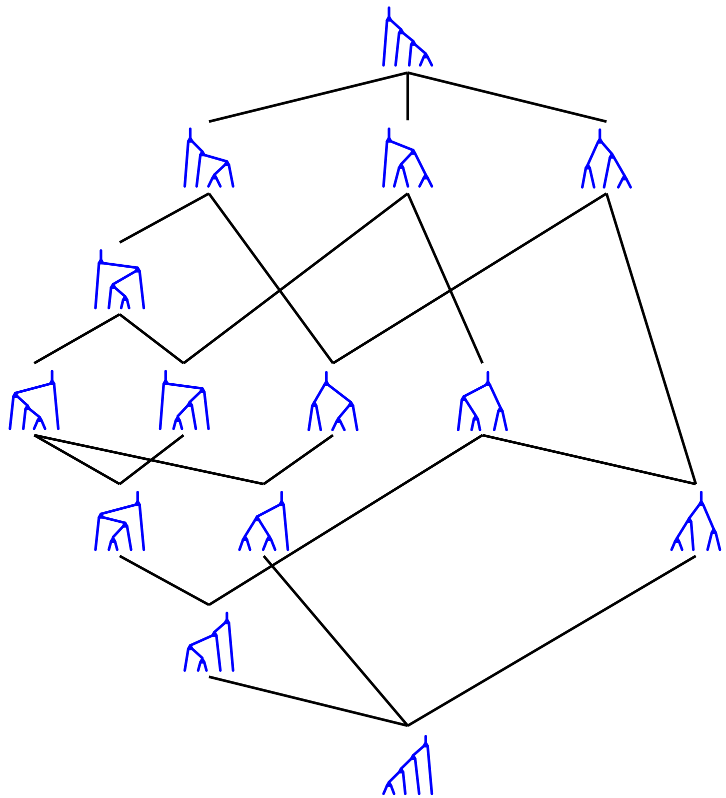

weak order = permutations of $\mathfrak{S}_n$
ordered by inclusion of inversion sets

Tamari lattice = binary trees on $[n]$
ordered by paths of right rotations

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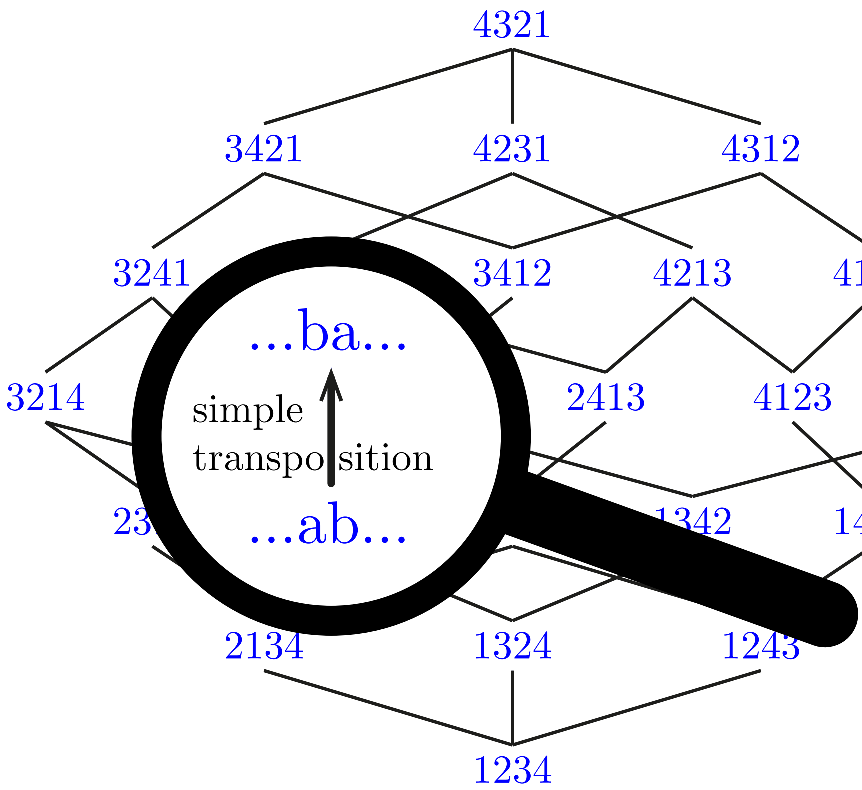

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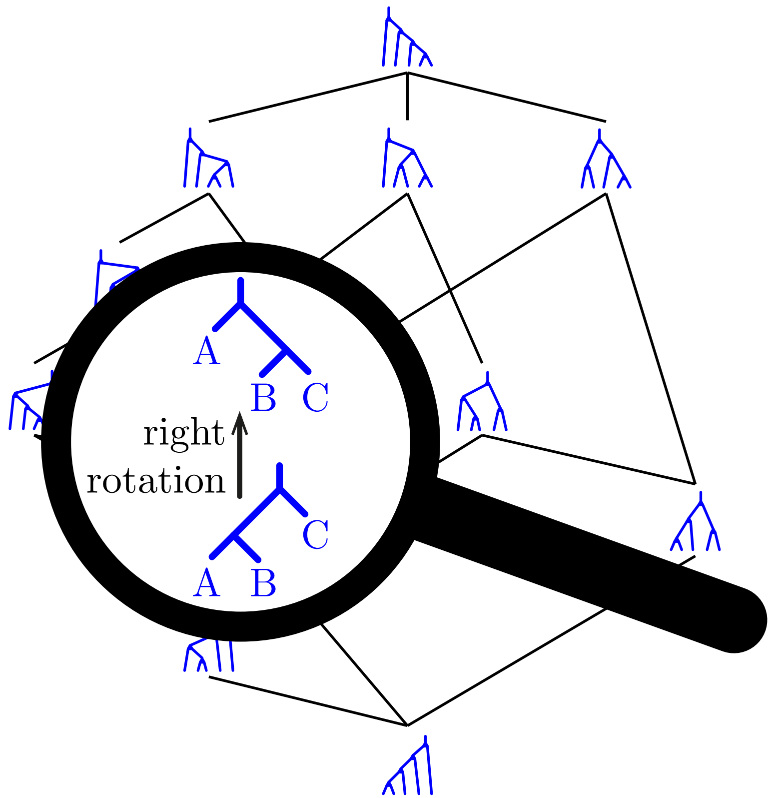

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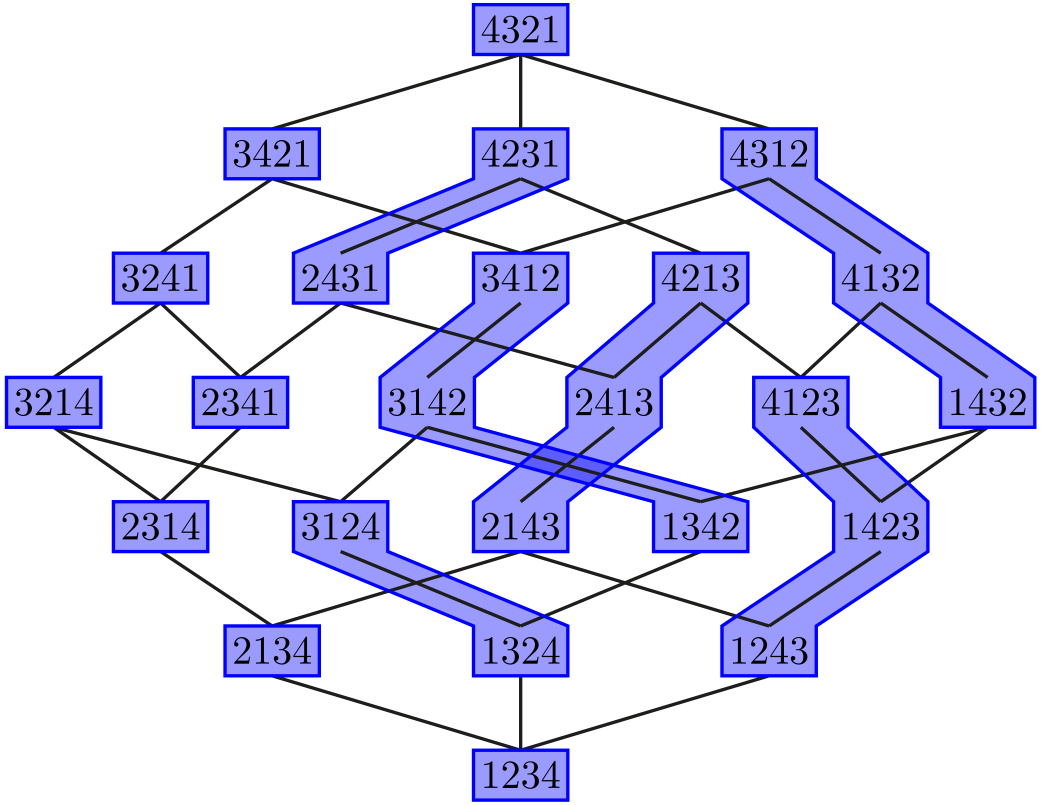

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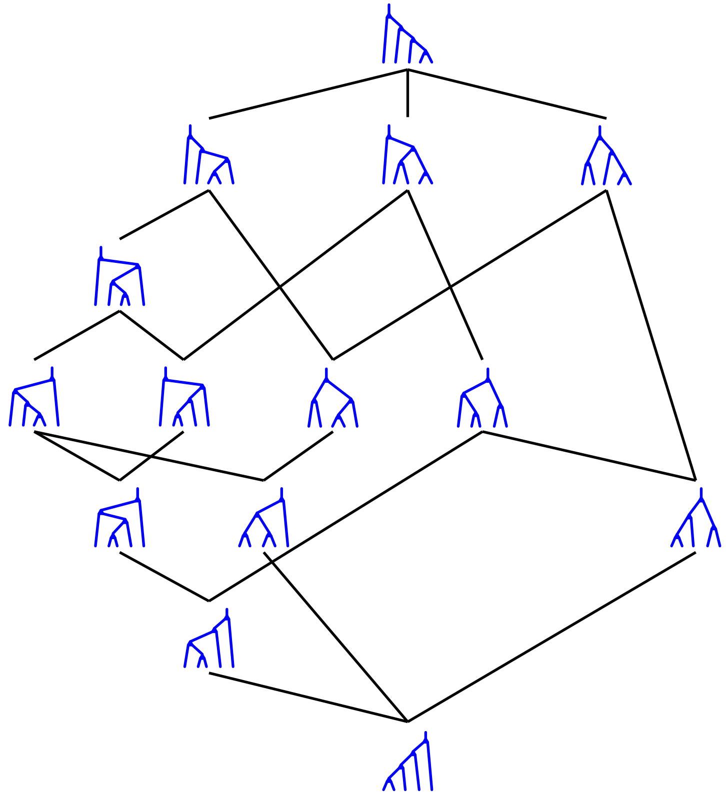

Tamari lattice $=$ binary trees on $[n]$
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sylvester congruence $=$ equivalence classes are fibers of BST insertion
$=$ rewriting rule $UacVbW \equiv_{\text{sylv}} UcaVbW$ with $a < b < c$

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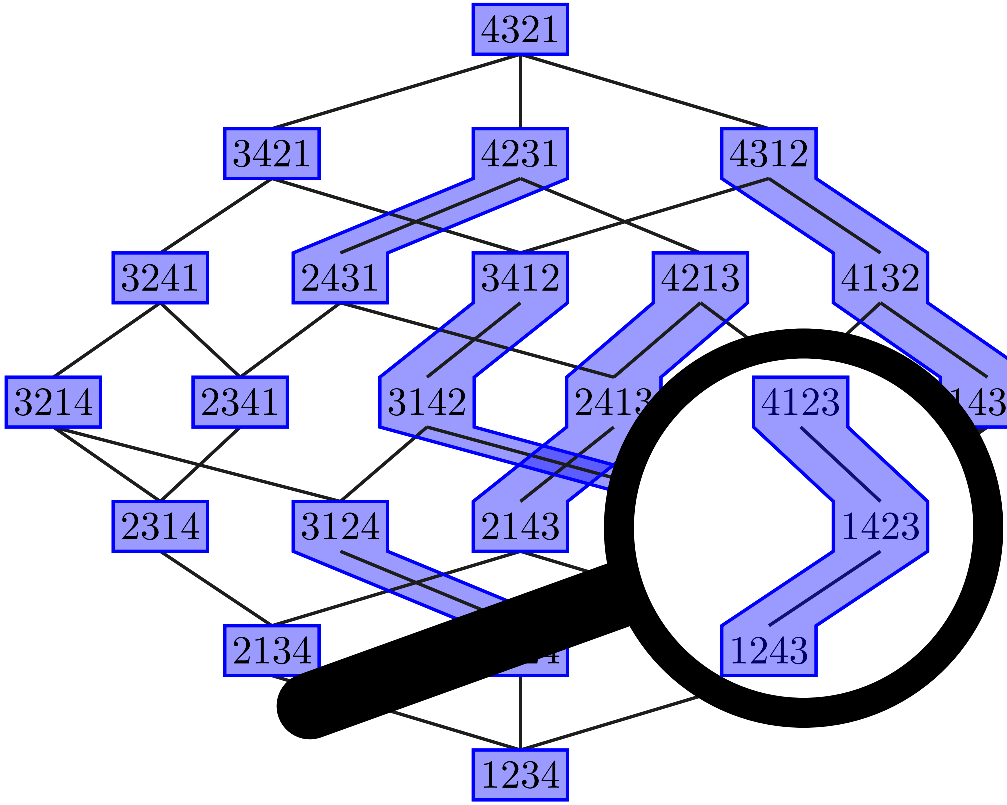

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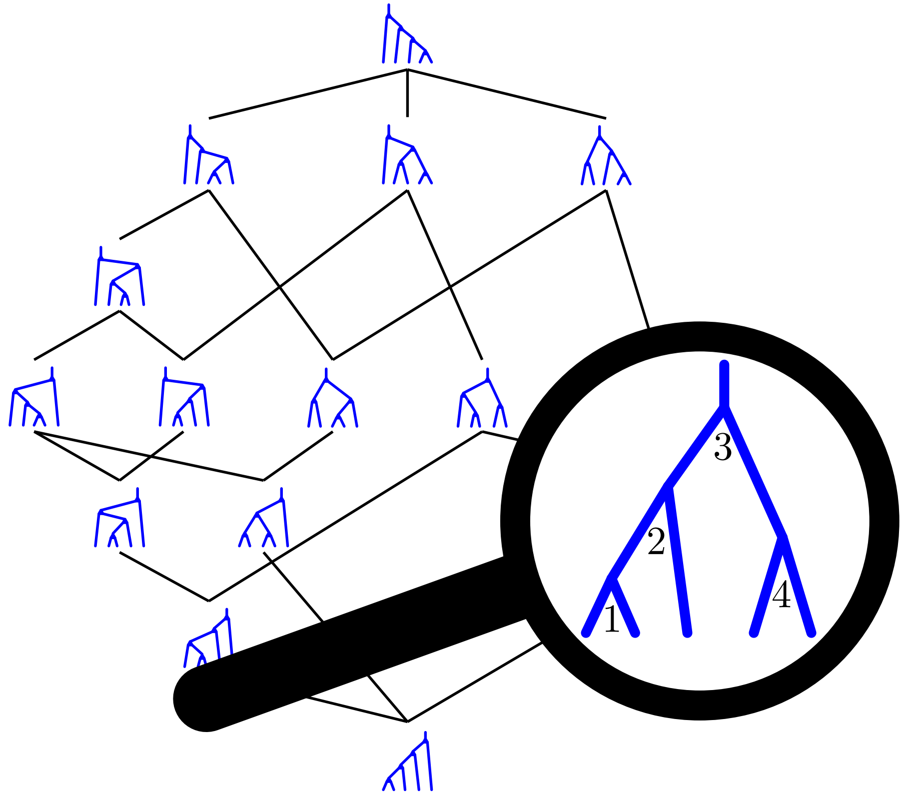

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POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON

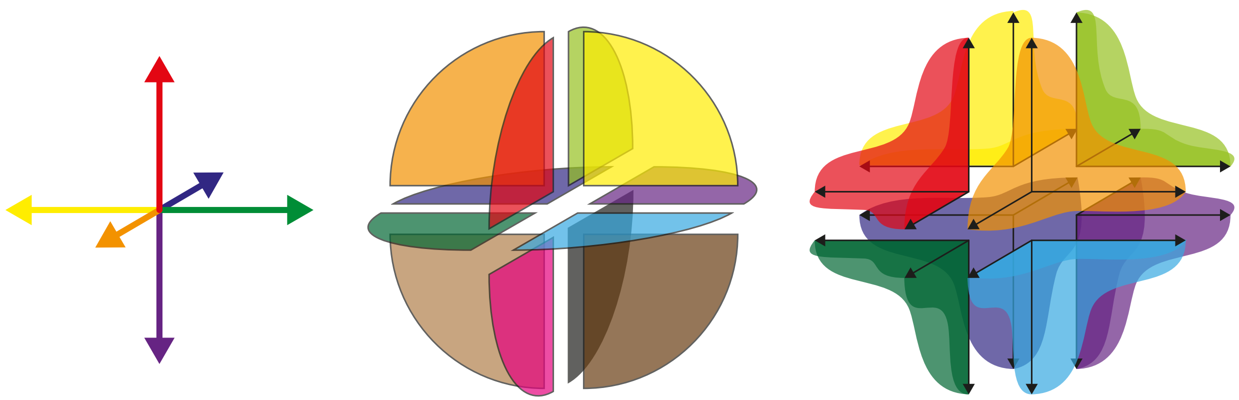
fan = collection of polyhedral cones closed by faces and intersecting along faces

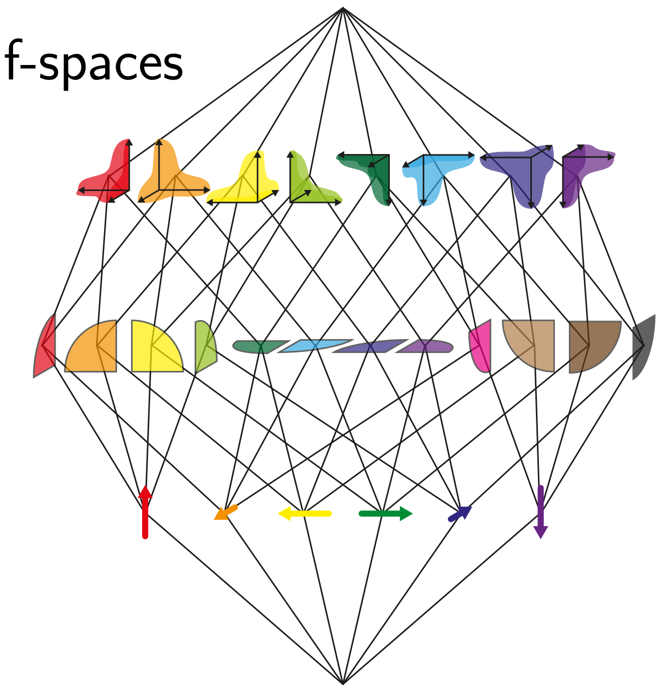
polytope = convex hull of a finite set = intersection of finitely many affine half-space

POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON

<u>polyhedral cone</u> = positive span of a finite set of $\mathbb{R}^n$
= intersection of finitely many linear half-spaces

<u>fan</u> = collection of polyhedral cones closed by faces
and where any two cones intersect along a face

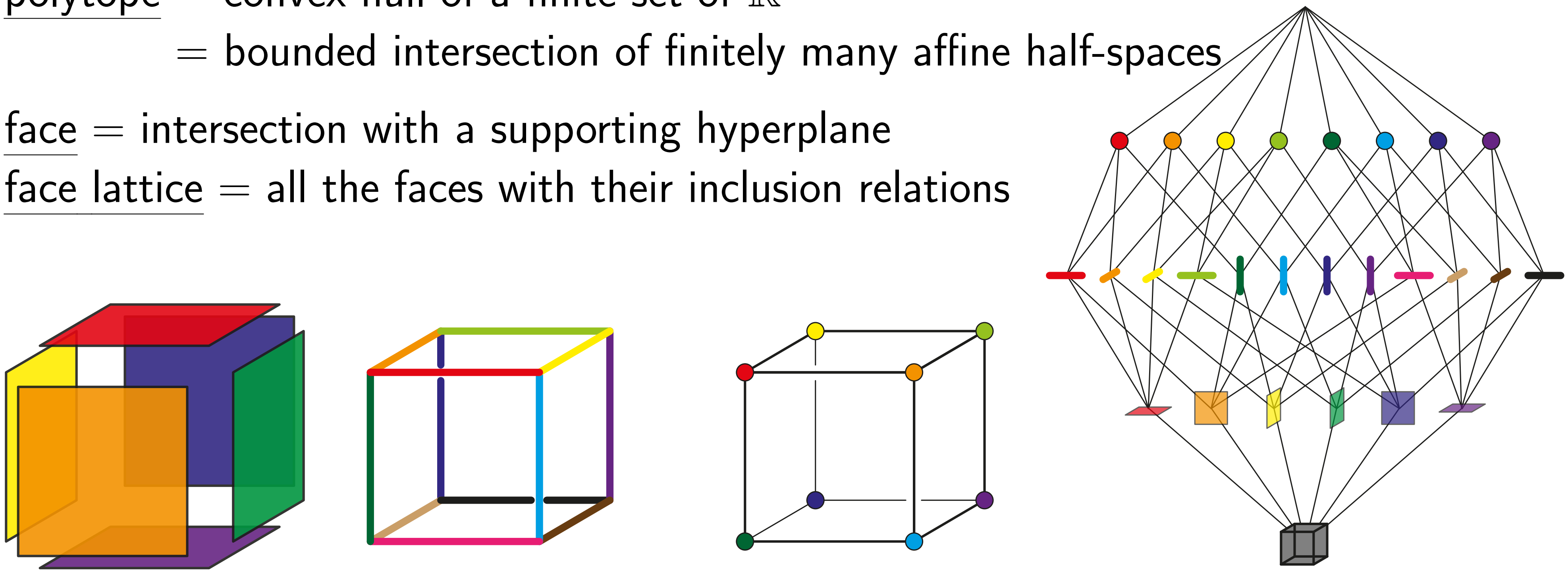




POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON

polytope $=$ convex hull of a finite set of $\mathbb{R}^n$
$=$ bounded intersection of finitely many affine half-spaces

face $=$ intersection with a supporting hyperplane
face lattice $=$ all the faces with their inclusion relations



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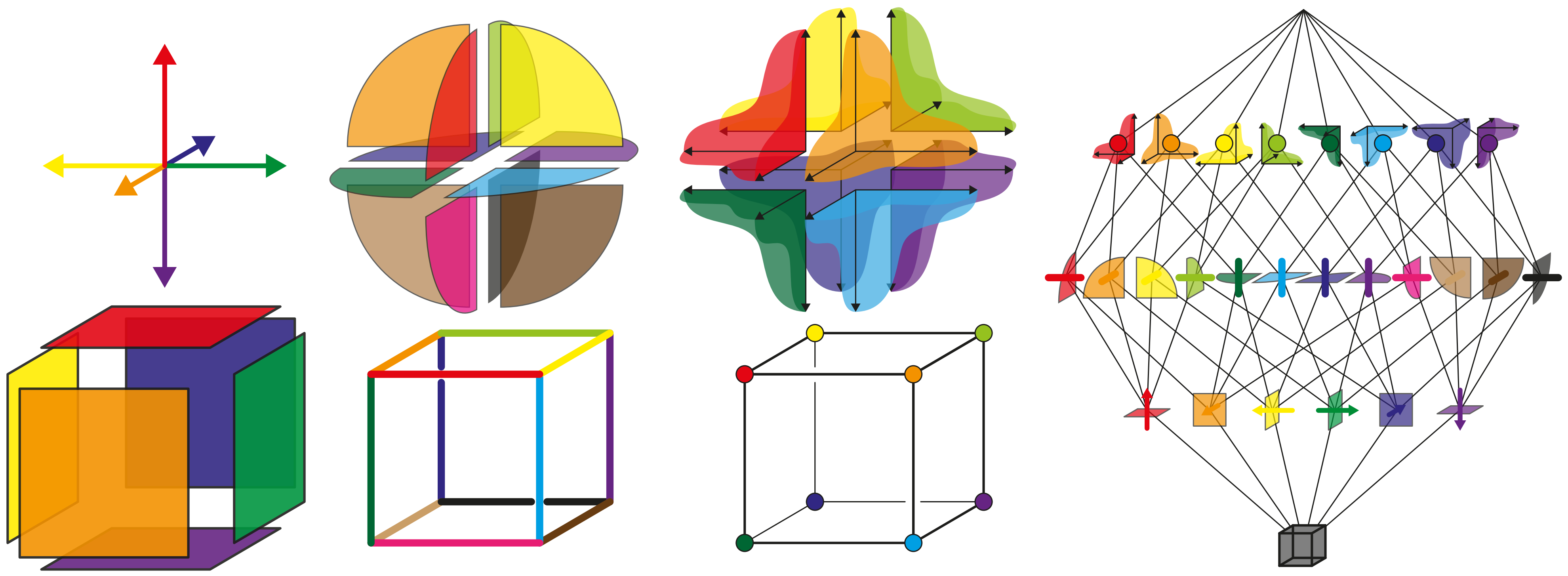

face $\mathbb{F}$ of polytope $\mathbb{P}$

normal cone of $\mathbb{F}$ = positive span of the outer normal vectors of the facets containing $\mathbb{F}$

normal fan of $\mathbb{P} = \{$ normal cone of $\mathbb{F} \mid \mathbb{F}$ face of $\mathbb{P}$ $\}$

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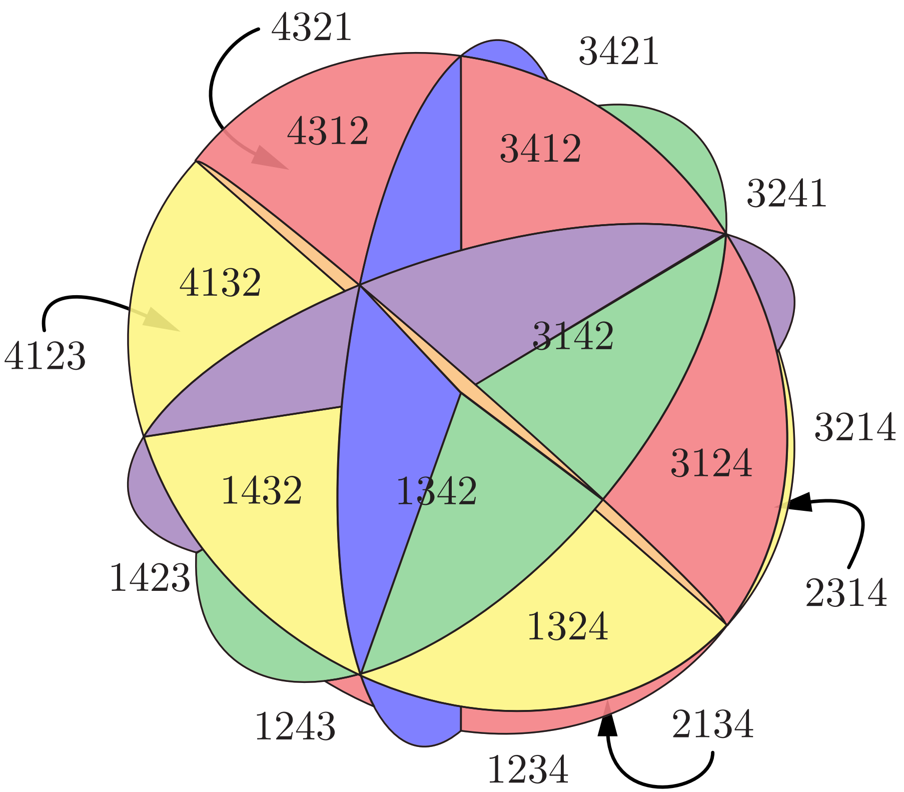

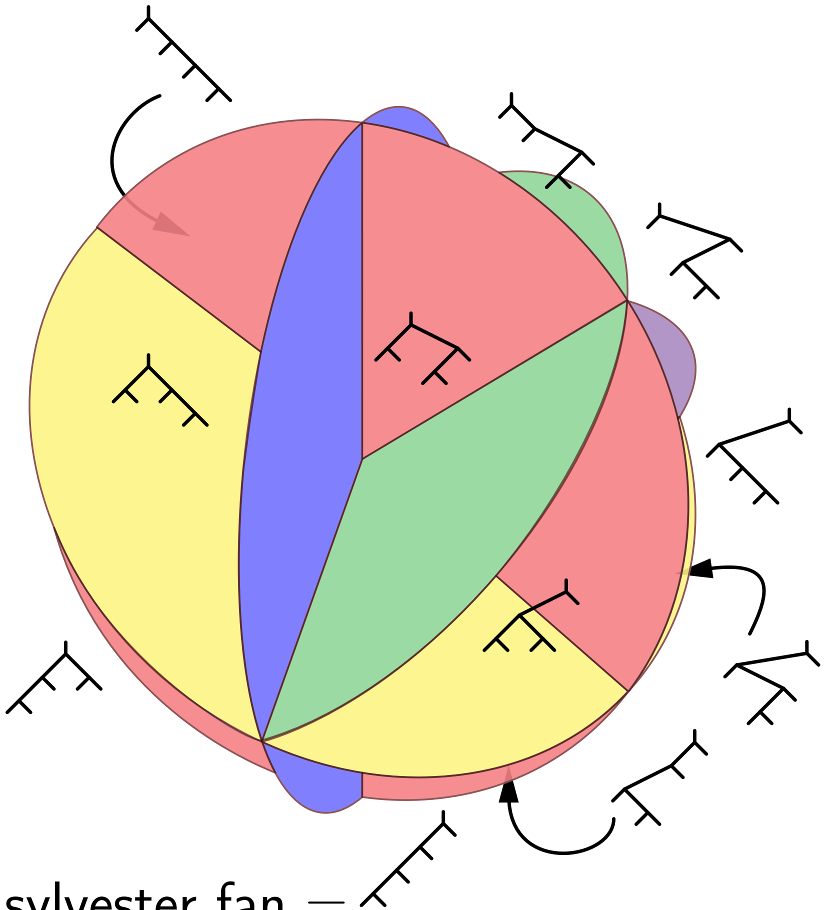

braid fan =
$$\mathbb{C}(\sigma) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_{\sigma(1)} \le \cdots \le x_{\sigma(n)} \right\}$$

sylvester fan =
$$\mathbb{C}(T) = \{ \boldsymbol{x} \in \mathbb{R}^n \mid x_i \le x_j \text{ if } i \to j \text{ in } T \}$$

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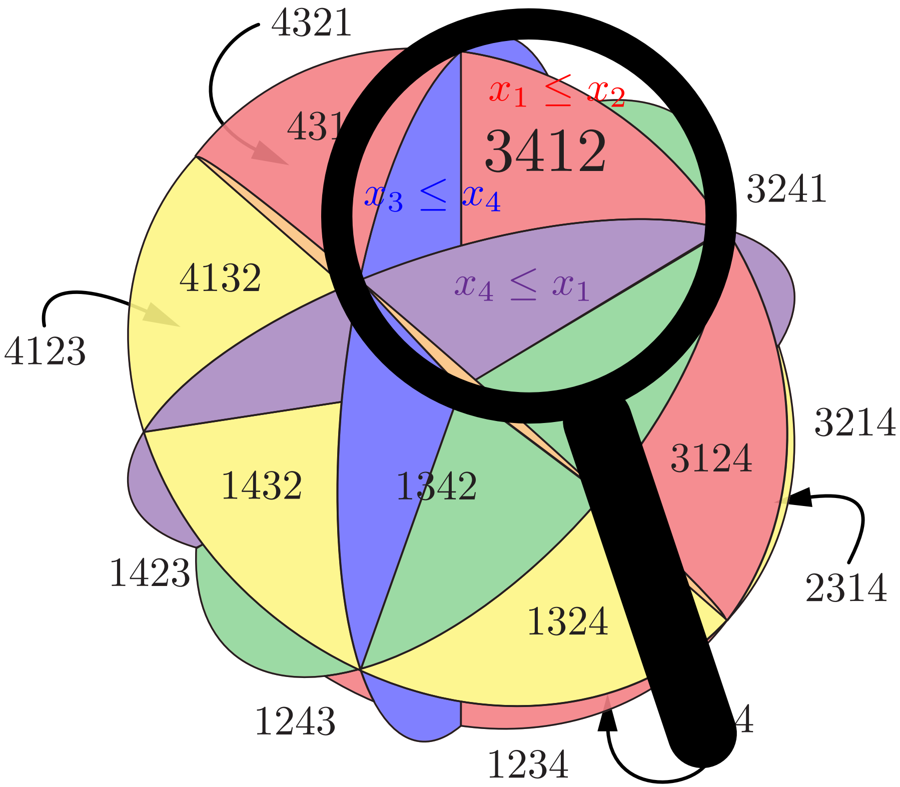

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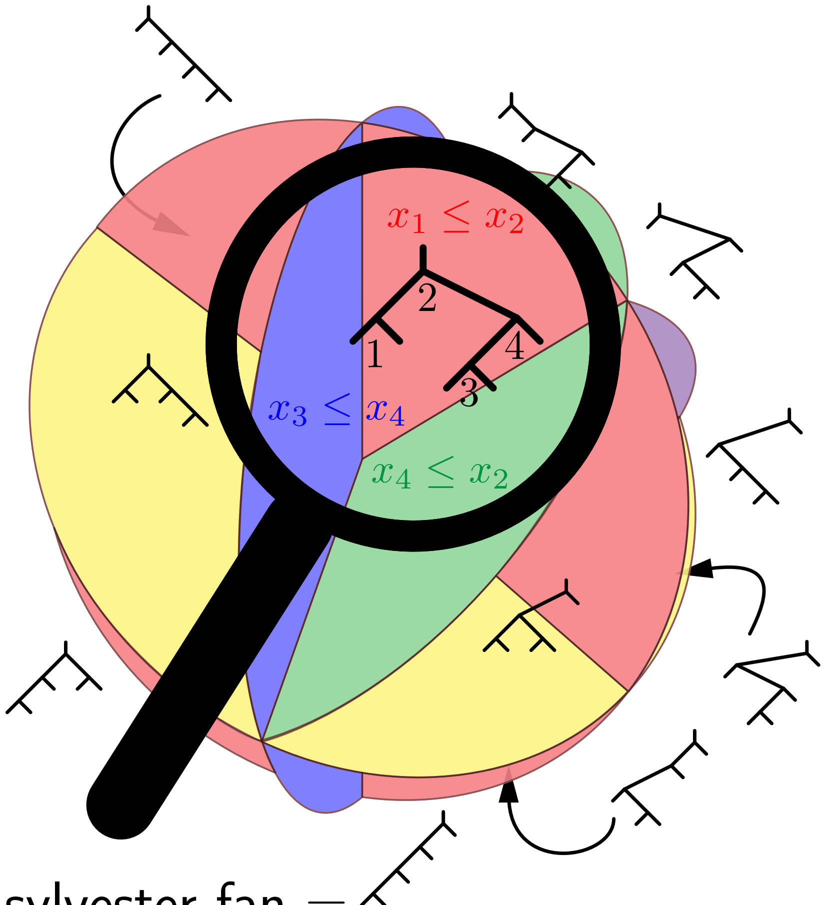

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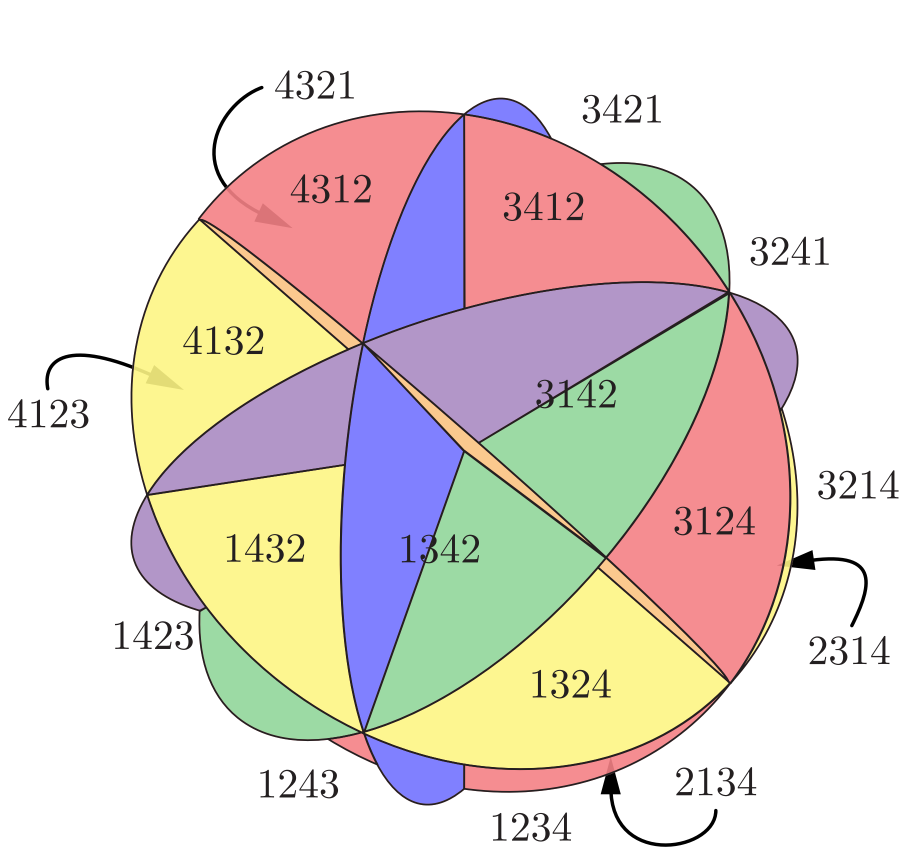

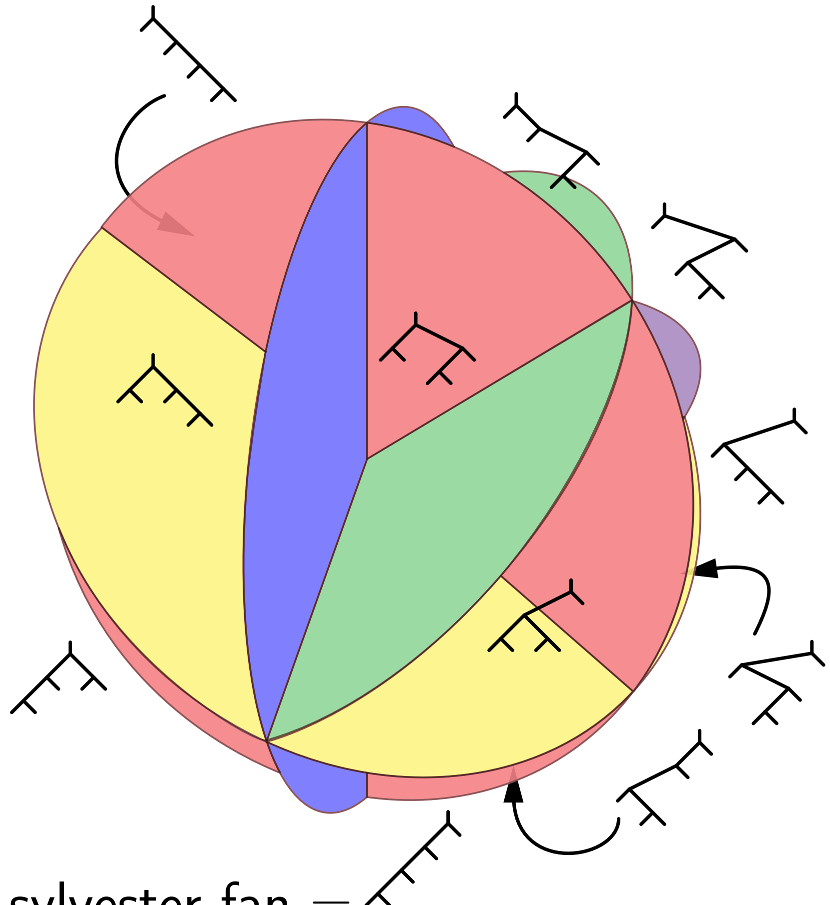

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quotient fan = $\mathbb{C}(T)$ obtained by glueing $\mathbb{C}(\sigma)$ for all σ in the same BST insertion fiber

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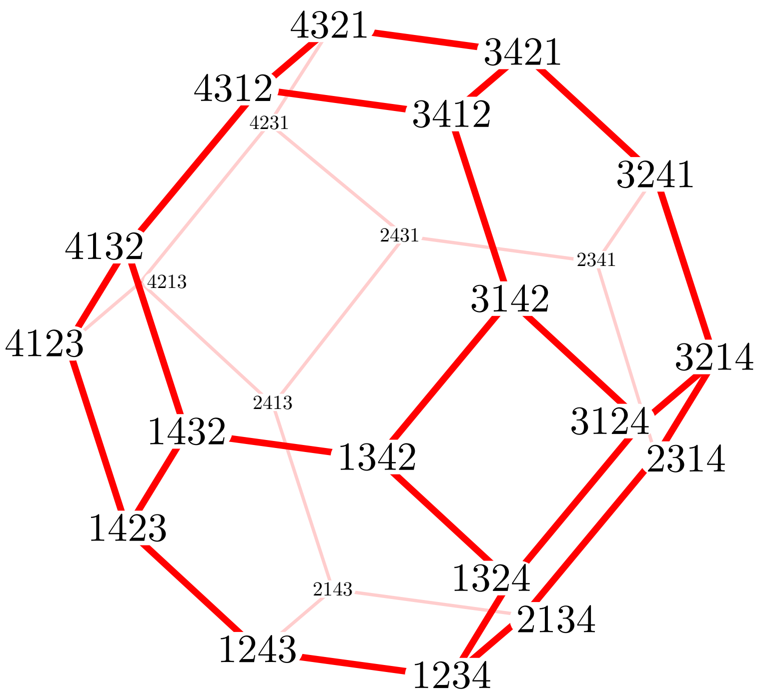

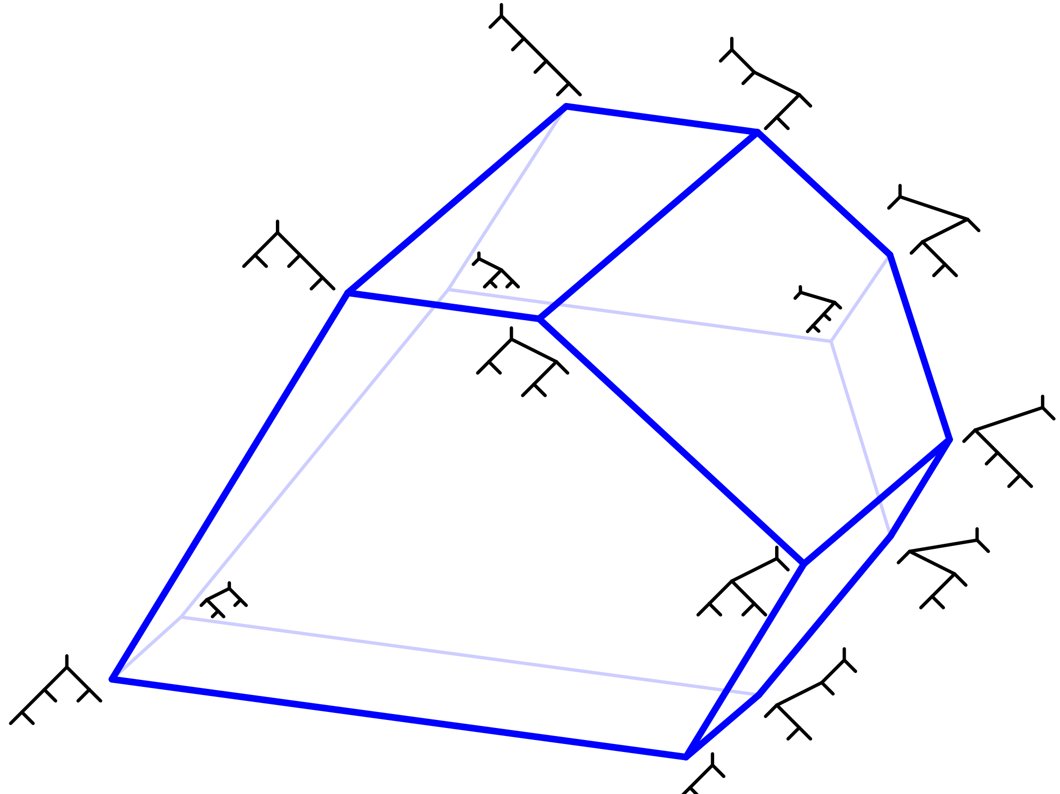

permutahedron $\mathbb{P}\mathrm{erm}(n)$

$$= \operatorname{conv}\left\{ [\sigma^{-1}(i)]_{i\in[n]} \;\middle|\; \sigma \in \mathfrak{S}_n \right\}$$

$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n]} \mathbb{H}_J$$

where $\mathbb{H}_J = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \sum_{j\in J} x_j \geq \binom{|J|+1}{2} \right\}$

associahedron $\mathbb{A}\mathrm{sso}(n)$

$$= \operatorname{conv}\left\{ [\ell(T,i)\cdot r(T,i)]_{i\in[n]} \;\middle|\; T \text{ binary tree} \right\}$$

$$= \mathbb{H} \cap \bigcap_{1\leq i<j\leq n} \mathbb{H}_{[i,j]}$$

Stasheff ('63)
Shnider–Sternberg ('93)
Loday ('04)

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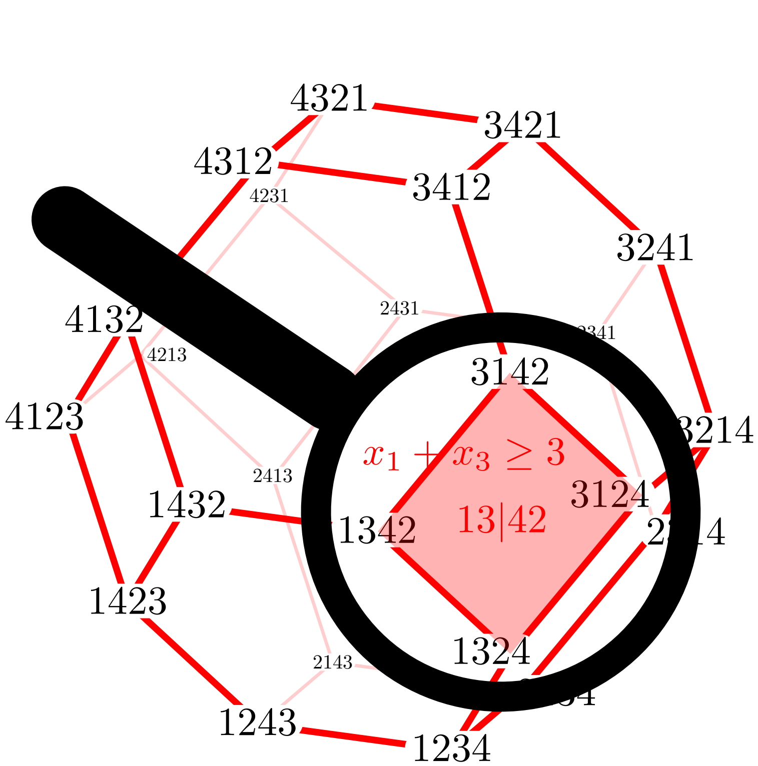

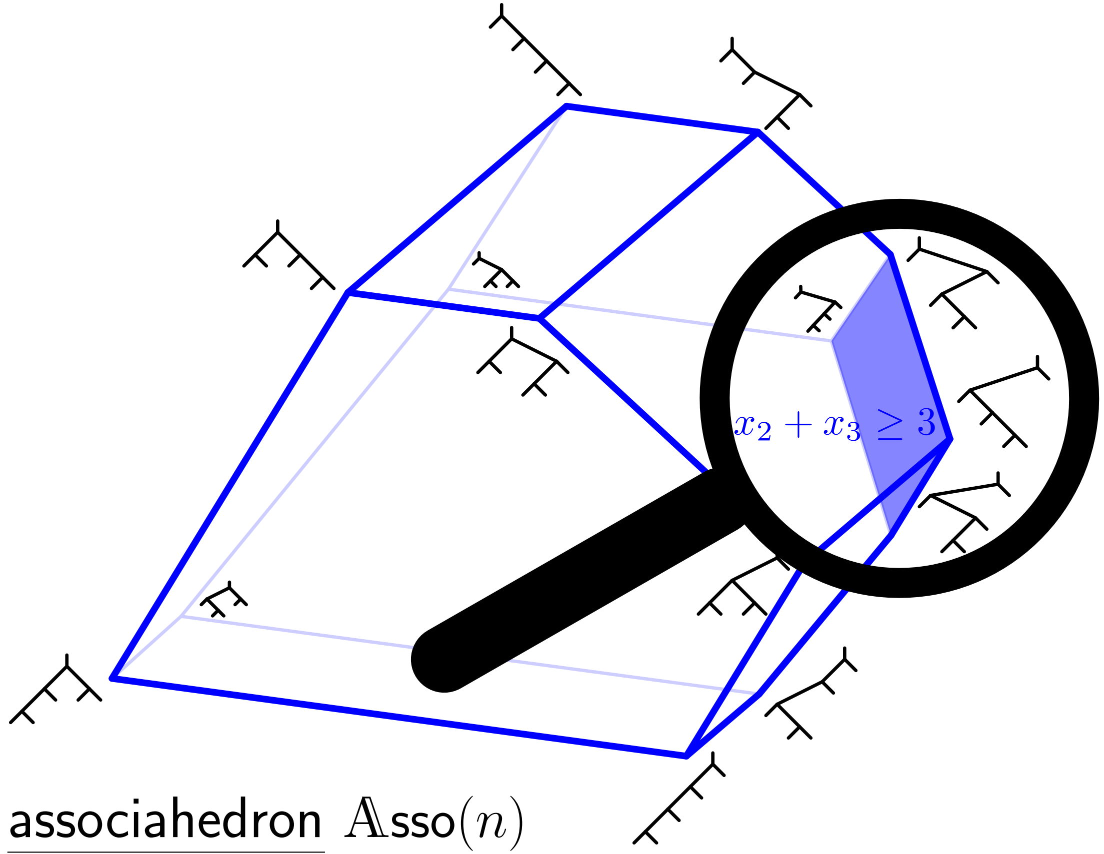

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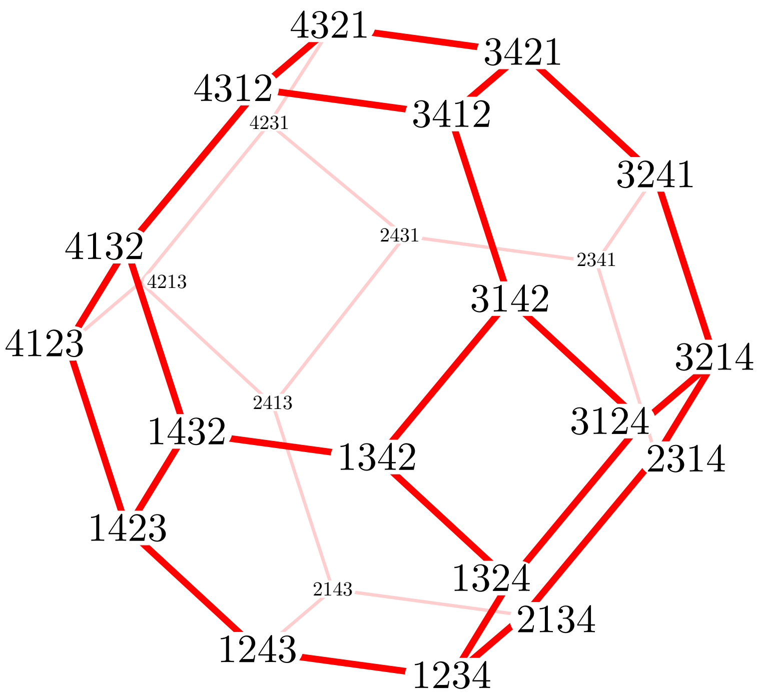

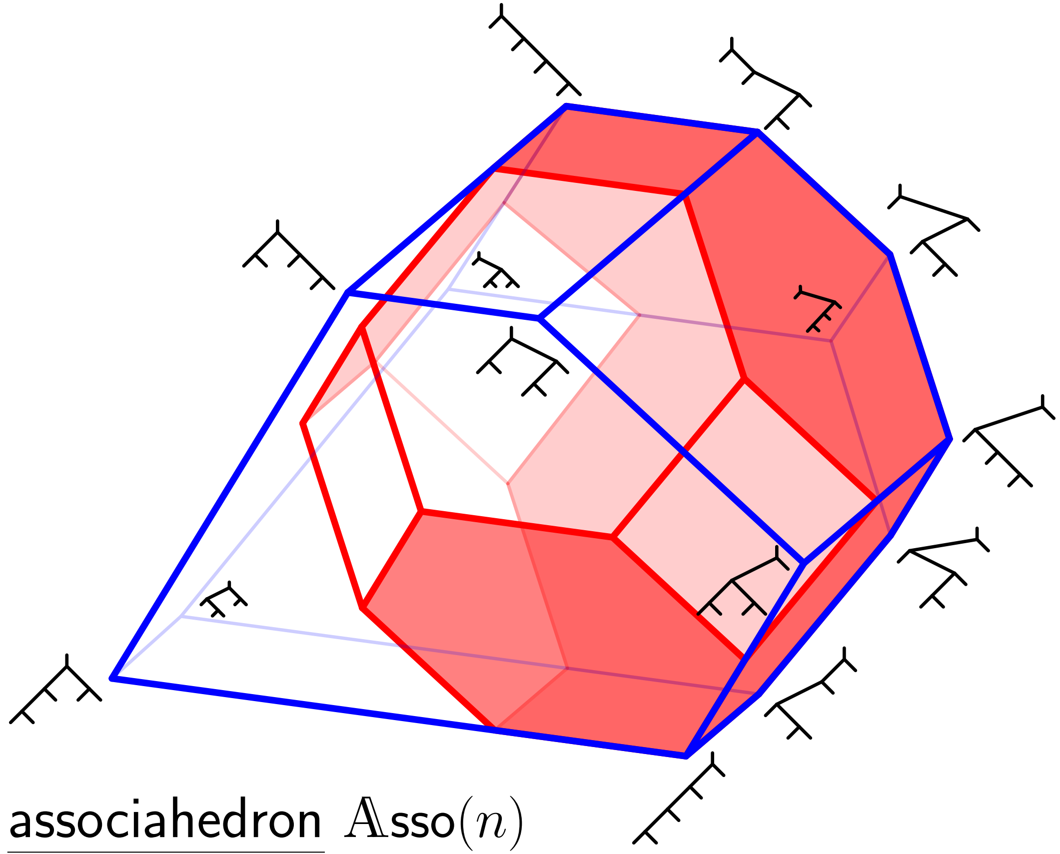

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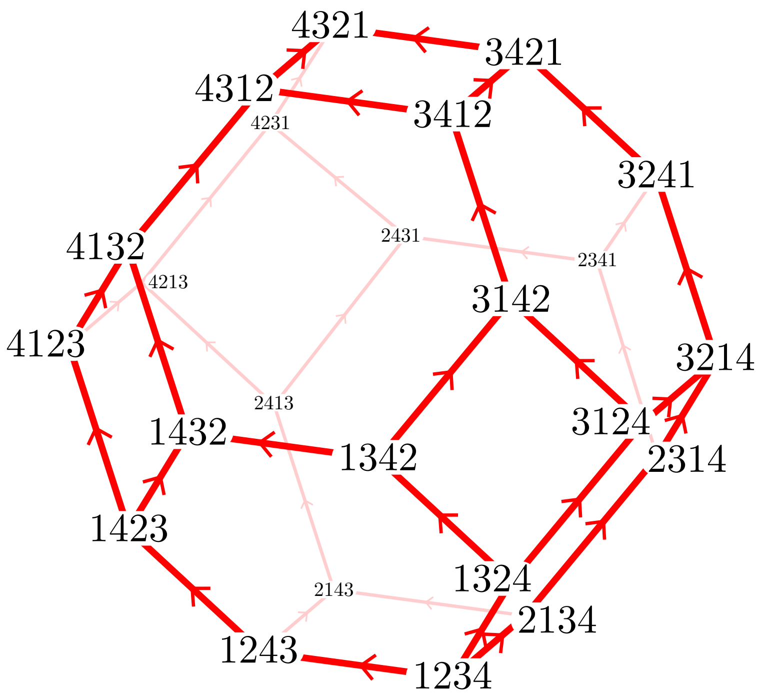

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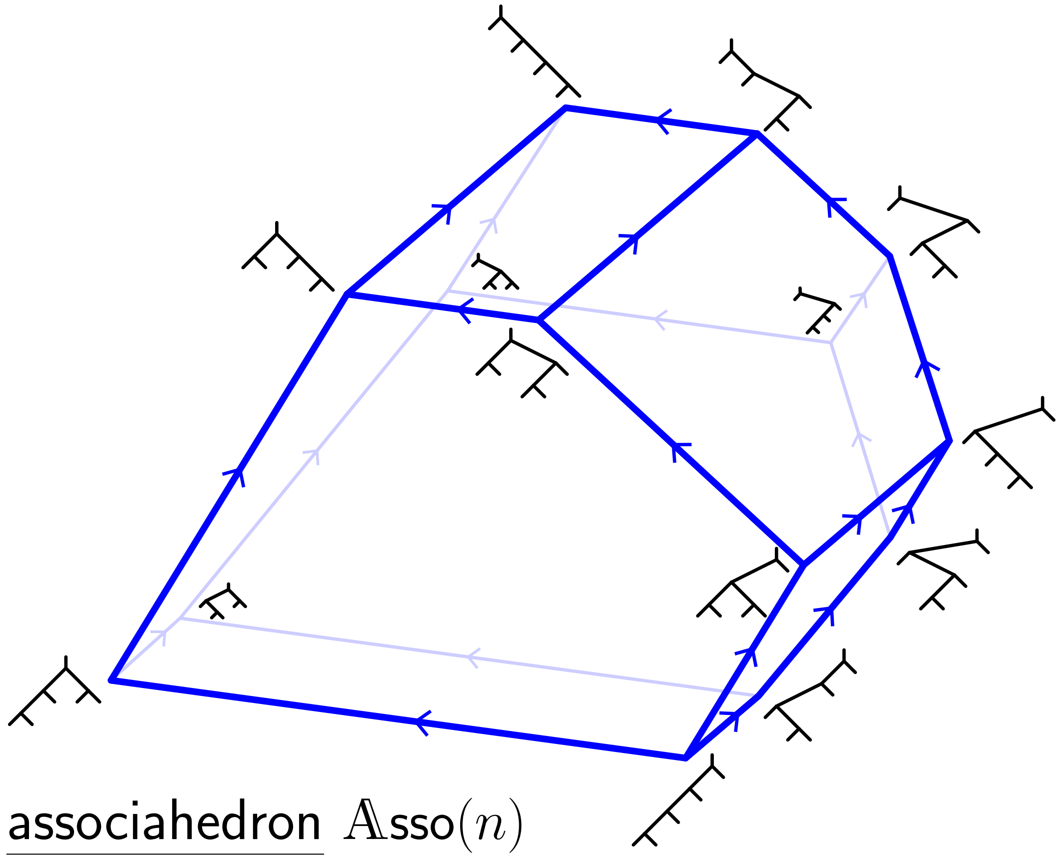
$\Longrightarrow$ weak order on permutations



associahedron $\mathbb{A}\text{sso}(n)$

$\Longrightarrow$ Tamari lattice on binary trees

Hasse diagram of	weak order = graph of Tamari lattice	permutahedron oriented associahedron	$12\dots n \to n\dots 21$ left $\to$ right comb

HOPF ALGEBRAS: MALVENUTO–REUTENAUER AND LODAY–RONCO

product $=$ linear map $\cdot : V \otimes V \to V$ $=$ a tool to combine two elements (glue)
coproduct $=$ linear map $\triangle : V \to V \otimes V$ $=$ a tool to decompose an element (scisors)
Hopf algebra $= (V, \cdot, \triangle)$ such that $\triangle(a \cdot b) = \triangle(a) \cdot \triangle(b)$

Two operations on permutations:

shuffle $12 \sqcup\!\sqcup 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$

convol. $12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$

	Malvenuto–Reutenauer	$\supseteq$	Loday–Ronco
vector space	$\langle\, \mathbb{F}_\sigma \mid \sigma$ permutation of any size $\rangle$		$\langle\, \mathbb{P}_T \mid T$ binary tree of any size $\rangle$
product	$\mathbb{F}_\rho \cdot \mathbb{F}_\sigma = \sum_{\tau \in \rho \sqcup\!\sqcup \sigma} \mathbb{F}_\tau = \sum_{\rho \backslash \sigma \le \tau \le \rho / \sigma} \mathbb{F}_\tau$		$\mathbb{P}_R \cdot \mathbb{P}_S = \sum_{R \backslash S \le \tau \le R / S} \mathbb{P}_T$
coproduct	$\triangle(\mathbb{F}_\tau) = \sum_{\tau \in \rho \star \sigma} \mathbb{F}_\rho \otimes \mathbb{F}_\sigma$		$\triangle(\mathbb{P}_T) = \sum_{\substack{R_1 \cdots R_k \| S \\ \text{cut of } T}} \Big(\prod_{i \in [k]} \mathbb{P}_{R_i}\Big) \otimes \mathbb{P}_S$

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Hopf subalgebra $=$ define $\mathbb{P}_T = \sum_\tau \mathbb{F}_\tau$ over all permutations τ in the BST fiber of T

LATTICE THEORY OF THE WEAK ORDER

DISTRIBUTIVE AND SEMIDISTRIBUTIVE LATTICES

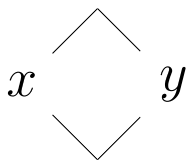
lattice $=$ poset $(L, \leq)$ with a meet $\wedge$ and a join $\vee$

$(L, \leq, \wedge, \vee)$ finite lattice is

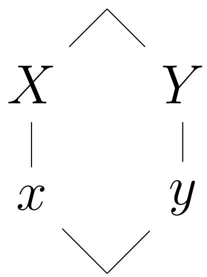
- distributive if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any $x, y, z \in L$

- join semidistributive if $x \vee y = x \vee z$ implies $x \vee (y \wedge z) = x \vee y$ for any $x, y, z \in L$

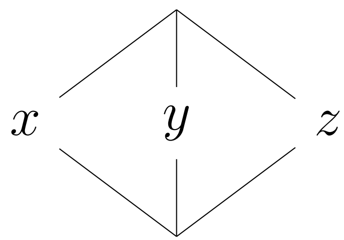
- semidistributive if both join and meet semidistributive



distributive



semidistributive



not semidistributive

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$(L, \leq, \wedge, \vee)$ finite lattice is

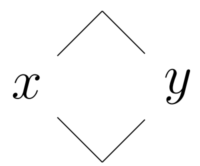
- distributive if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any $x, y, z \in L$

  $\Longrightarrow$ any $y \in L$ is represented as $y = \bigvee_{j \in J} j$ where $J = \{$join irreducibles below $y\}$

- join semidistributive if $x \vee y = x \vee z$ implies $x \vee (y \wedge z) = x \vee y$ for any $x, y, z \in L$

  $\Longrightarrow$ any $y \in L$ admits a canonical join representation $y = \bigvee_{x \lessdot y} k_\vee(x, y)$
  where $k_\vee(x, y)$ is the unique minimal element of $\{z \in L \mid x \vee z = y\}$

- semidistributive if both join and meet semidistributive



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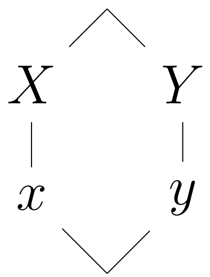
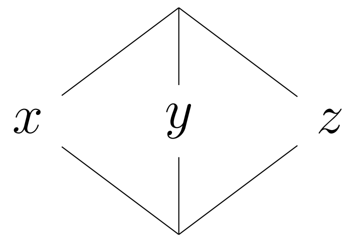


semidistributive



not semidistributive

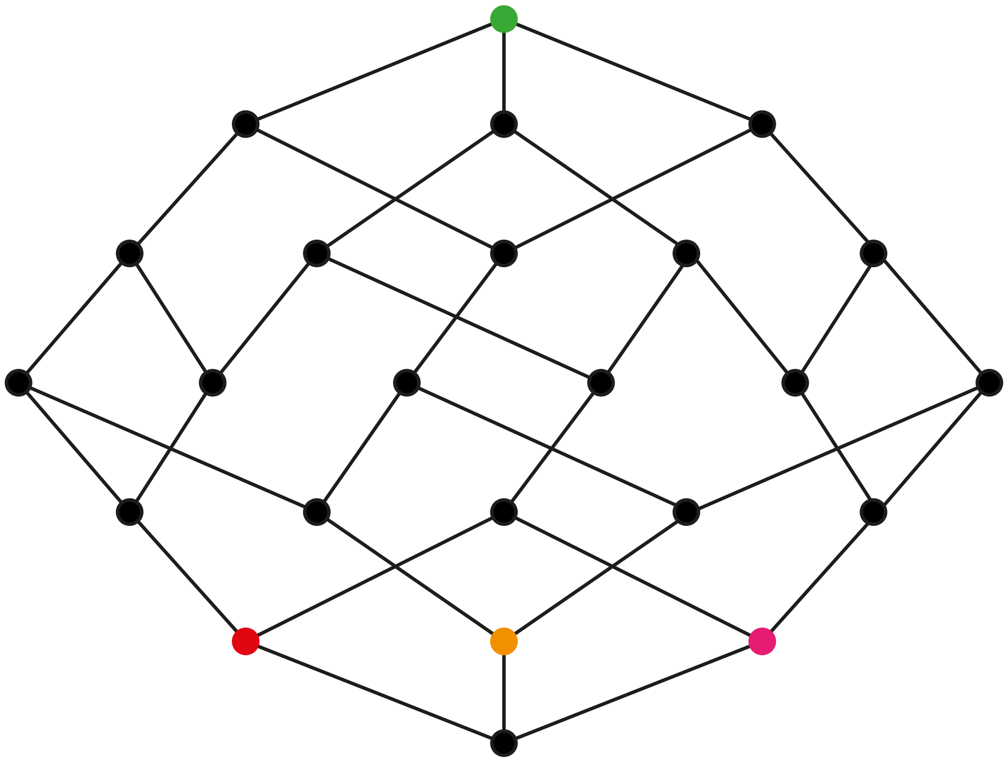
CANONICAL JOIN REPRESENTATIONS

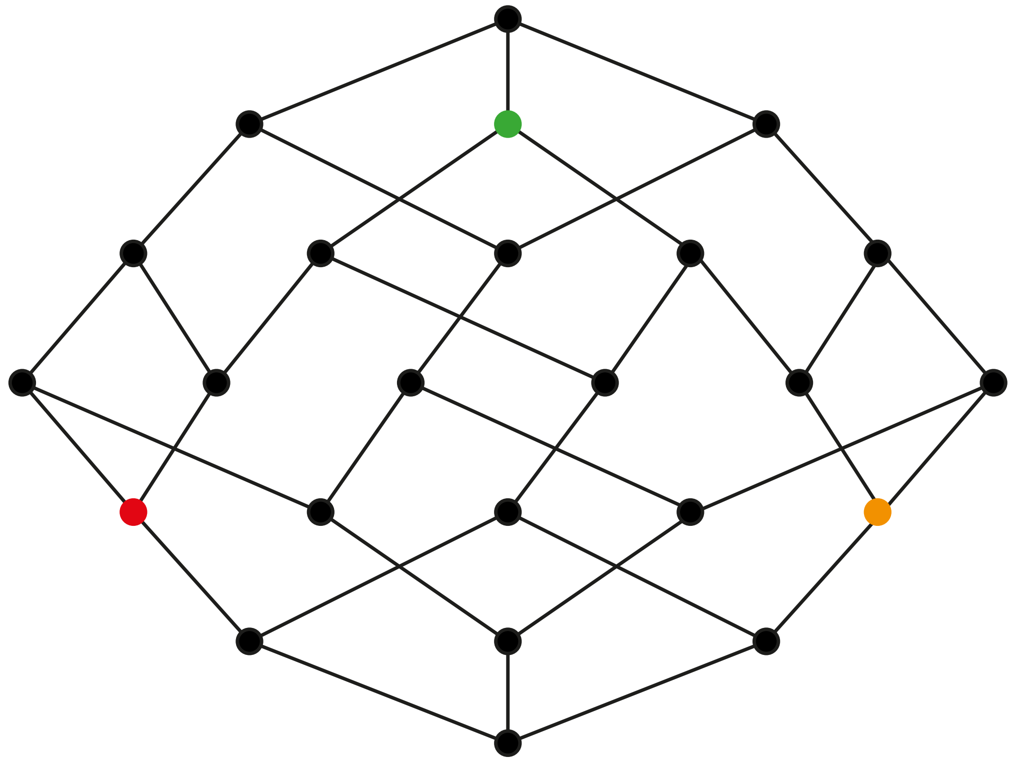
join representation of $y \in L$ = subset $J \subseteq L$ such that $y = \bigvee J$.

$y = \bigvee J$ irredundant if $\nexists\, J' \subsetneq J$ with $y = \bigvee J'$

JR are ordered by containement of order ideals: $J \leq J' \iff \forall\, z \in J,\ \exists\, z' \in J',\ z \leq z'$

canonical join representation of y = minimal irred. join representation of y (if it exists)





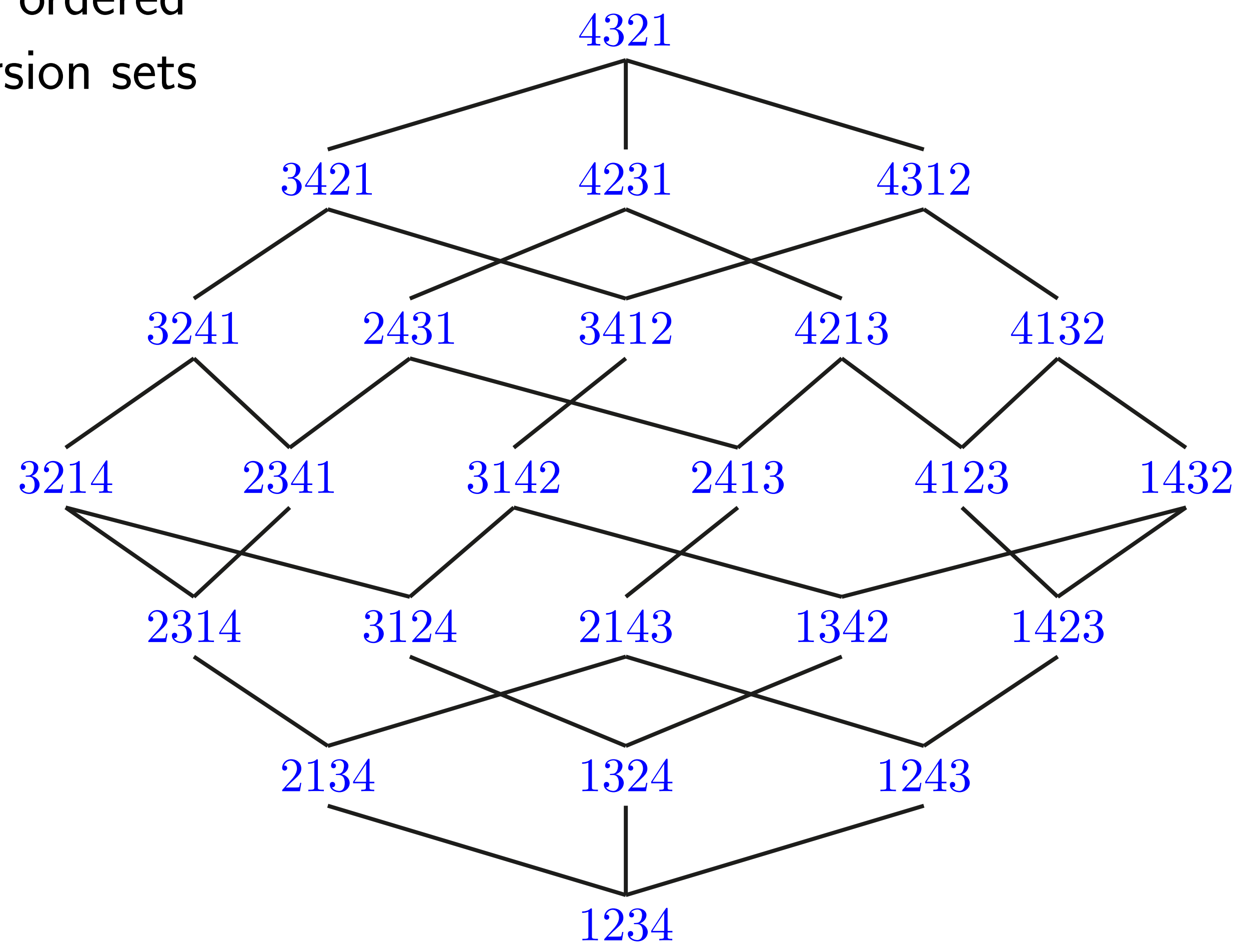
$\Longrightarrow$ "lowest way to write y as a join"

CANONICAL JOIN REPRESENTATIONS

σ permutation

inversions of σ = pair (σ_i, σ_j) such that $i < j$ and $\sigma_i > \sigma_j$

weak order = permutations of $\mathfrak{S}_n$ ordered
by inclusion of inversion sets



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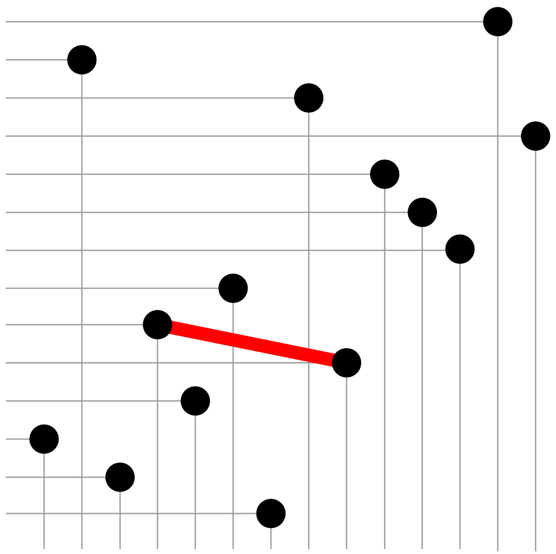
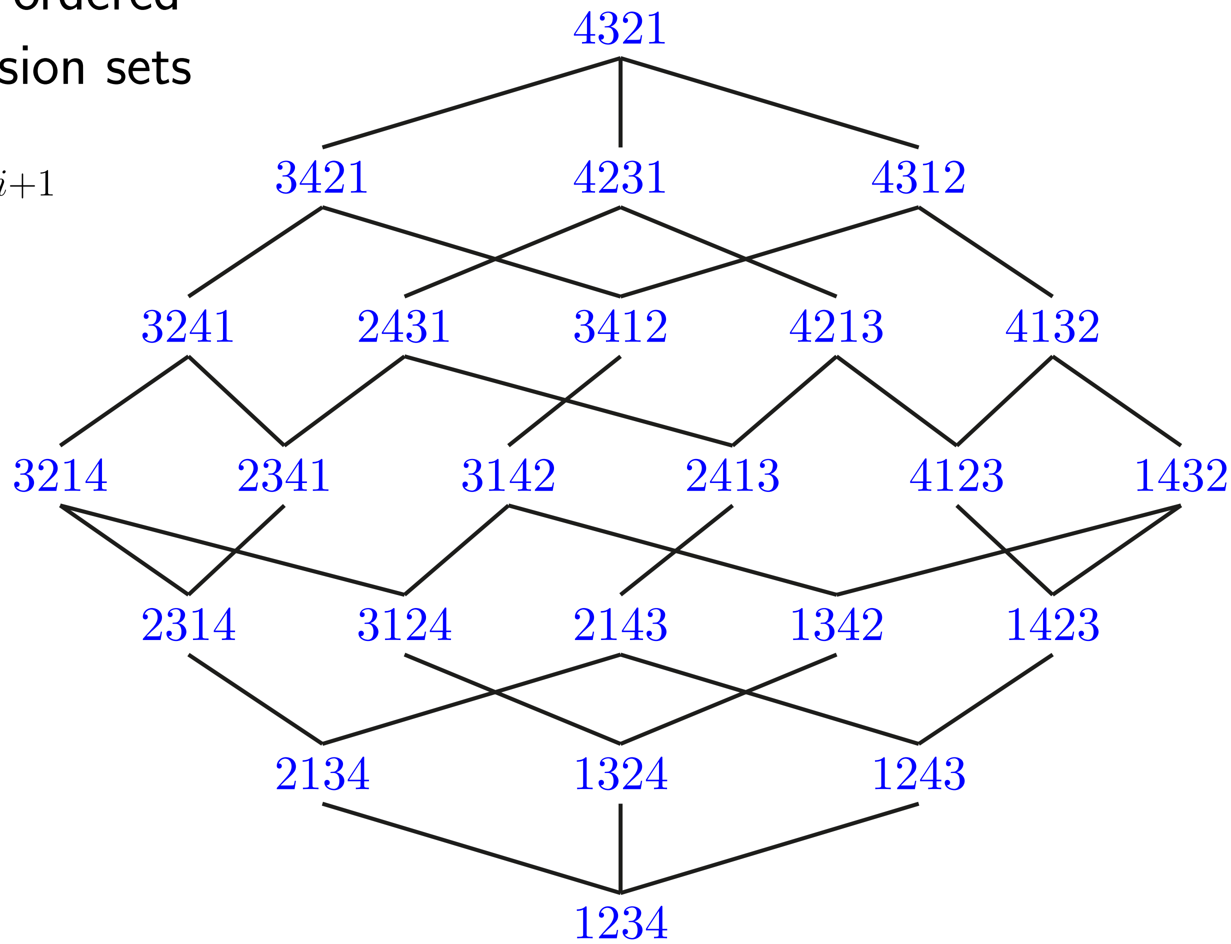
$\sigma =$ 



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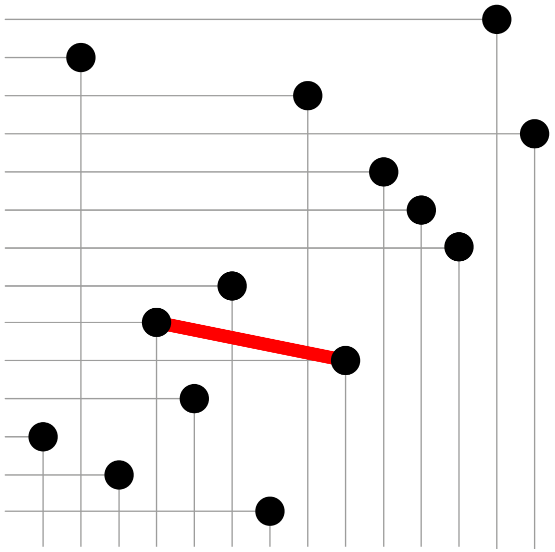
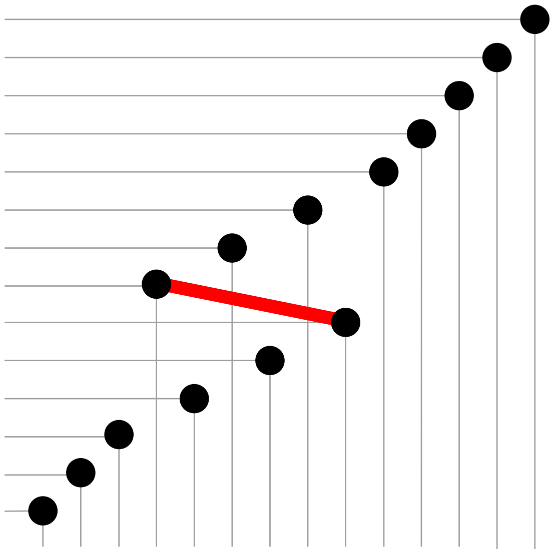
join-irreducible $\lambda(\sigma, i)$

$\sigma =$  $\longrightarrow$ $\lambda(\sigma, i) =$ 

THM. Canonical join representation of $\sigma = \bigvee_{\sigma_i > \sigma_{i+1}} \lambda(\sigma, i)$. Reading ('15)

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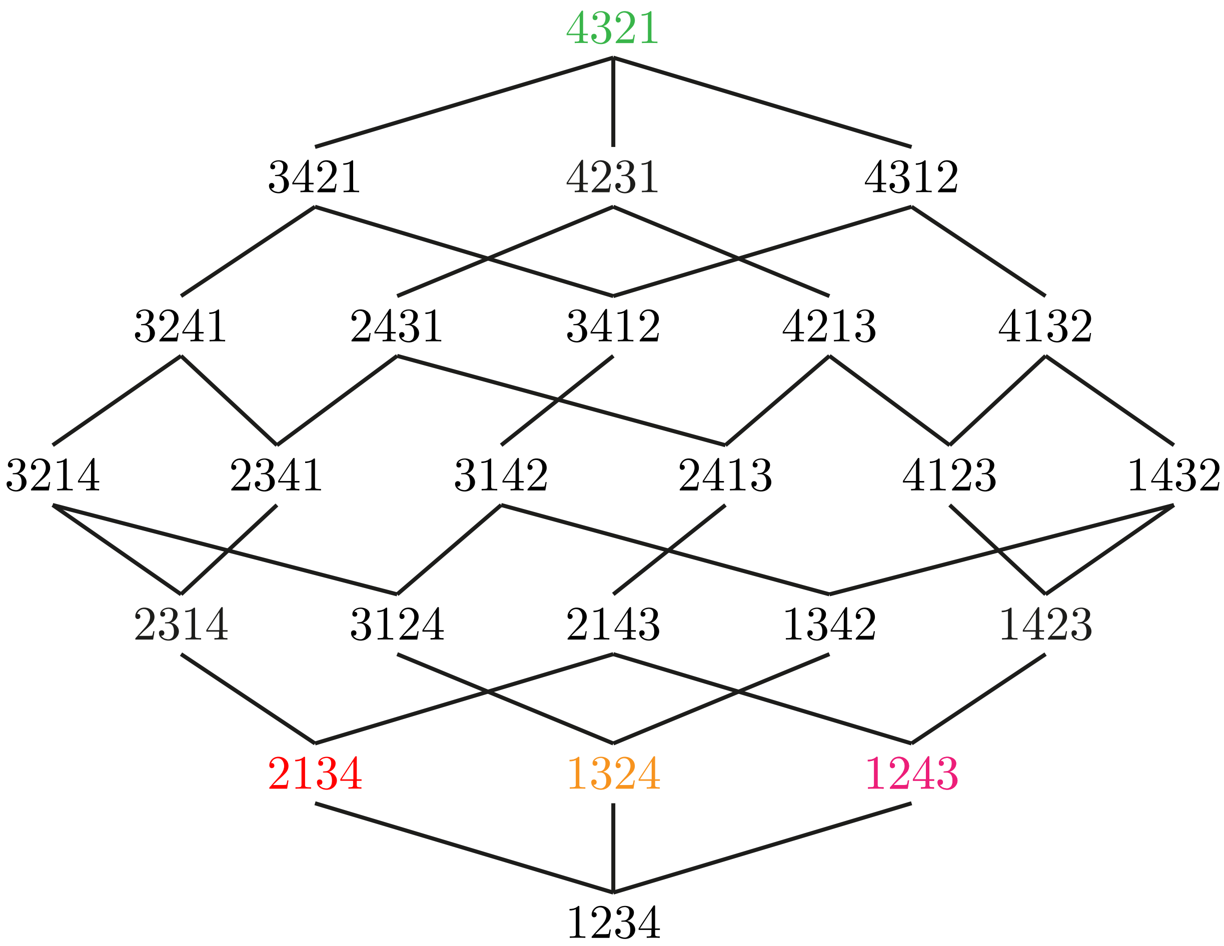

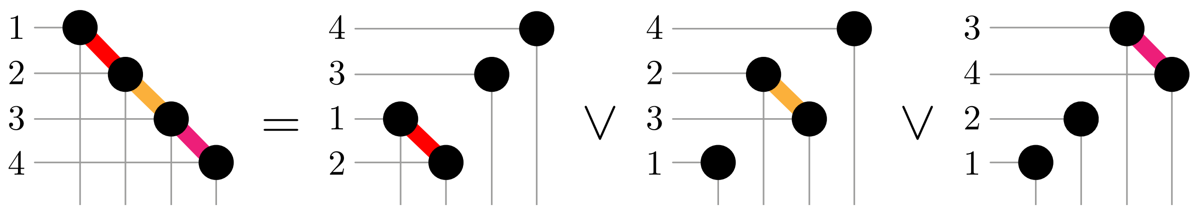

CANONICAL JOIN REPRESENTATIONS

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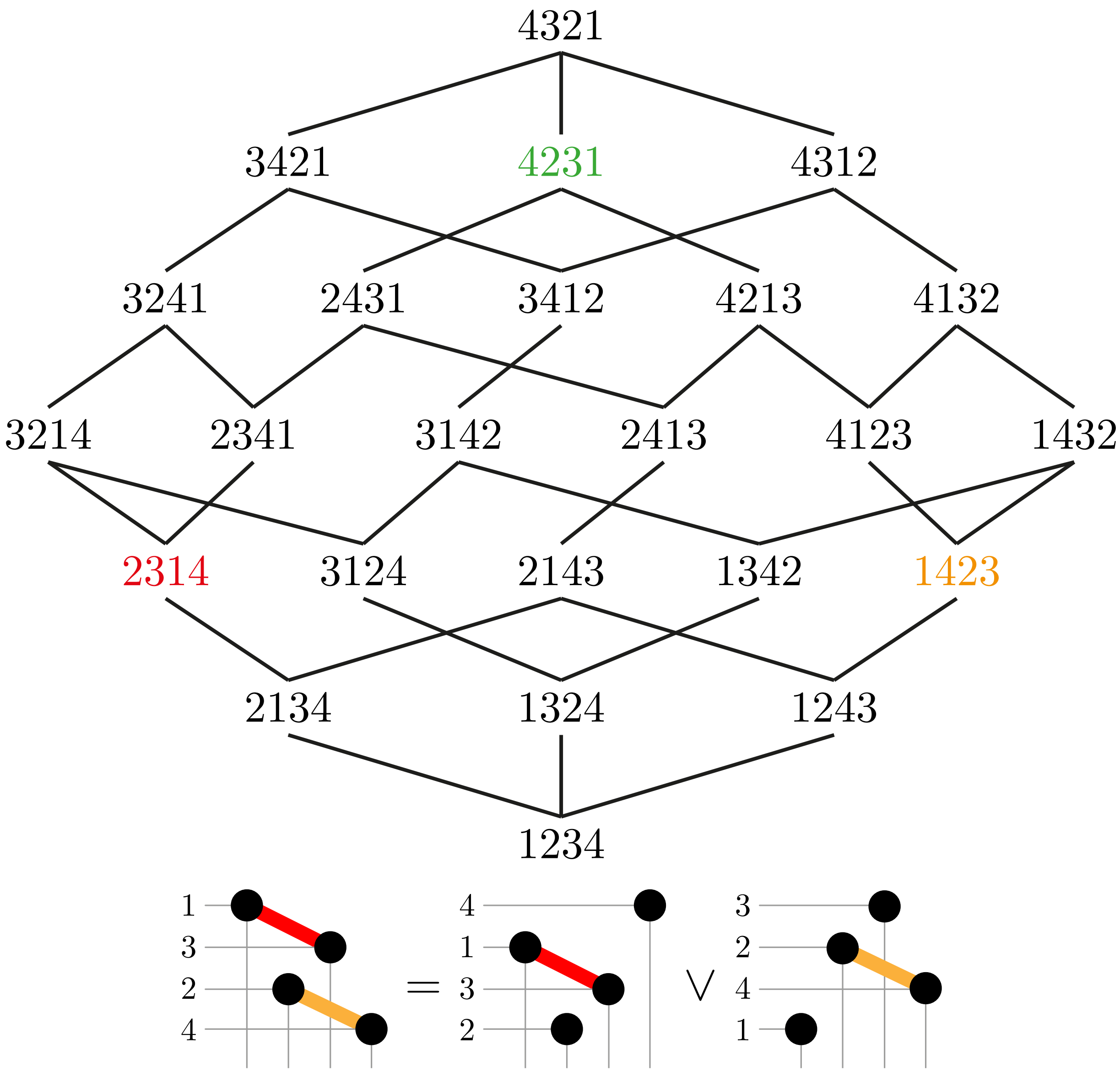
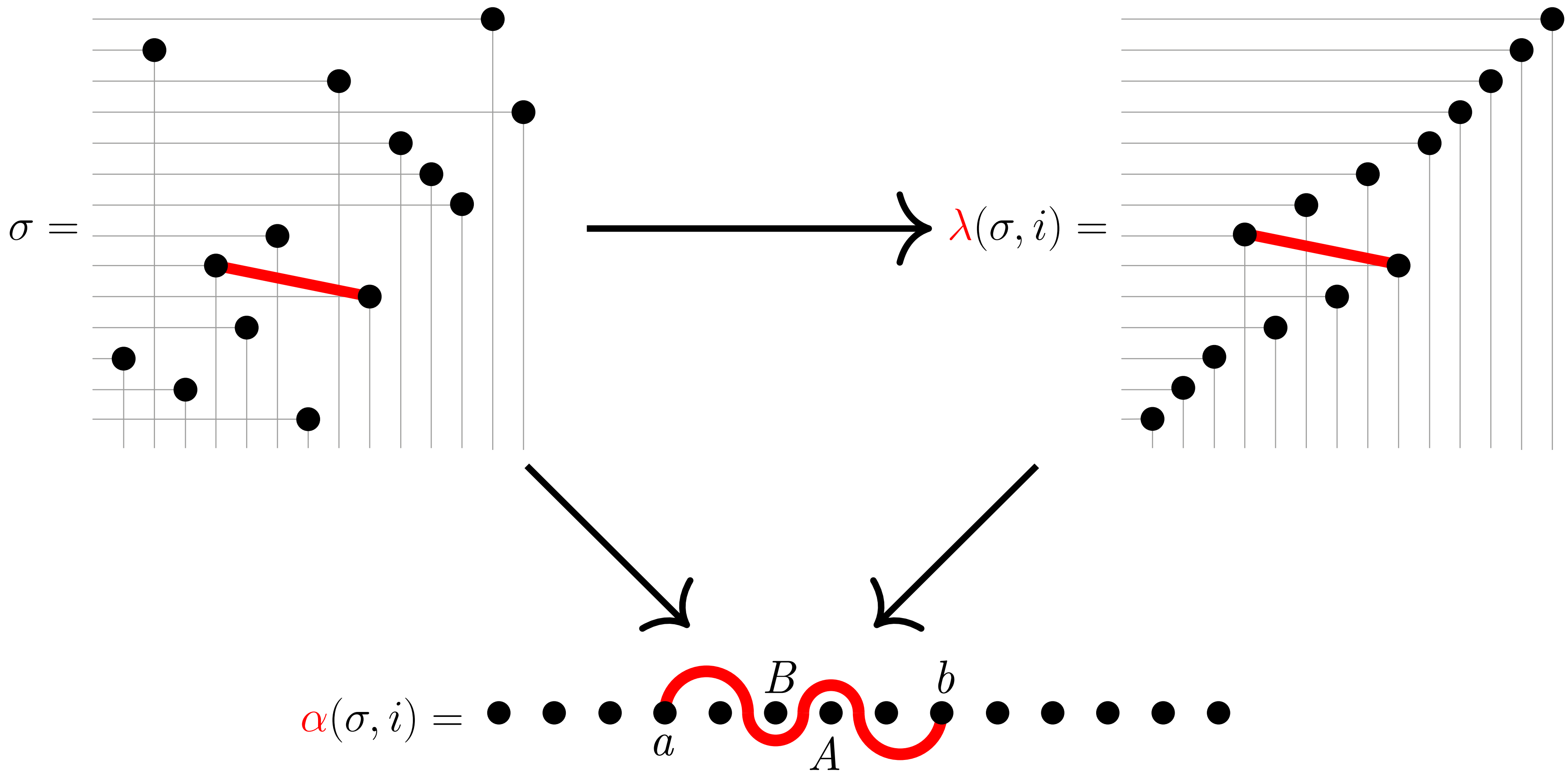


ARCS



$\underline{\text{arc}} = (a, b, A, B)$ with $1 \leq a < b \leq n$ and $A \sqcup B =]a, b[$

Reading ('15)

FROM PERMUTATIONS TO NONCROSSING ARC DIAGRAMS

$\sigma = 2537146$

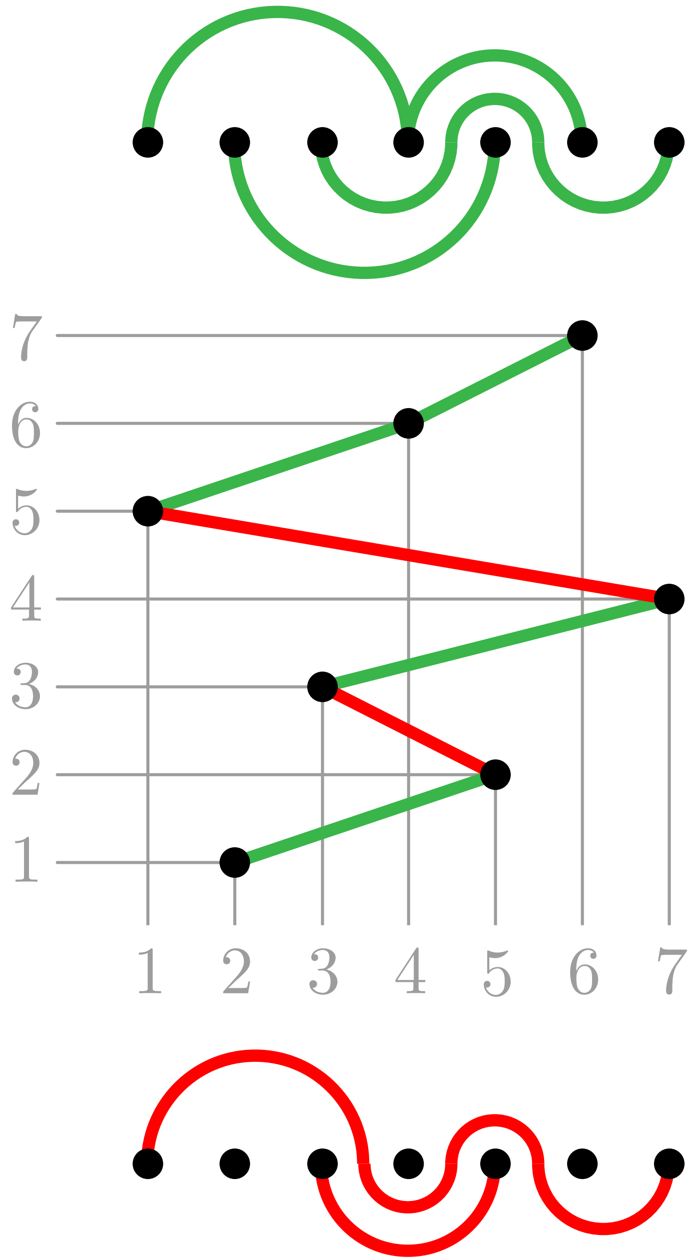
draw the table of points (σ_i, i)
draw all arcs (σ_i, i) — $(\sigma_{i+1}, i+1)$ with
descents in red and ascent in green

project down the red arcs and up the green arcs
allowing arcs to bend but not to cross or pass points

$\delta(\sigma)$ = projected red arcs
$\delta(\sigma)$ = projected green arcs

noncrossing arc diagrams = set $\mathcal{D}$ of arcs st. $\forall\, \alpha, \beta \in \mathcal{D}$:

- left$(\alpha) \neq$ left(β) and right$(\alpha) \neq$ right(β),
- α and β are not crossing.



THM. $\sigma \to \delta(\sigma)$ and $\sigma \to \delta(\sigma)$ are bijections from permutations to noncrossing arc diagrams.

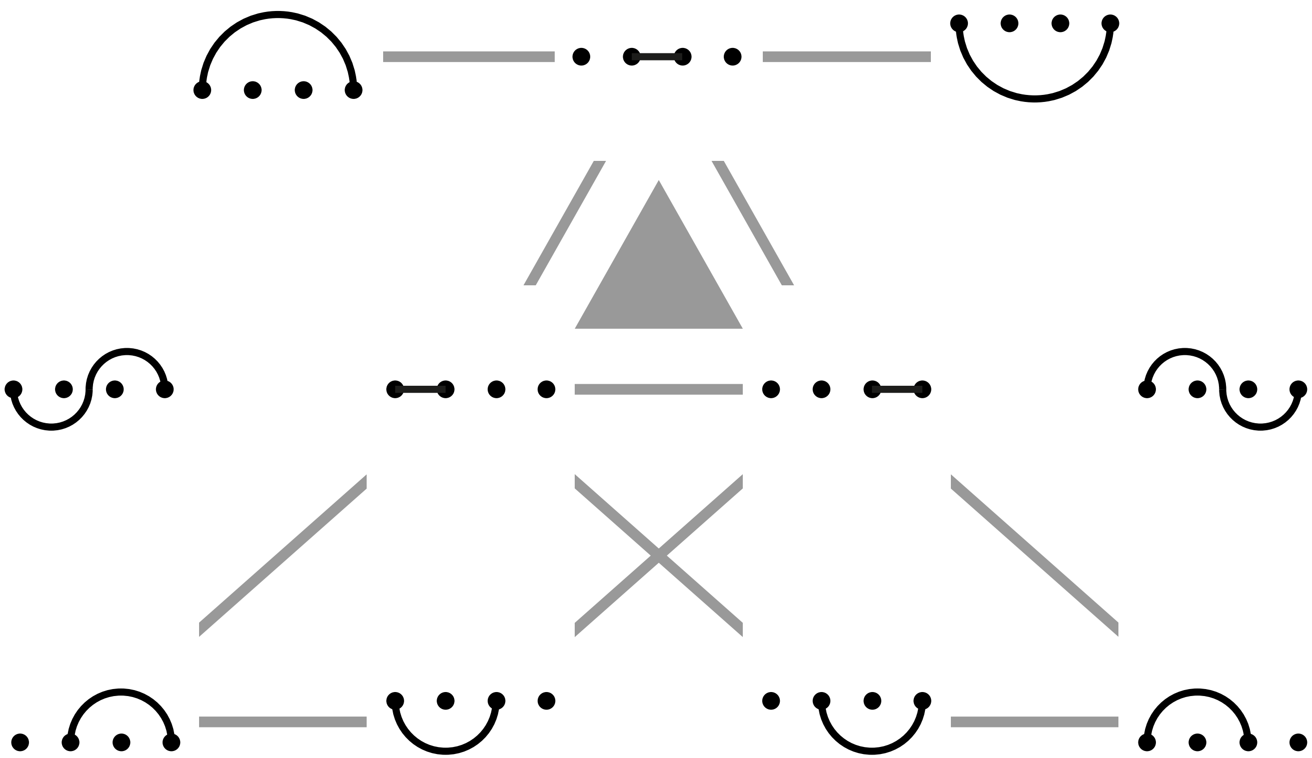
Reading ('15)

CANONICAL JOIN COMPLEX

canonical join complex of a join semidistributive lattice L = simplicial complex with

- vertices = join irreducibles of L
- faces = canonical join representations in L

THM. canonical join complex of the weak order $\longleftrightarrow$ non-crossing complex on arcs



Reading ('15)

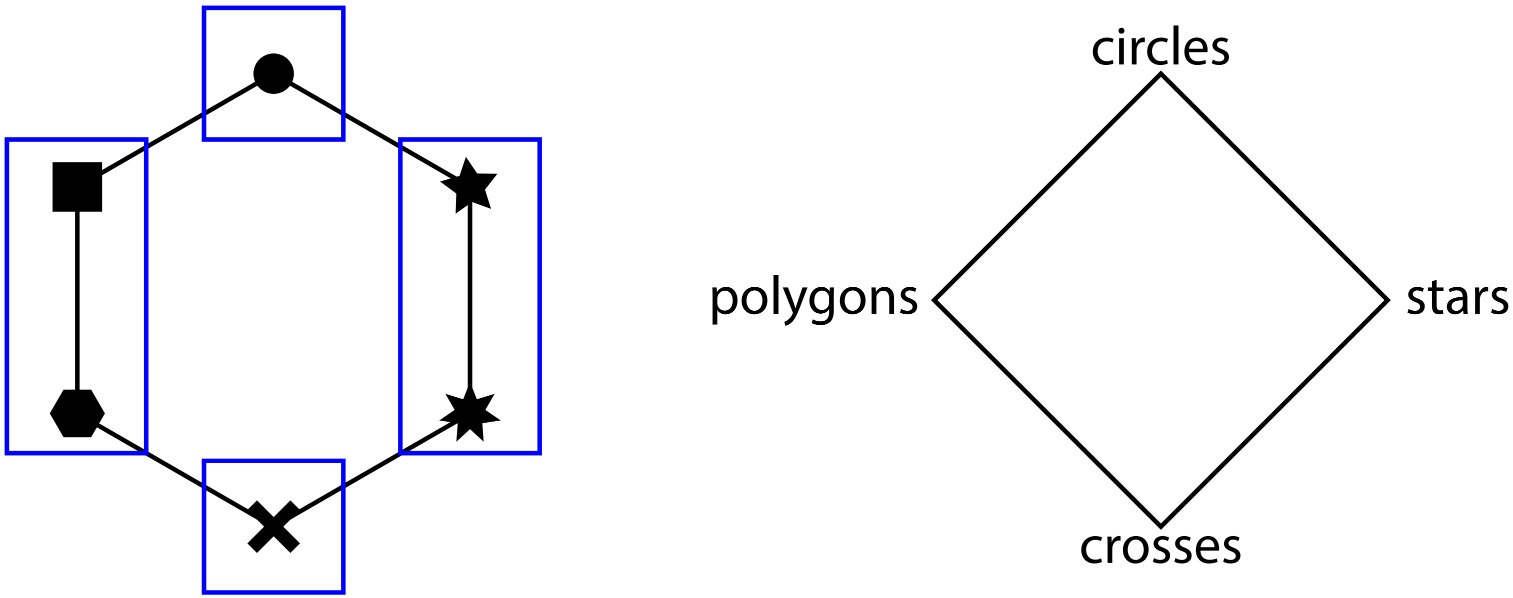
LATTICE CONGRUENCES

lattice congruence of L = equivalence relation $\equiv$ which respects meets and joins

$$x \equiv x' \text{ and } y \equiv y' \implies x \wedge y \equiv x' \wedge y' \text{ and } x \vee y \equiv x' \vee y'$$

lattice quotient of $L/\!\equiv$ = lattice on equivalence classes of L under $\equiv$ where

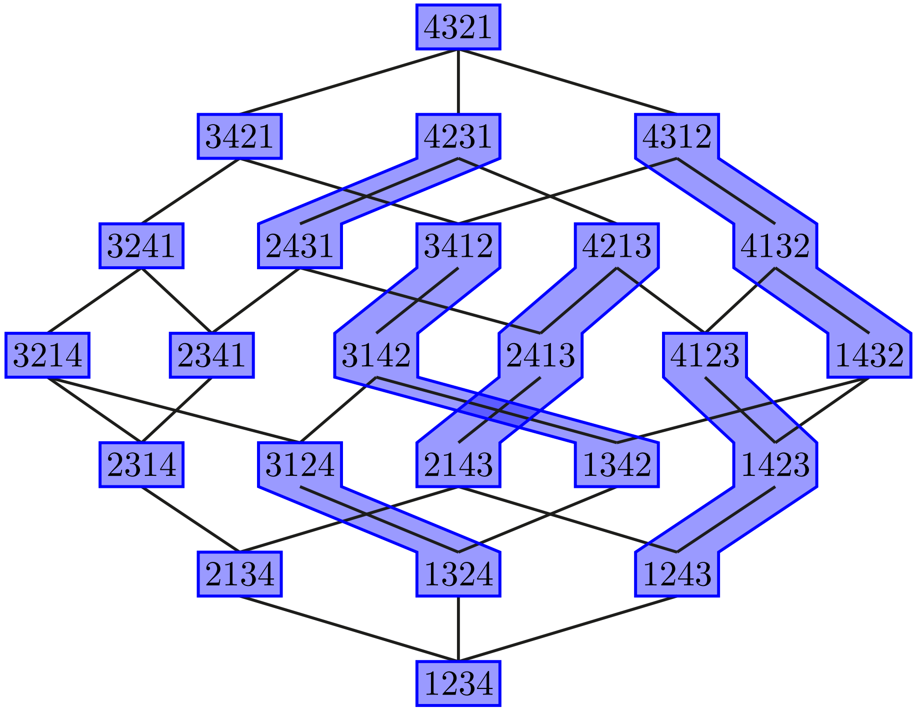
- $X \leq Y \iff \exists\, x \in X,\ y \in Y,\quad x \leq y$
- $X \wedge Y$ = equiv. class of $x \wedge y$ for any $x \in X$ and $y \in Y$
- $X \vee Y$ = equiv. class of $x \vee y$ for any $x \in X$ and $y \in Y$



LATTICE QUOTIENTS AND CANONICAL JOIN REPRESENTATIONS

$\equiv$ lattice congruence on L, then

- each class X is an interval $[\pi_\downarrow(X), \pi^\uparrow(X)]$
- $L/\equiv$ is isomorphic to $\pi_\downarrow(L)$ (as poset)
- canonical join representations in $L/\equiv$ are canonical join representations in L that only involve join irreducibles j with $\pi_\downarrow(j) = j$.



THM. $\equiv$ lattice congruence of the weak order on $\mathfrak{S}_n$
Let $\mathcal{I}_\equiv$ = arcs corresponding to join irreducibles σ with $\pi_\downarrow(\sigma) = \sigma$
Then

- $\pi_\downarrow(\sigma) = \sigma \iff \delta(\sigma) \subseteq \mathcal{I}_\equiv$.
- the map $\mathfrak{S}_n/\equiv \longrightarrow$ {nc arc diagrams in $\mathcal{I}_\equiv$} is a bijection.

$$X \longmapsto \delta(\pi_\downarrow(X))$$

Reading ('15)

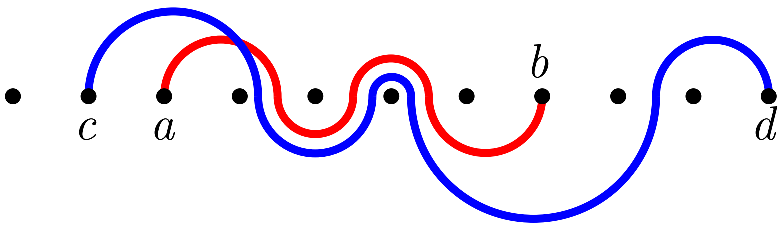
FORCING AND ARC IDEALS

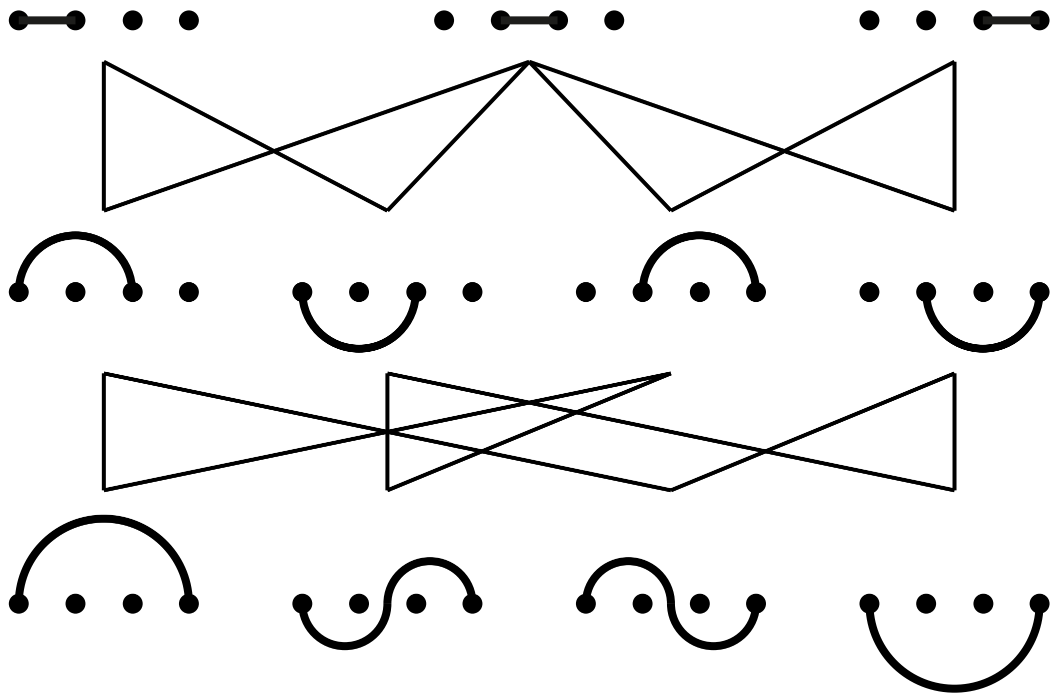
THM. $\mathcal{I}_{\equiv}$ = arcs corresponding to join irreducibles σ with $\pi_{\downarrow}(\sigma) = \sigma$.
Bijection $\mathfrak{S}_n/{\equiv} \longleftrightarrow$ {nc arc diagrams in $\mathcal{I}_{\equiv}$}.

THM. The following are equivalent for a set of arcs $\mathcal{I}$:

- there exists a lattice congruence $\equiv$ on $\mathfrak{S}_n$ with $\mathcal{I} = \mathcal{I}_{\equiv}$
- $\mathcal{I}$ is an upper ideal of the forcing order

(a, b, A, B) forces (c, d, C, D) = $c \le a < b \le d$ and $A \subseteq C$ and $B \subseteq D$

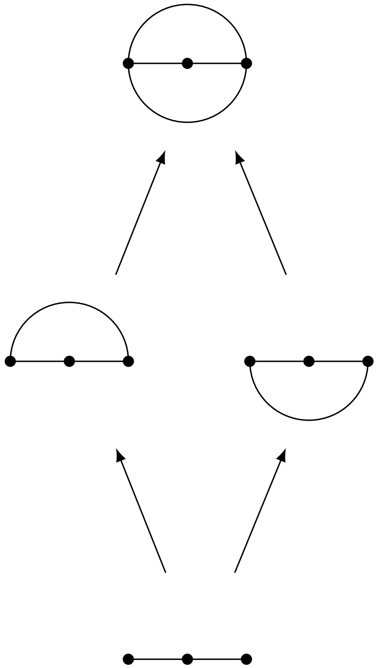




Reading ('15)

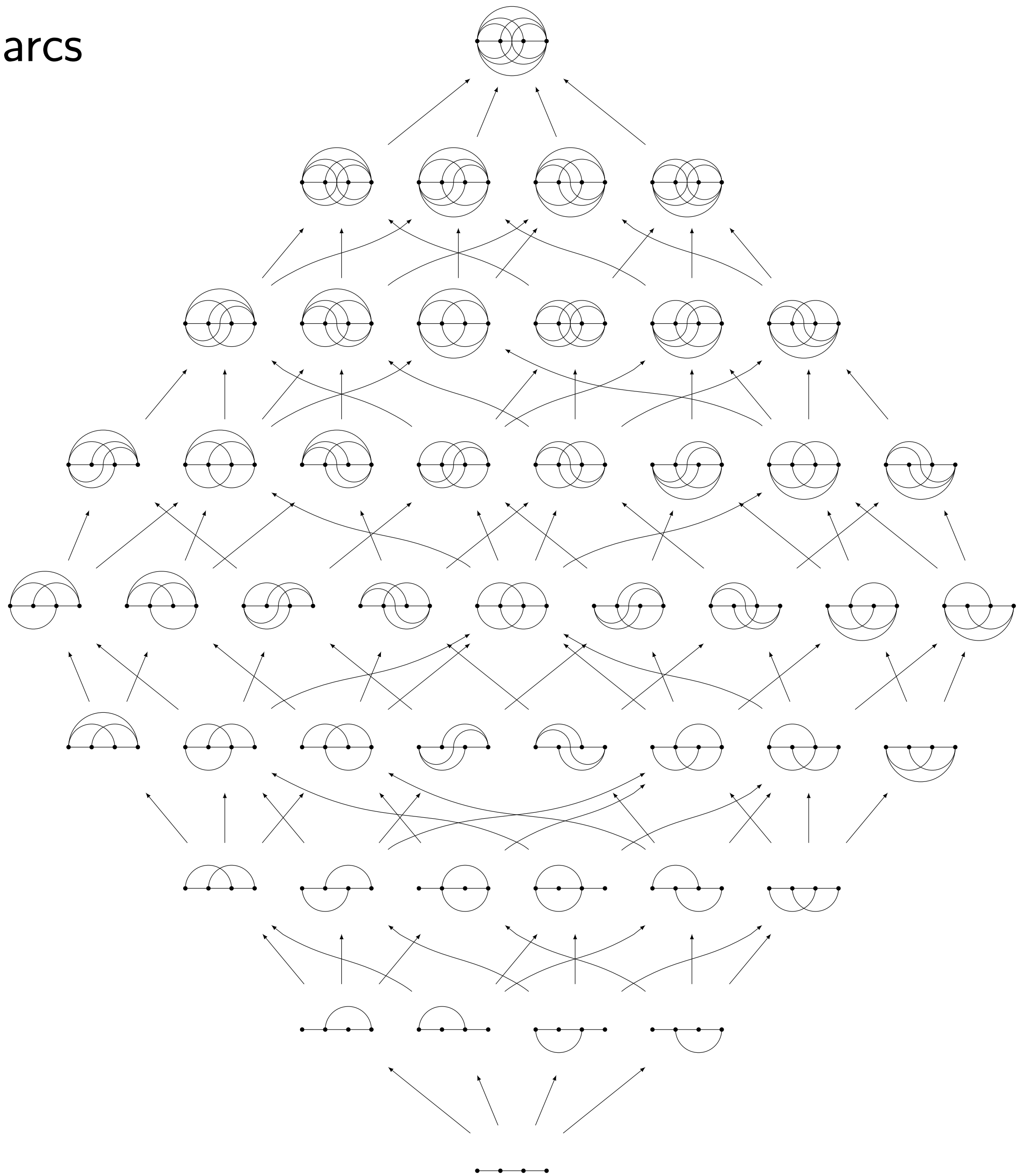
ARC IDEALS

arc ideal = ideal of the forcing poset on arcs



essential congruences:
1, 1, 4, 47, 3322, ...
OEIS A330039

all congruences
1, 2, 7, 60, 3444, ...
OEIS A091687



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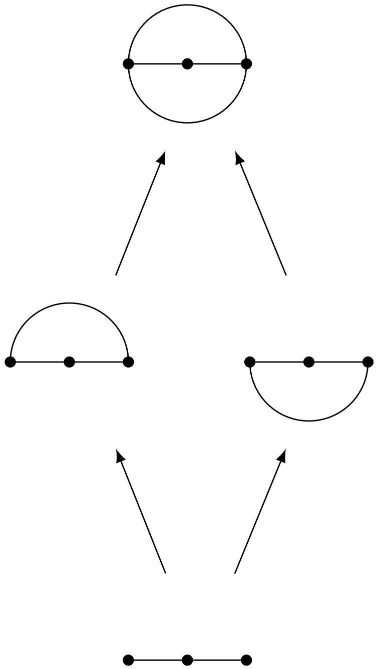

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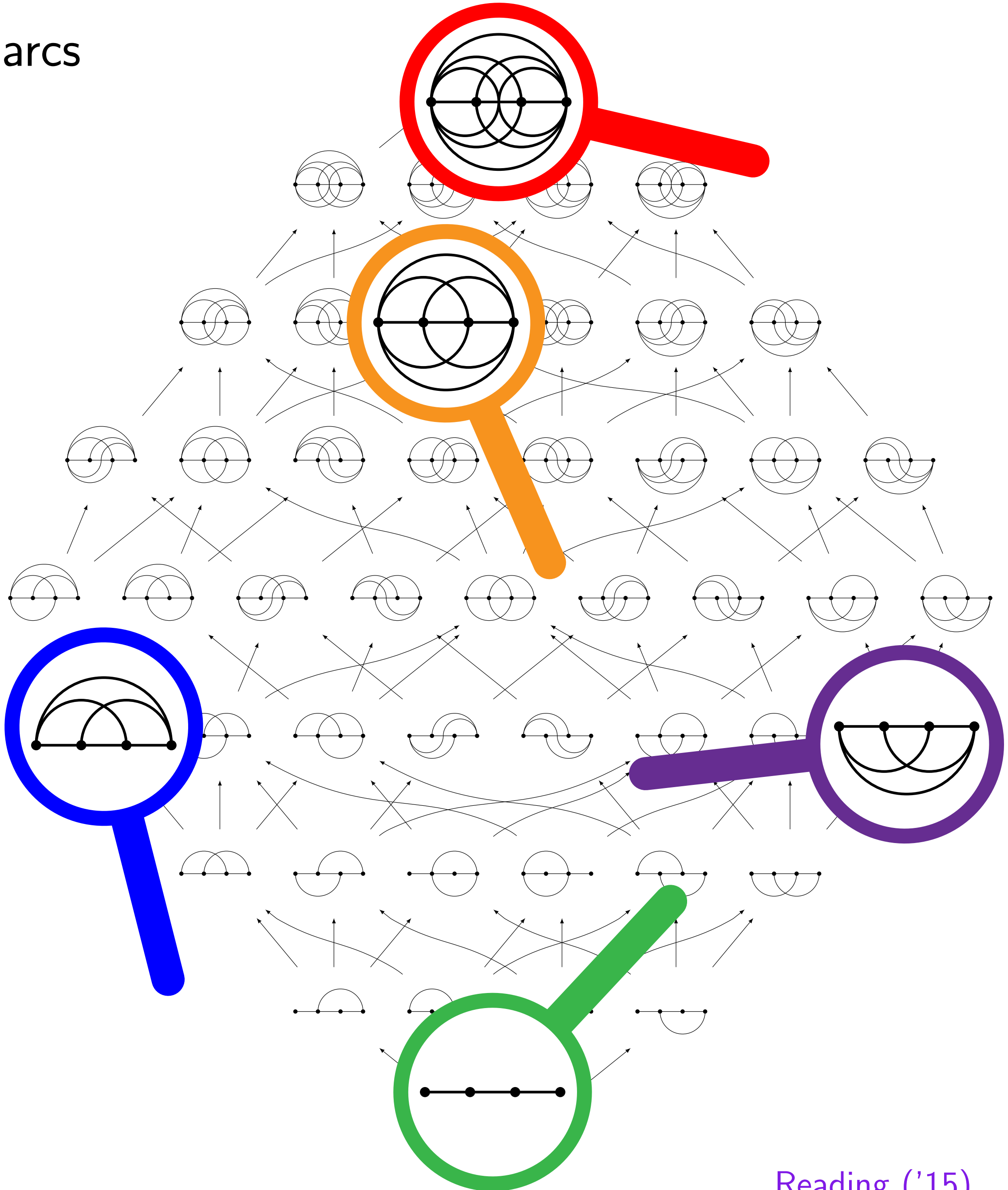
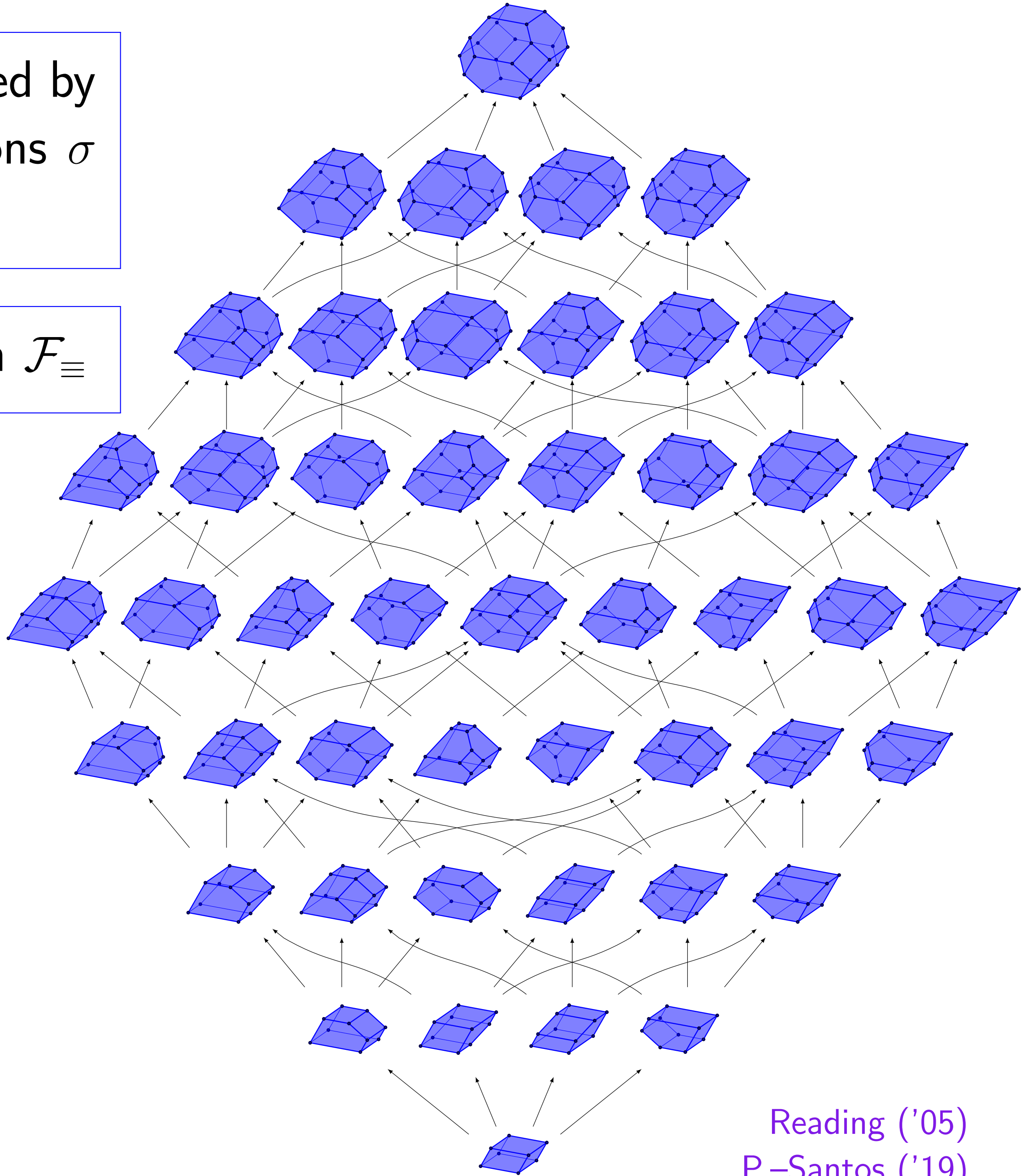


Reading ('15)

QUOTIENT FANS & QUOTIENTOPES

quotient fan $\mathcal{F}_{\equiv}$ = chambers are obtained by glueing the chambers of the permutations σ in the same congruence class of $\equiv$

quotientope = polytope with normal fan $\mathcal{F}_{\equiv}$



Reading ('05)
P.–Santos ('19)
Padrol–P.–Ritter ('20^{+})

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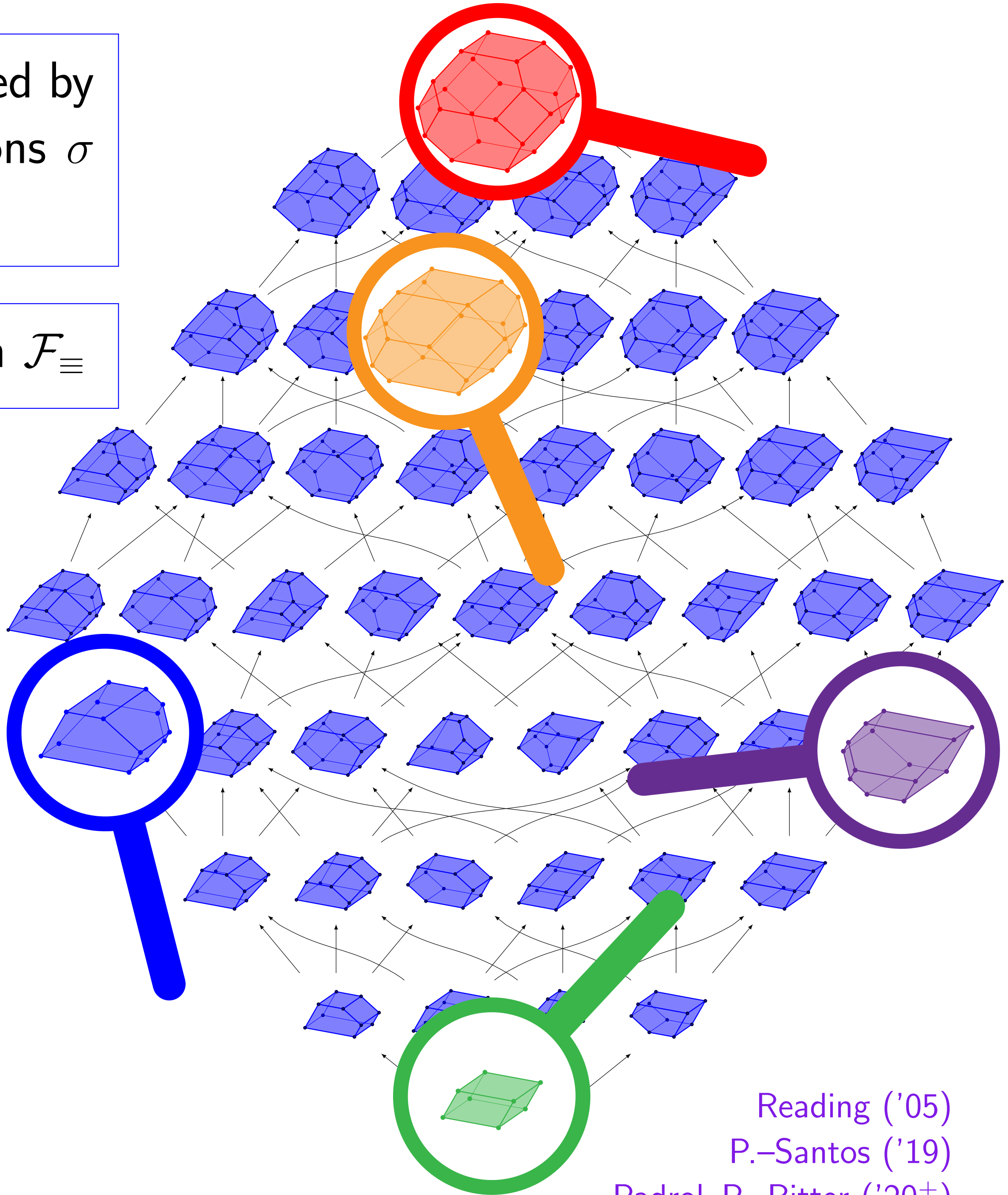

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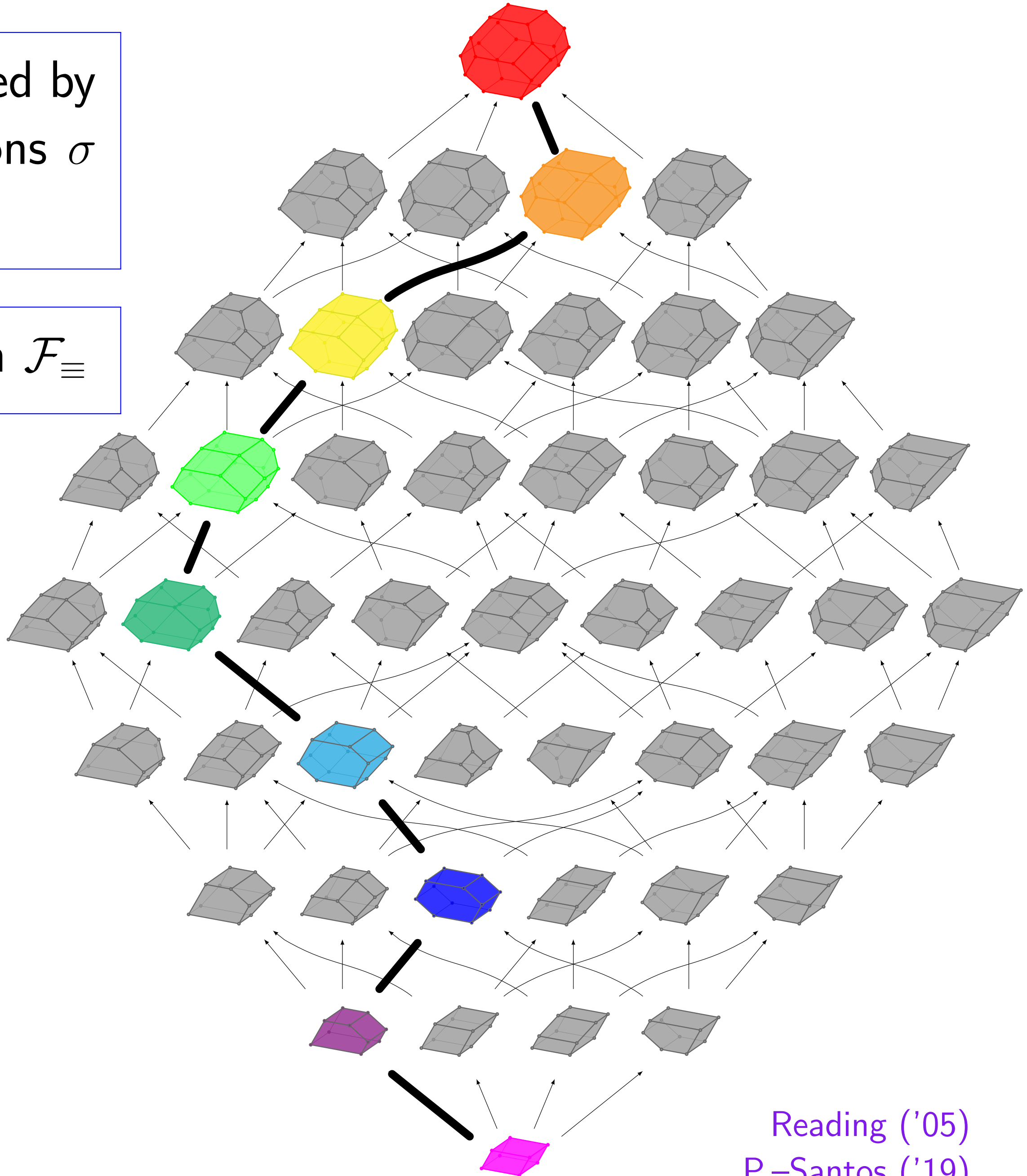


POLYWOOD

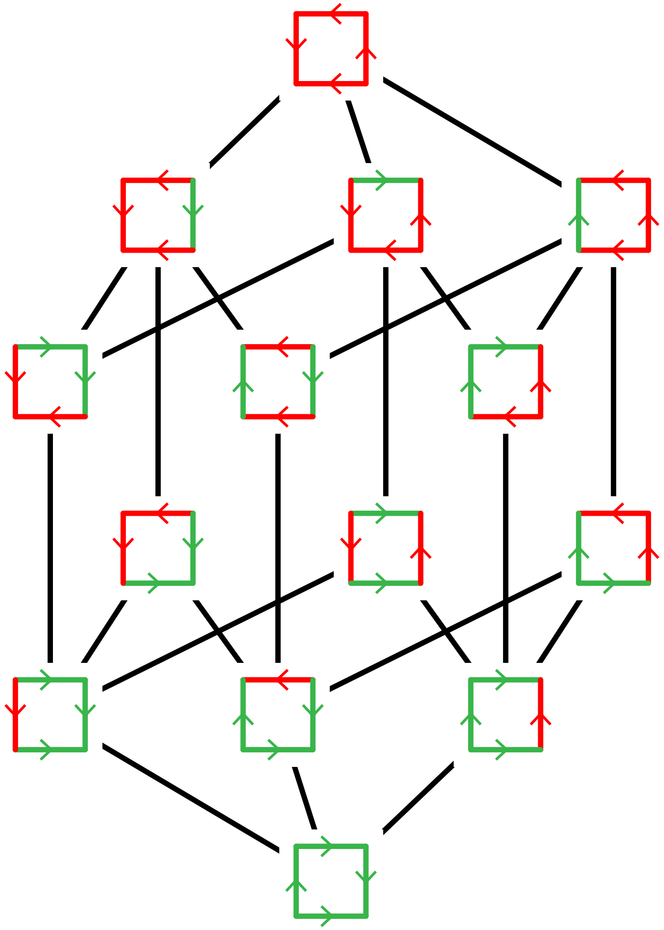
Reading ('05)
P.–Santos ('19)
Padrol–P.–Ritter ('20^{+})

ACYCLIC REORIENTATION LATTICES

ACYCLIC REORIENTATION POSETS

D directed acyclic graph

$\mathcal{AR}_D =$ all acyclic reorientations of D, ordered by inclusion of their sets of reversed arcs



minimal element D

maximal element $\bar{D}$

self-dual under reversing all arcs

cover relations $=$ flipping a single arc

flippable arcs of E $=$ <u>transitive reduction</u> of E

$= E \smallsetminus \{(u,v) \in E \mid \exists$ directed path $u \rightsquigarrow v$ in $E\}$

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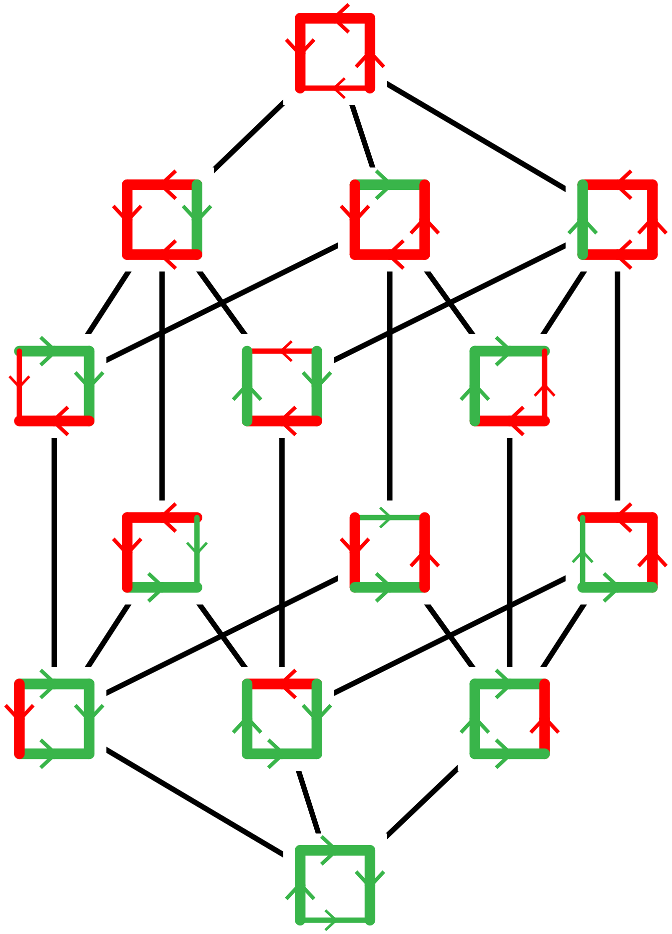

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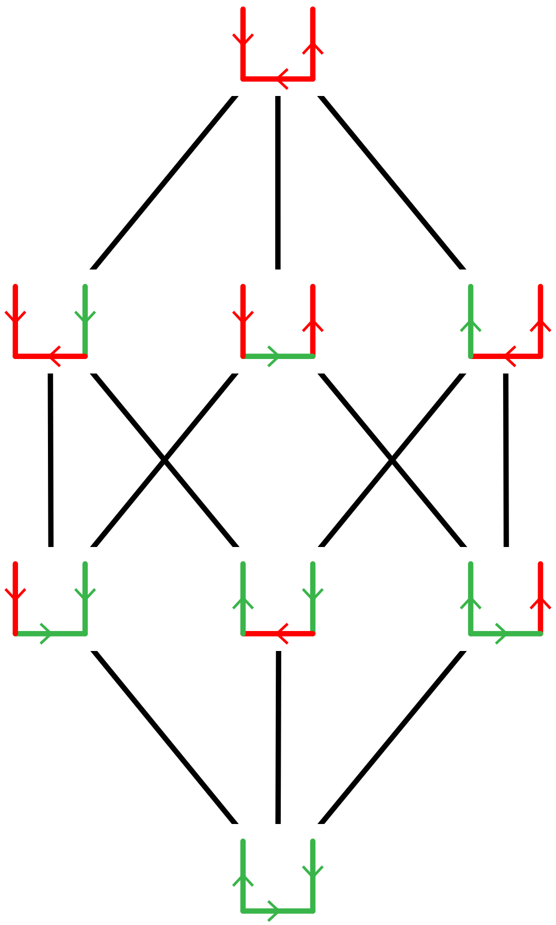
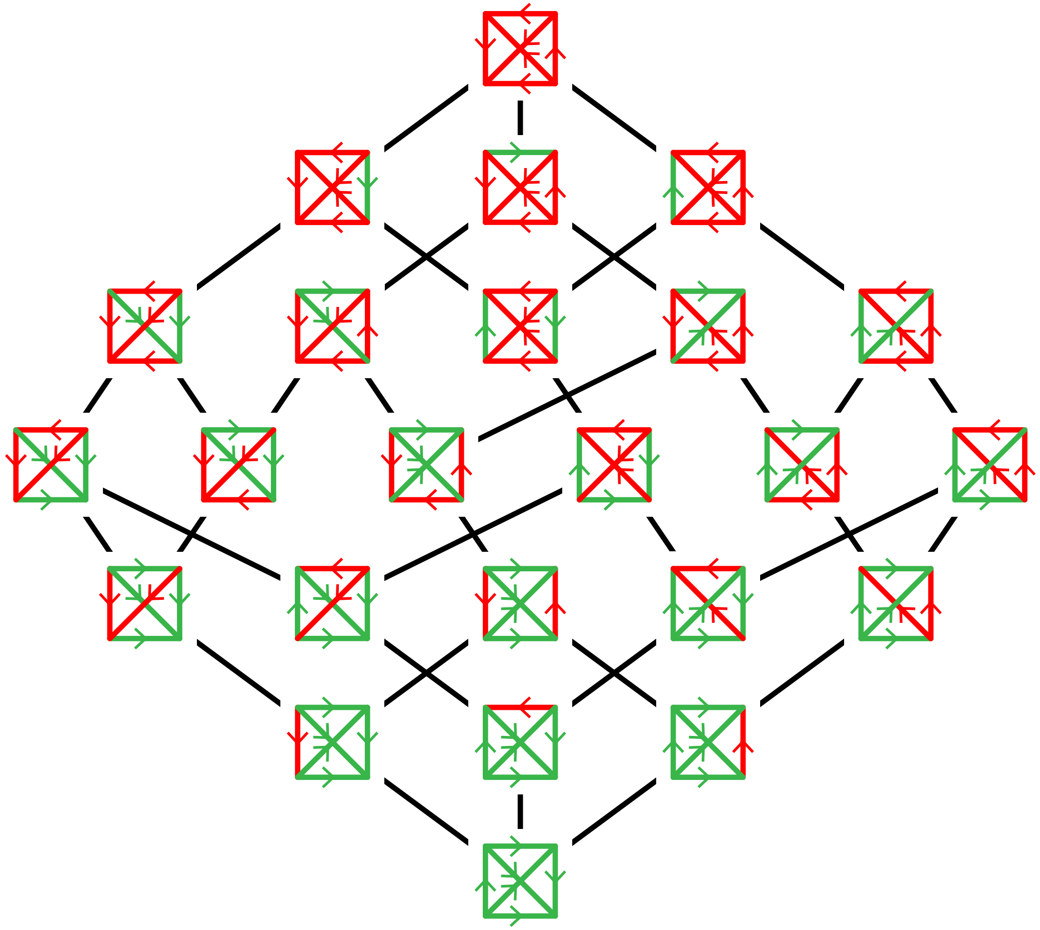
D forest



boolean lattice

D tournament

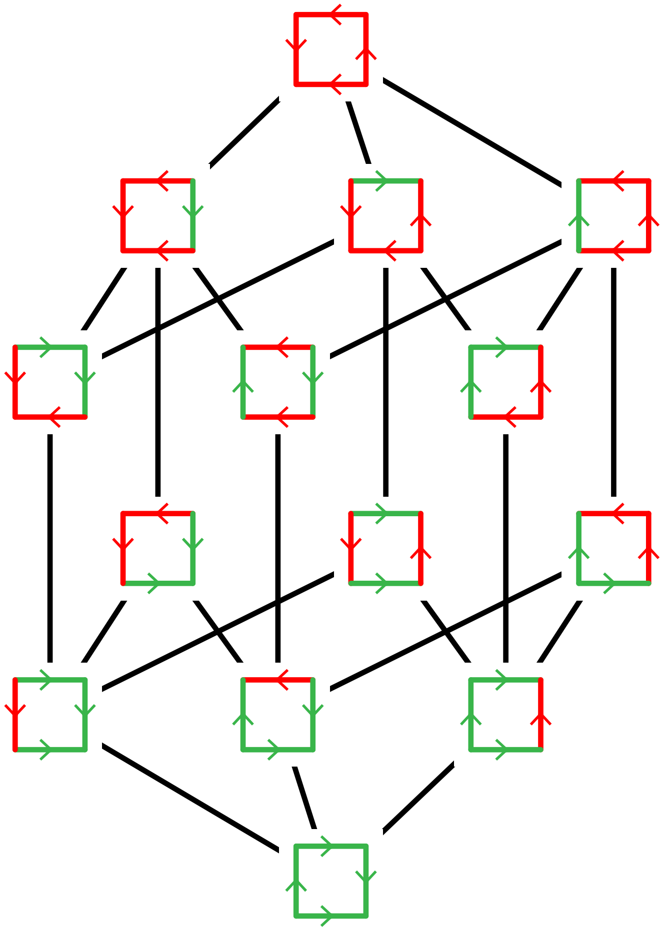


weak order

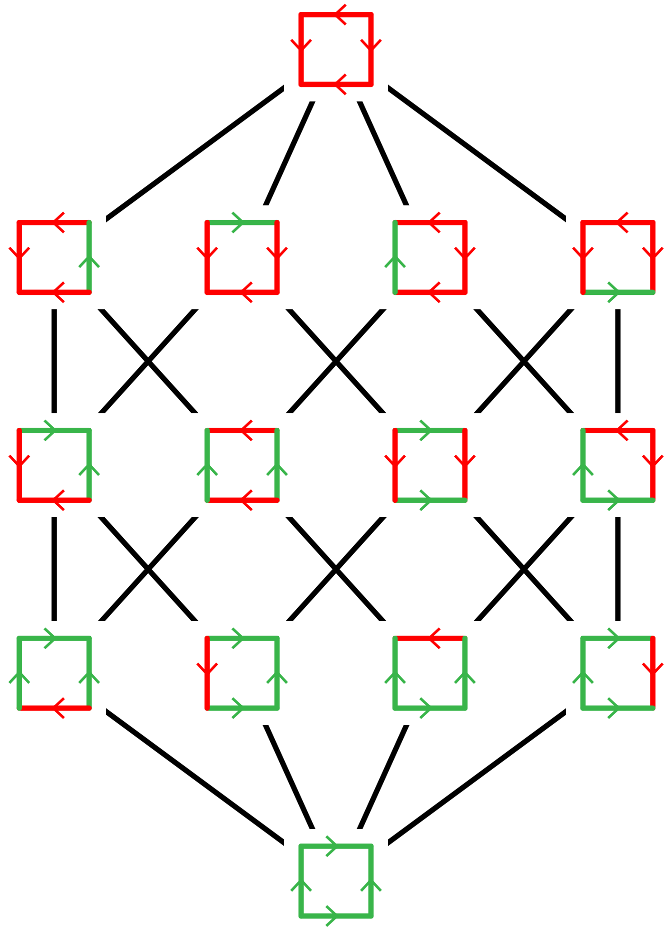
ACYCLIC REORIENTATION LATTICES

D vertebrate $=$ transitive reduction of any induced subgraph of D is a forest

THM. $\mathcal{AR}_D$ lattice $\iff$ D vertebrate



lattice



not lattice

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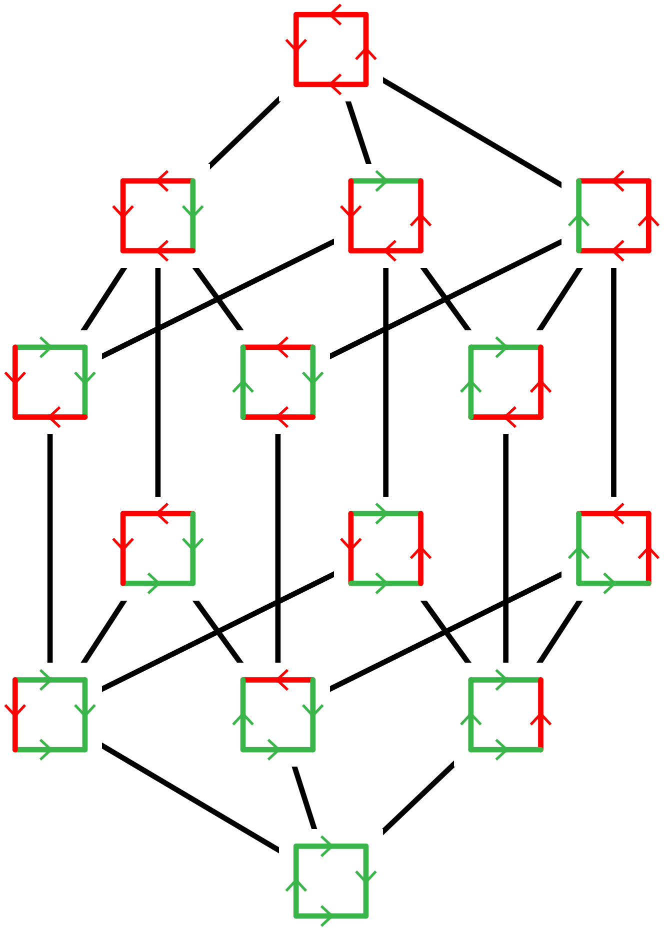

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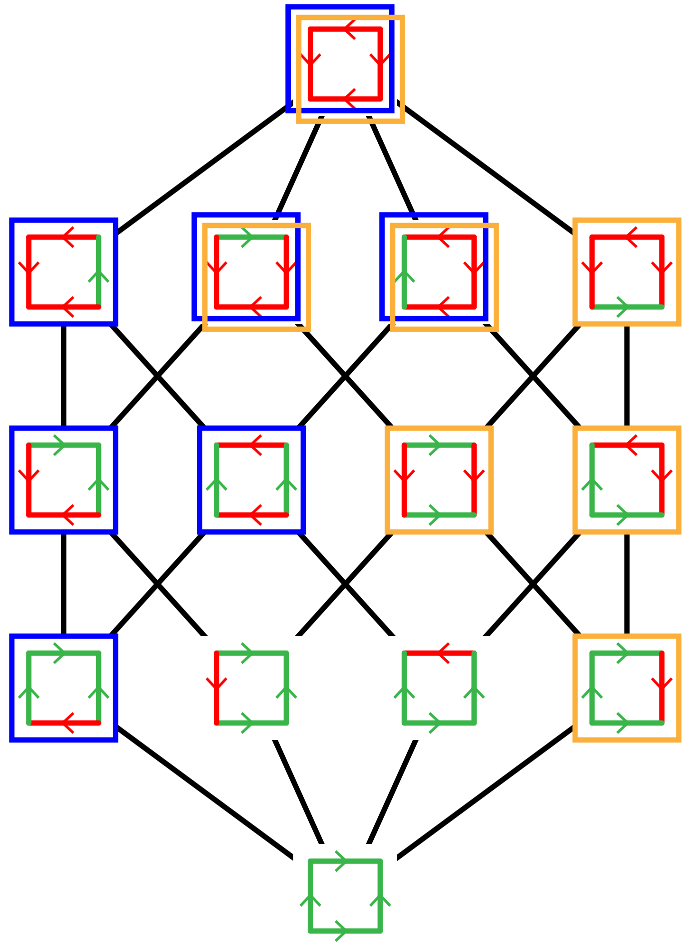

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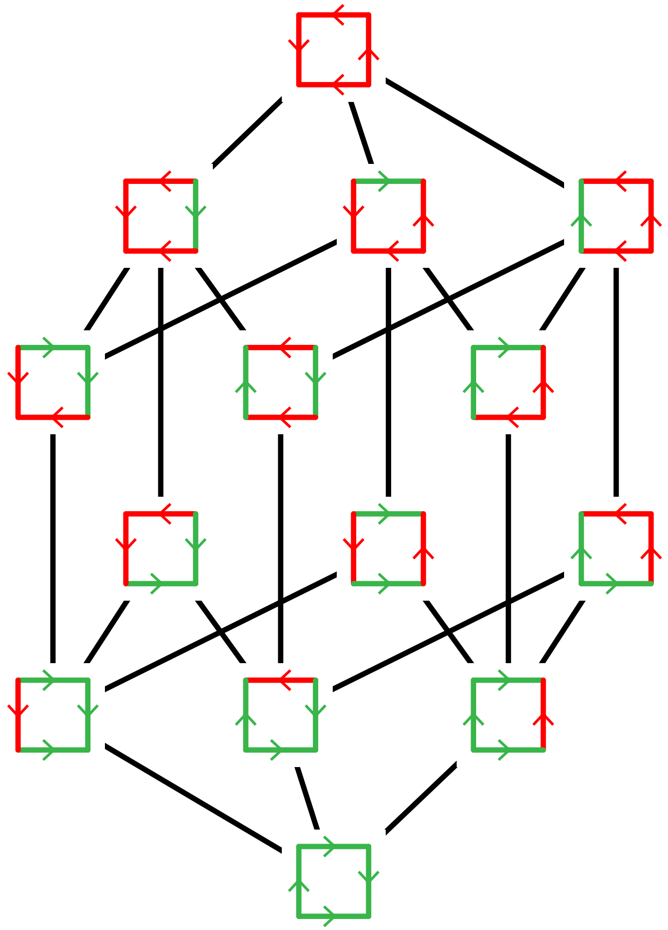


X subset of arcs of D is

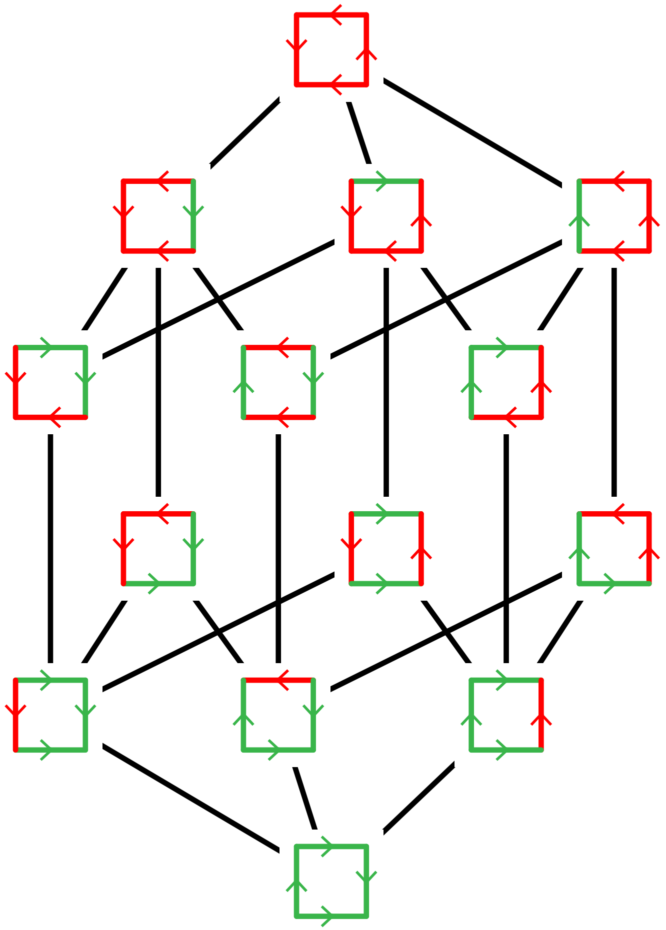
- closed if all arcs of D in the transitive closure of X also belong to X
- coclosed if its complement is closed
- biclosed if it is closed and coclosed

PROP. If D vertebrate,
X biclosed $\iff$ the reorientation of X is acyclic

ACYCLIC REORIENTATION LATTICES

D vertebrate $=$ transitive reduction of any induced subgraph of D is a forest

THM. $\mathcal{AR}_D$ lattice $\iff$ D vertebrate



PROP. If D vertebrate,

$$\mathrm{bwd}(E_1 \vee \ldots \vee E_k) = \text{transitive closure of } \mathrm{bwd}(E_1) \cup \cdots \cup \mathrm{bwd}(E_k)$$

$$\mathrm{fwd}(E_1 \wedge \ldots \wedge E_k) = \text{transitive closure of } \mathrm{fwd}(E_1) \cup \cdots \cup \mathrm{fwd}(E_k)$$

DISTRIBUTIVITY & SEMIDISTRIBUTIVITY

DISTRIBUTIVE AND SEMIDISTRIBUTIVE LATTICES

lattice $=$ poset $(L, \le)$ with a meet $\wedge$ and a join $\vee$

$(L, \le, \wedge, \vee)$ finite lattice is

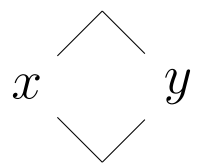
- distributive if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any $x, y, z \in L$

  $\Longrightarrow$ any $y \in L$ is represented as $y = \bigvee_{j \in J} j$ where $J = \{$join irreducibles below $y\}$

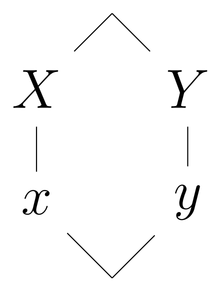
- join semidistributive if $x \vee y = x \vee z$ implies $x \vee (y \wedge z) = x \vee y$ for any $x, y, z \in L$

  $\Longrightarrow$ any $y \in L$ admits a canonical join representation $y = \bigvee_{x \lessdot y} k_{\vee}(x, y)$
  where $k_{\vee}(x, y)$ is the unique minimal element of $\{z \in L \mid x \vee z = y\}$

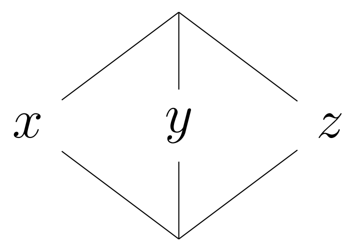
- semidistributive if both join and meet semidistributive



distributive



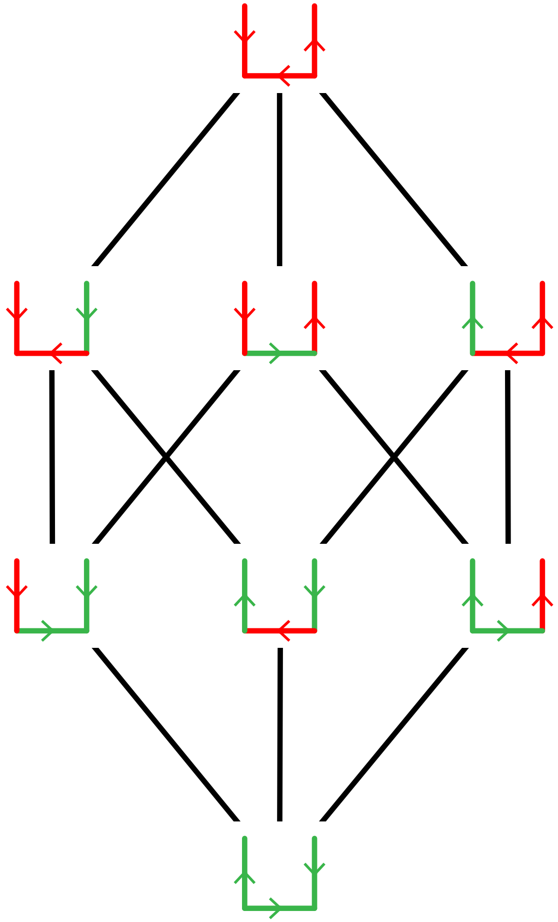
semidistributive



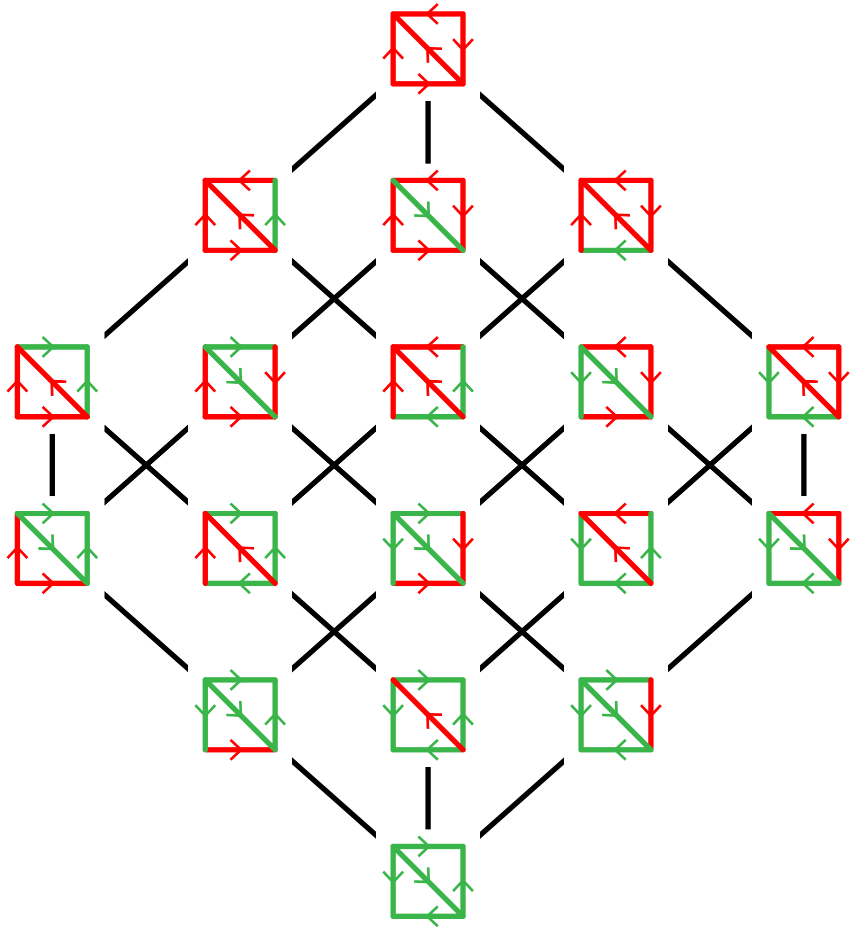
not semidistributive

DISTRIBUTIVE ACYCLIC REORIENTATION POSETS

THM. $\mathcal{AR}_D$ distributive lattice $\iff$ D forest $\iff$ $\mathcal{AR}_D$ boolean lattice



distributive



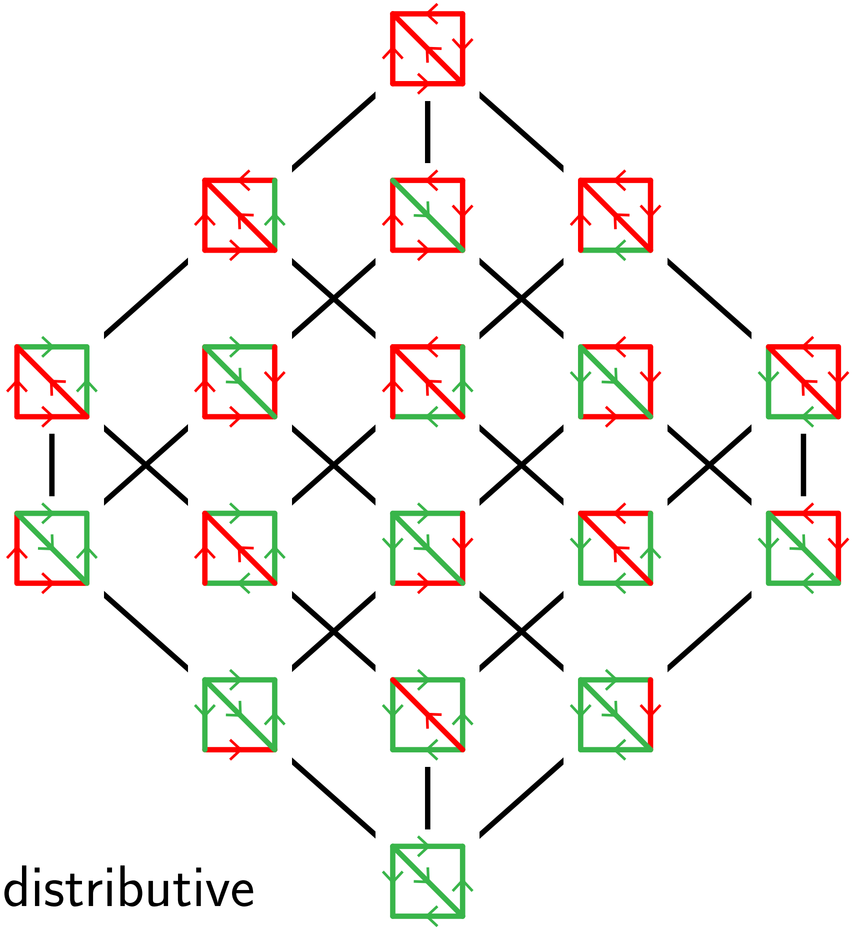
not distributive

SEMIDISTRIBUTIVE ACYCLIC REORIENTATION LATTICES

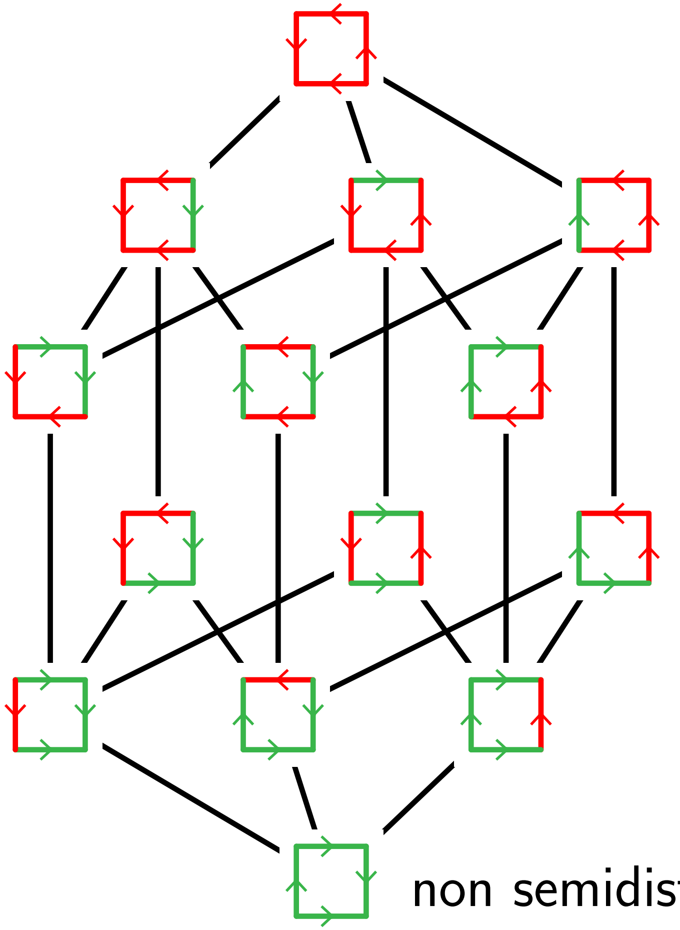
D skeletal $=$

- D vertebrate $=$ transitive reduction of any induced subgraph of D is a forest
- D filled $=$ any directed path joining the endpoints of an arc in D induces a tournament

THM. $\mathcal{AR}_D$ semidistributive lattice $\iff$ D is skeletal



semidistributive



non semidistributive

SEMIDISTRIBUTIVE ACYCLIC REORIENTATION LATTICES

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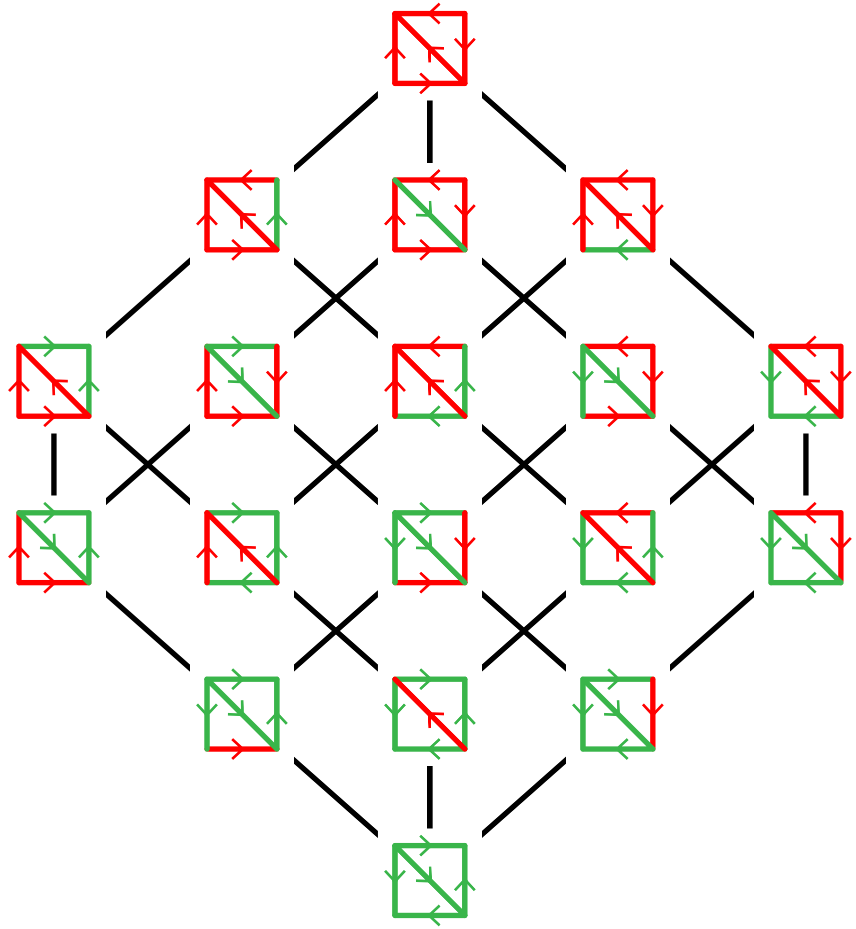


THM. If D skeletal, the canonical join representation of an acyclic reorientation E of D is $E = \bigvee_a E_a$ where

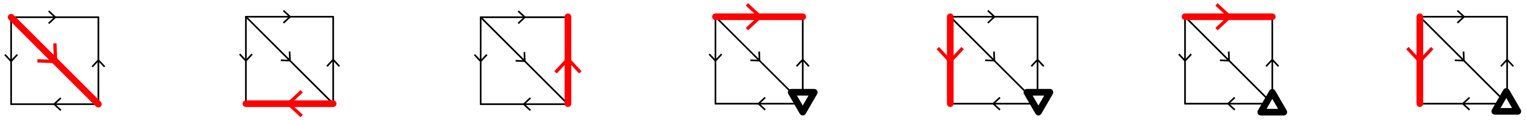
- a runs over the arcs of D reversed in the transitive reduction of E
- E_a is the acyclic reorientation of D where an arc is reversed iff it is the only arc reversed in E along a path in D joining the endpoints of a

ROPES

ROPES & NON-CROSSING ROPE DIAGRAMS

rope of D = quadruple $\rho = (u, v, \bigtriangledown, \bigtriangleup)$ where

- (u, v) is an arc of D
- $\bigtriangledown \sqcup \bigtriangleup$ partitions the transitive support of (u, v) minus $\{u, v\}$

ropes 

ROPES & NON-CROSSING ROPE DIAGRAMS

rope of D = quadruple $\rho = (u, v, \triangledown, \vartriangle)$ where

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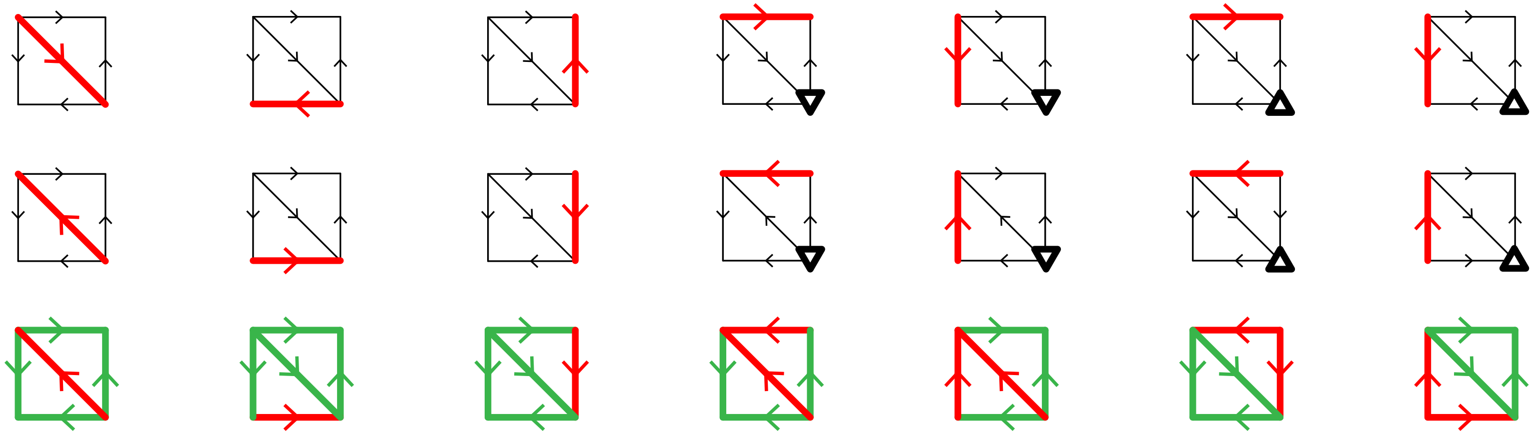
ropes

join irreducibles

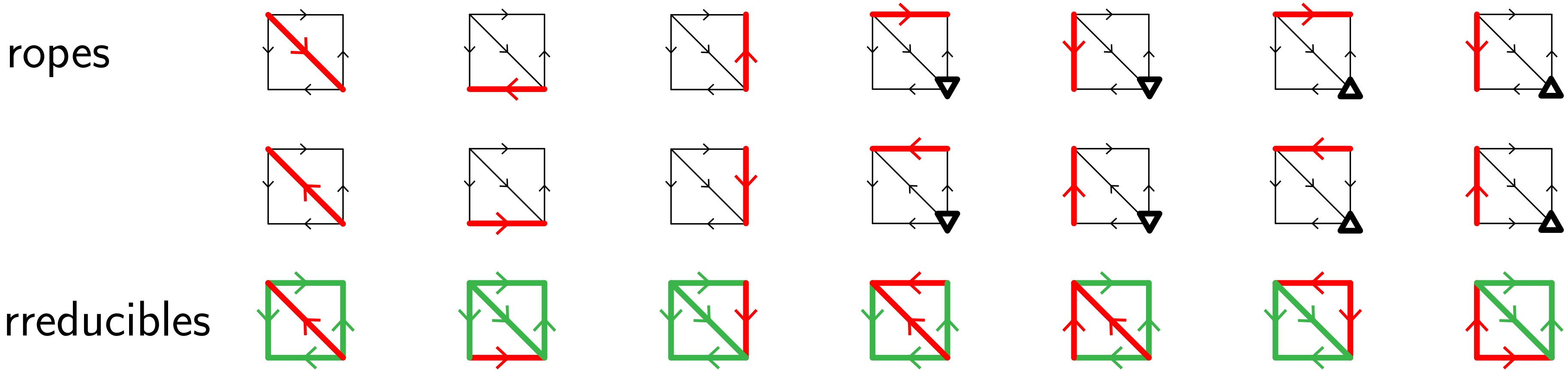


THM. join irreducibles of $\mathcal{AR}_D$ $\longleftrightarrow$ ropes of D

ROPES & NON-CROSSING ROPE DIAGRAMS

rope of D = quadruple $\rho = (u, v, \triangledown, \vartriangle)$ where

- (u, v) is an arc of D
- $\triangledown \sqcup \vartriangle$ partitions the transitive support of (u, v) minus $\{u, v\}$



THM. join irreducibles of $\mathcal{AR}_D$ $\longleftrightarrow$ ropes of D

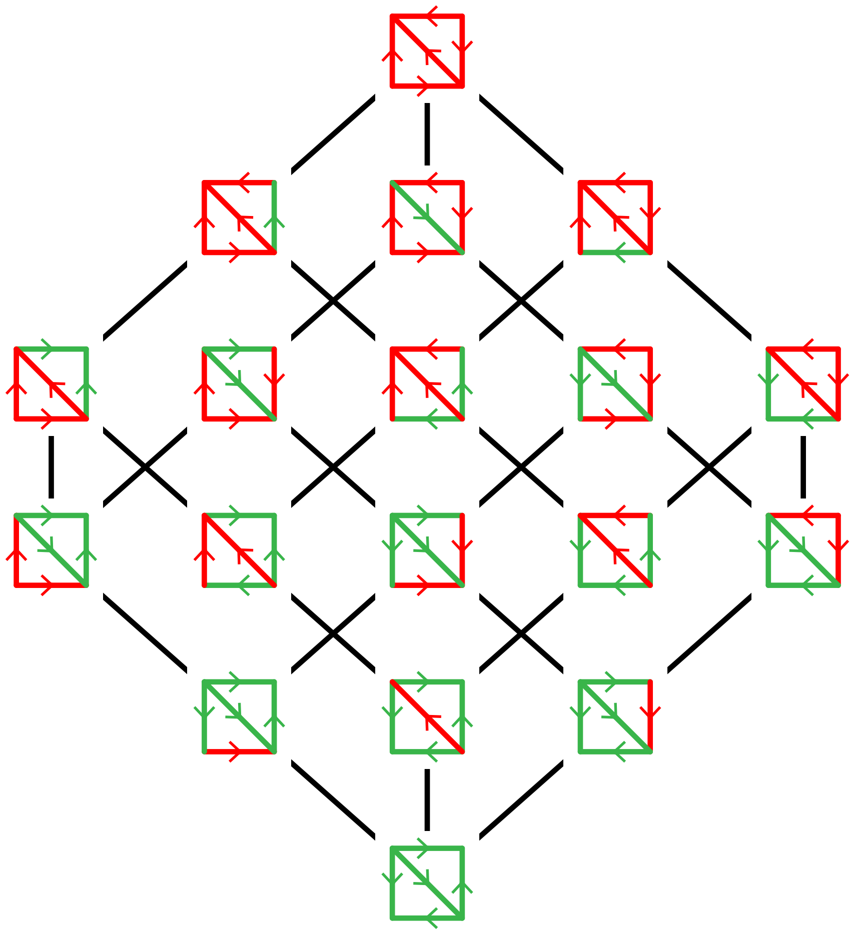
canonical join representations of $\mathcal{AR}_D$ $\longleftrightarrow$ non-crossing rope diagrams of $\mathcal{AR}_D$

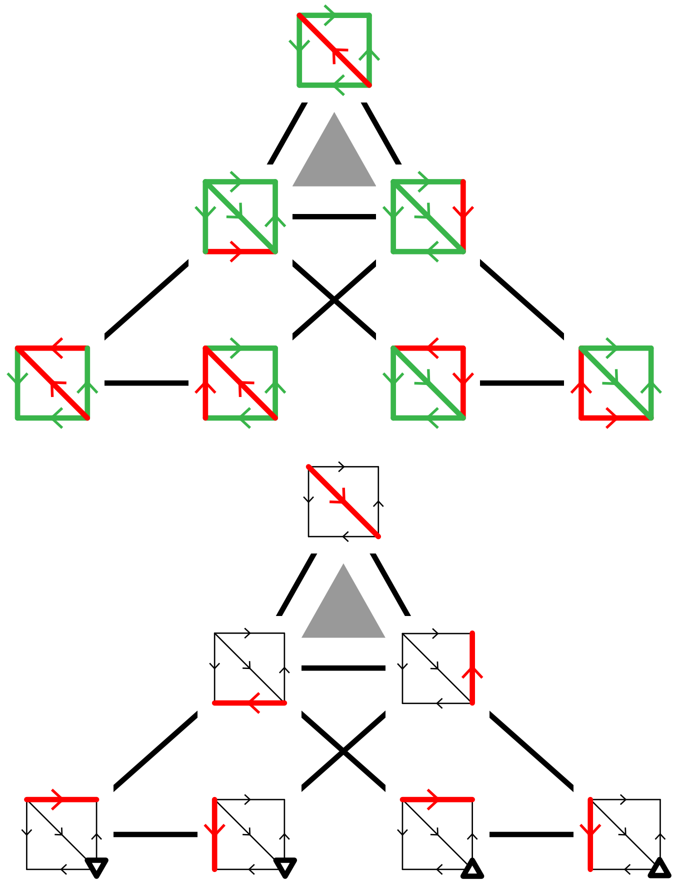
$(u, v, \triangledown, \vartriangle)$ and $(u', v', \triangledown', \vartriangle')$ are crossing if there are $w \neq w'$ such that

- $w \in (\triangledown \cup \{u, v\}) \cap (\vartriangle' \cup \{u', v'\})$
- $w' \in (\vartriangle \cup \{u, v\}) \cap (\triangledown' \cup \{u', v'\})$

NON-CROSSING ROPE DIAGRAMS & CANONICAL JOIN REPRESENTATIONS

PROP. The canonical join complex is isomorphic to the non-crossing rope complex





<u>rope</u> of $D = (u, v, \bigtriangledown, \bigtriangleup)$ where

- (u, v) is an arc of D
- $\bigtriangledown \sqcup \bigtriangleup =$ trans. supp. of (u, v)

$(u, v, \bigtriangledown, \bigtriangleup)$ and $(u', v', \bigtriangledown', \bigtriangleup')$ are <u>crossing</u> if there are $w \neq w'$ such that

- $w \in (\bigtriangledown \cup \{u, v\}) \cap (\bigtriangleup' \cup \{u', v'\})$
- $w' \in (\bigtriangleup \cup \{u, v\}) \cap (\bigtriangledown' \cup \{u', v'\})$

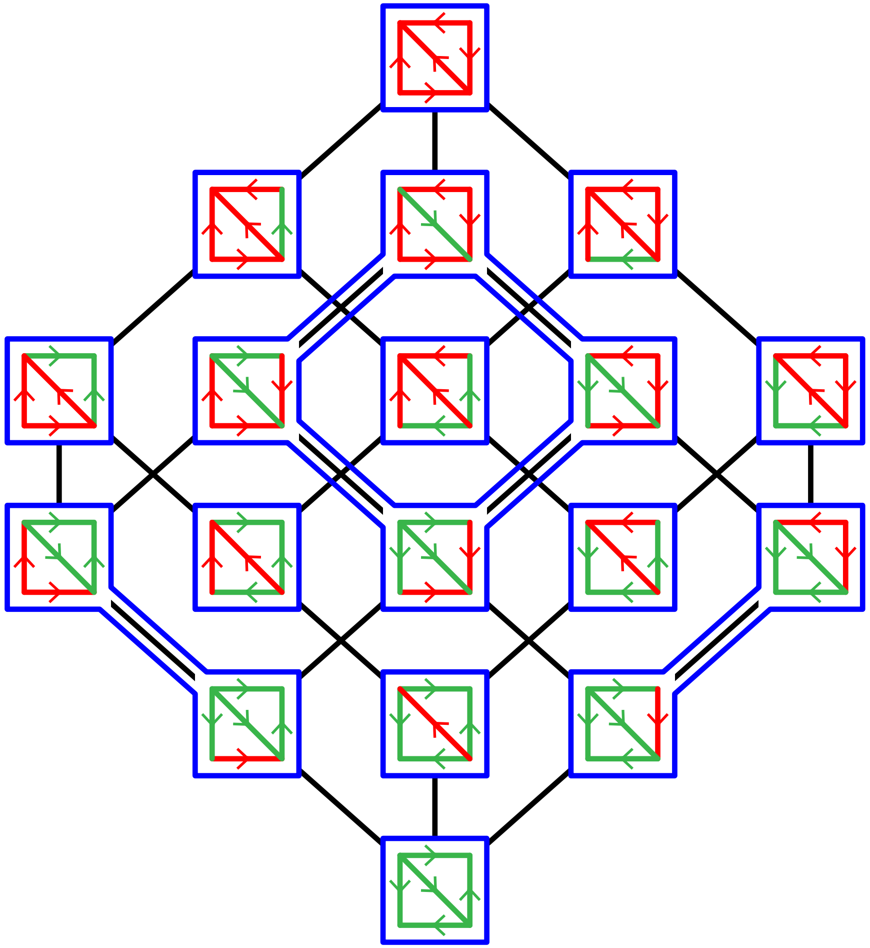
CONGRUENCES & QUOTIENTS

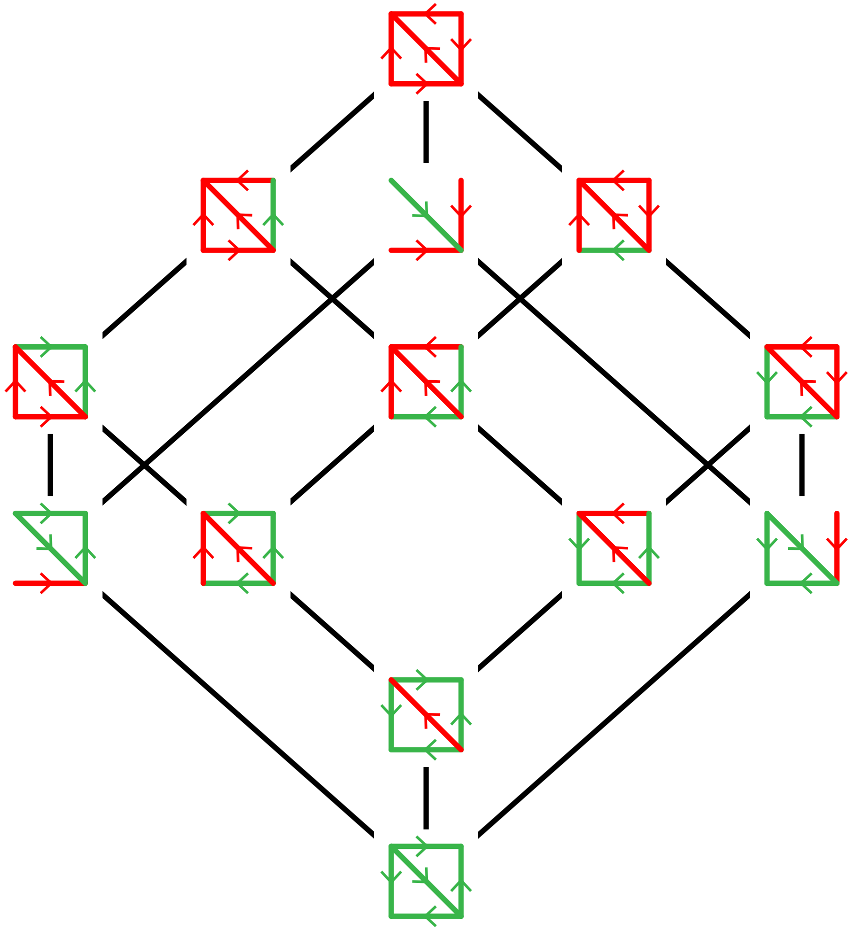
COHERENT CONGRUENCES

lattice congruence of L = equivalence relation $\equiv$ which respects meets and joins

$$x \equiv x' \text{ and } y \equiv y' \implies x \wedge y \equiv x' \wedge y' \text{ and } x \vee y \equiv x' \vee y'$$

lattice quotient $L/{\equiv}$ = lattice structure on the equivalence classes of $\equiv$



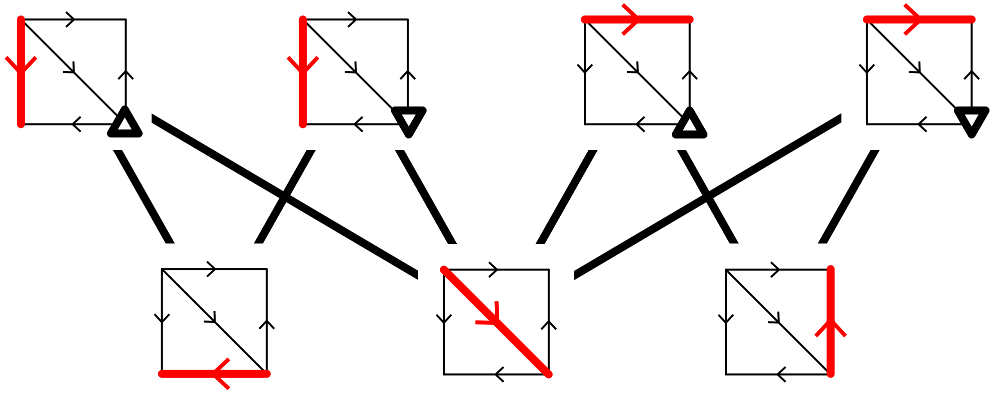


congruence lattice of L = lattice of all lattice congruences of L ordered by refinement

SUBROPES & FORCING

THM. $\mathcal{AR}_D$ congruence uniform lattice $\iff$ D is skeletal

$(u, v, \triangledown, \triangle)$ subrope of $(u', v', \triangledown', \triangle') = u, v \in \{u', v'\} \cup \triangledown' \cup \triangle'$ and $\triangledown \subseteq \triangledown'$ and $\triangle \subseteq \triangle'$



PROP. congruence lattice of $\mathcal{AR}_D \simeq$ lower ideal lattice of subrope order

CORO. $\equiv$ lattice congruence of $\mathcal{AR}_D$

- E minimal in its $\equiv$-class $\iff$ $\delta(E) \subseteq \mathbb{I}_\equiv$
- quotient $\mathcal{AR}_D/\equiv \;\simeq$ subposet of $\mathcal{AR}_D$ induced by $\{E \in \mathcal{AR}_D \mid \delta(E) \subseteq \mathbb{I}_\equiv\}$

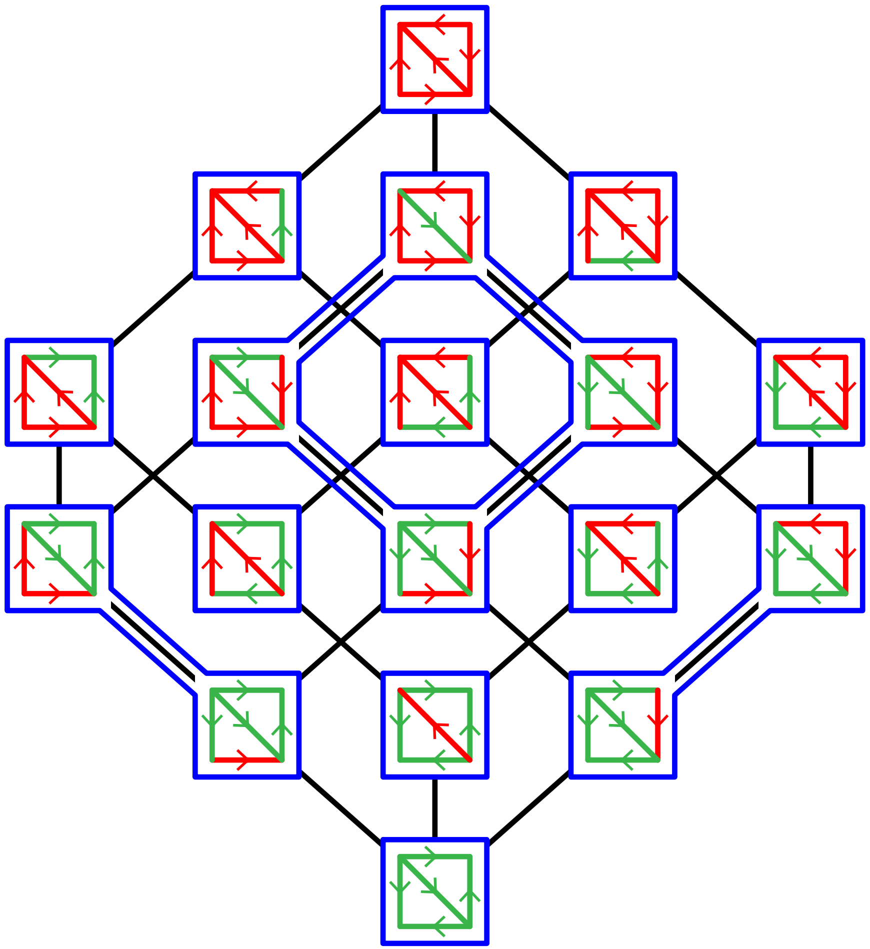
COHERENT CONGRUENCES

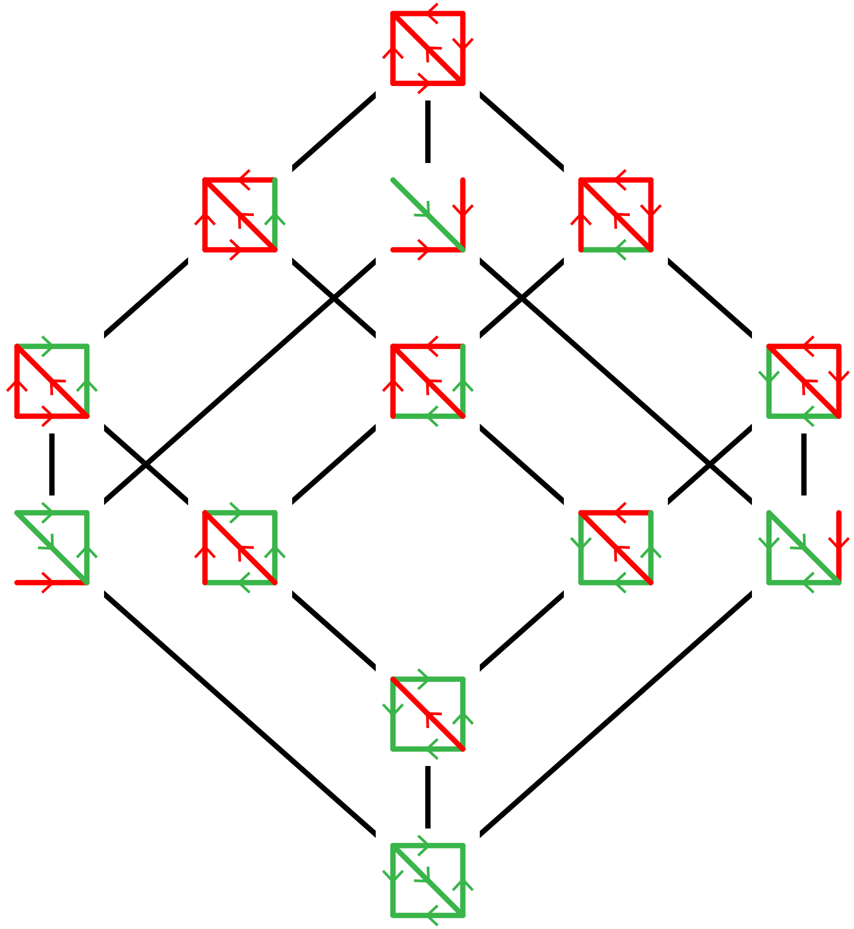
$(\mho, \Omega)$ = two of arbitrary subsets of V
$\mathbb{I}_{(\mho,\Omega)}$ = lower ideal of ropes $(u, v, \triangledown, \triangle)$ of D such that $\triangledown \subseteq \mho$ and $\triangle \subseteq \Omega$
coherent congruence $\equiv_{(\mho,\Omega)}$ = congruence with subrope ideal $\mathbb{I}_{(\mho,\Omega)}$

P.–Pons ('18)

examples:

- sylvester congruence = subrope ideal contains only ropes $(u, v, \triangledown, \varnothing)$





COHERENT CONGRUENCES

$(\mho, \Omega)$ = two of arbitrary subsets of V

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coherent congruence $\equiv_{(\mho,\Omega)}$ = congruence with subrope ideal $\mathbb{I}_{(\mho,\Omega)}$

P.–Pons ('18)

examples:

- sylvester congruence = subrope ideal contains only ropes $(u, v, \bigtriangledown, \varnothing)$
- Cambrian congruences = when $\mho \sqcup \Omega = V$

Reading ('06)

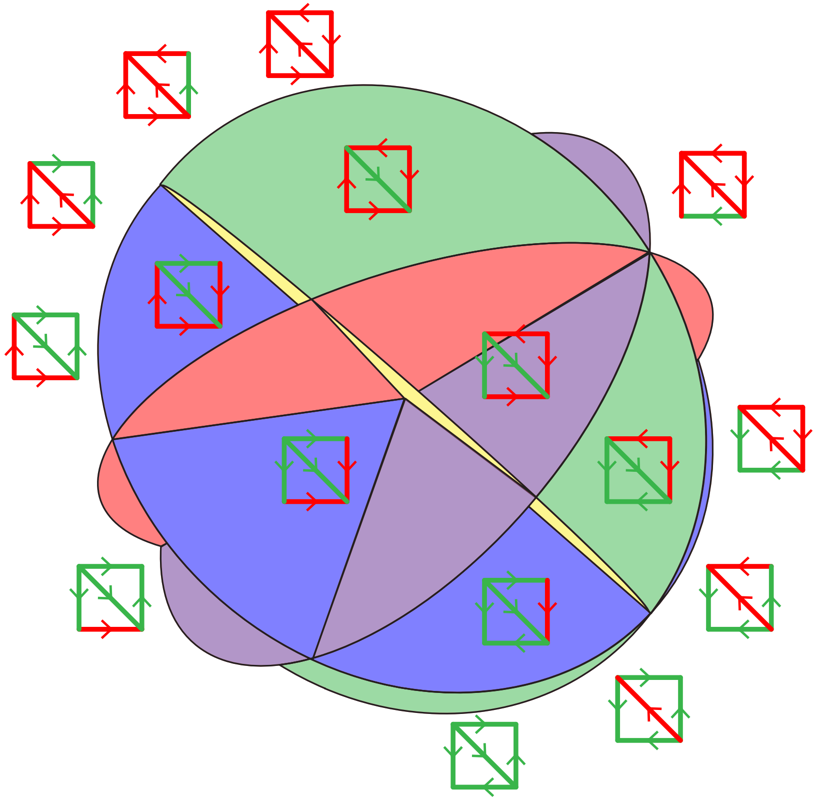
QUOTIENT FANS & QUOTIENTOPES

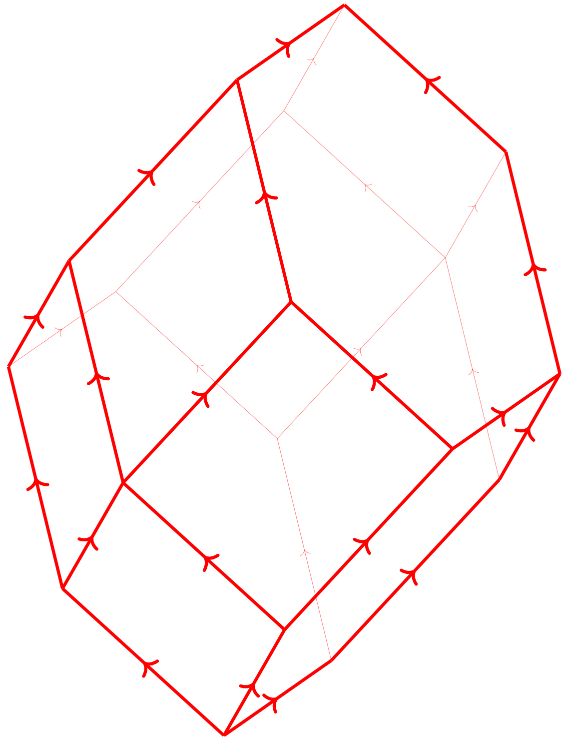
GRAPHICAL ARRANGEMENT & GRAPHICAL ZONOTOPE

D directed acyclic graph

graphical arrangement $\mathcal{H}_D$ = arrangement of hyperplanes $x_u = x_v$ for all arcs $(u, v) \in D$

graphical zonotope $\mathcal{Z}_D$ = Minkowski sum of $[\boldsymbol{e}_u, \boldsymbol{e}_v]$ for all arcs $(u, v) \in D$

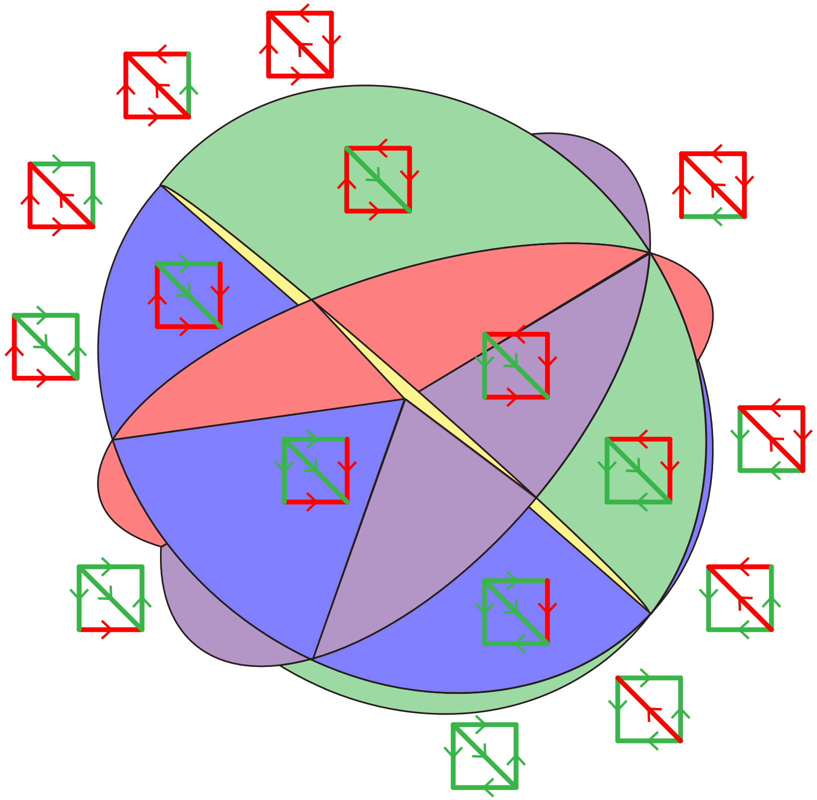


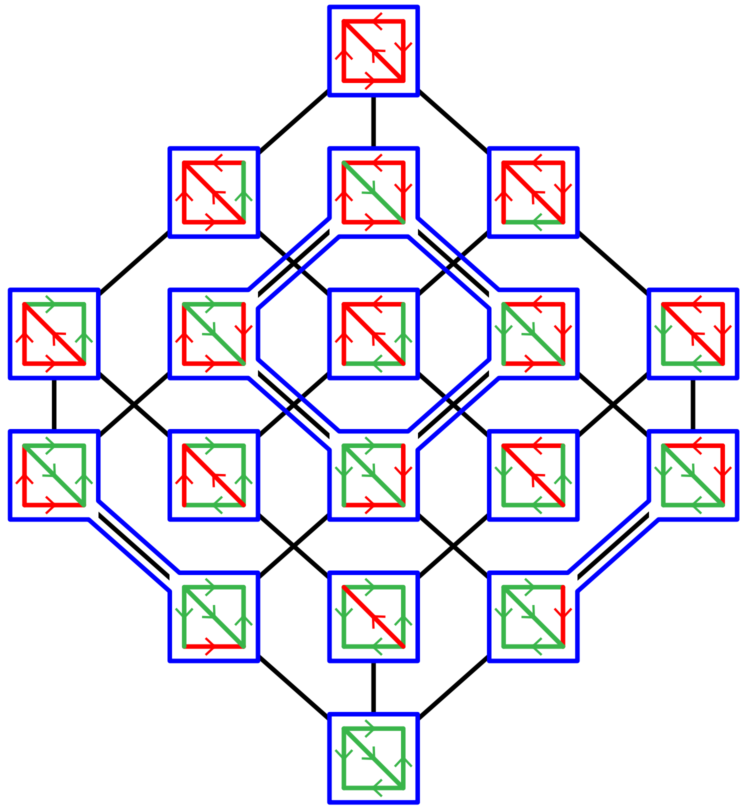


hyperplanes of $\mathcal{H}_D$	$\longleftrightarrow$	summands of $\mathcal{Z}_D$	$\longleftrightarrow$	arcs of D
regions of $\mathcal{H}_D$	$\longleftrightarrow$	vertices of $\mathcal{Z}_D$	$\longleftrightarrow$	acyclic reorientations of D
poset of regions of $\mathcal{H}_D$	$\longleftrightarrow$	oriented graph of $\mathcal{Z}_D$	$\longleftrightarrow$	acyclic reorientation poset of D

QUOTIENT FAN

THM. A lattice congruence $\equiv$ of $\mathcal{AR}_D$ defines a quotient fan $\mathcal{F}_\equiv$ where the chambers of $\mathcal{F}_\equiv$ are obtained by glueing the chambers of $\mathcal{H}_D$ corresponding to acyclic reorientations in the same equivalence class of $\equiv$





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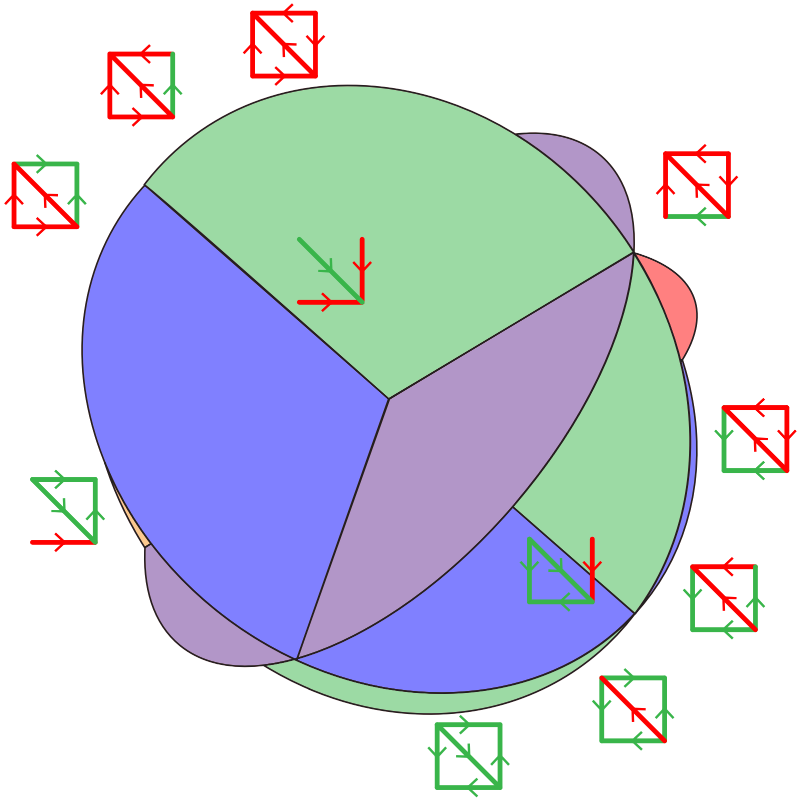
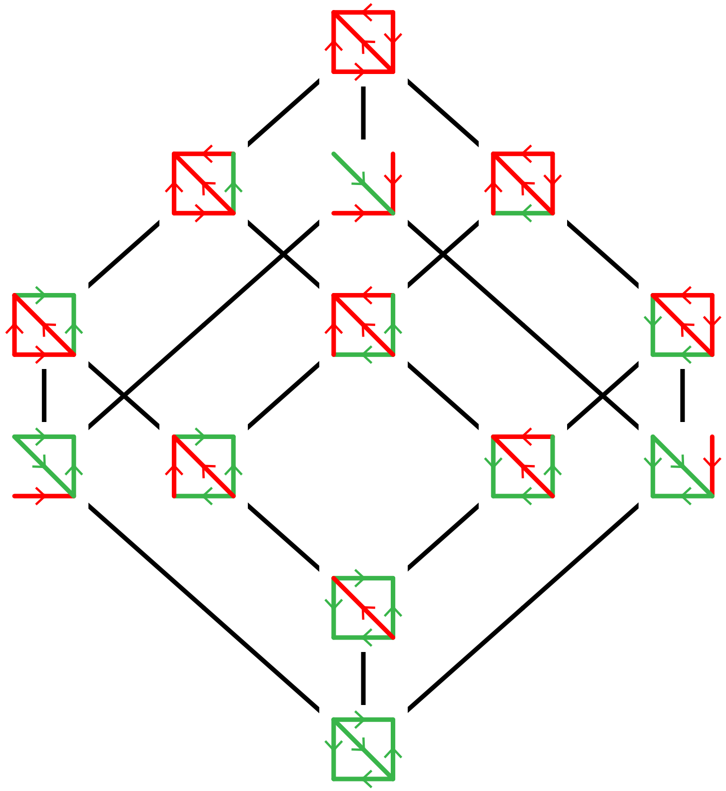
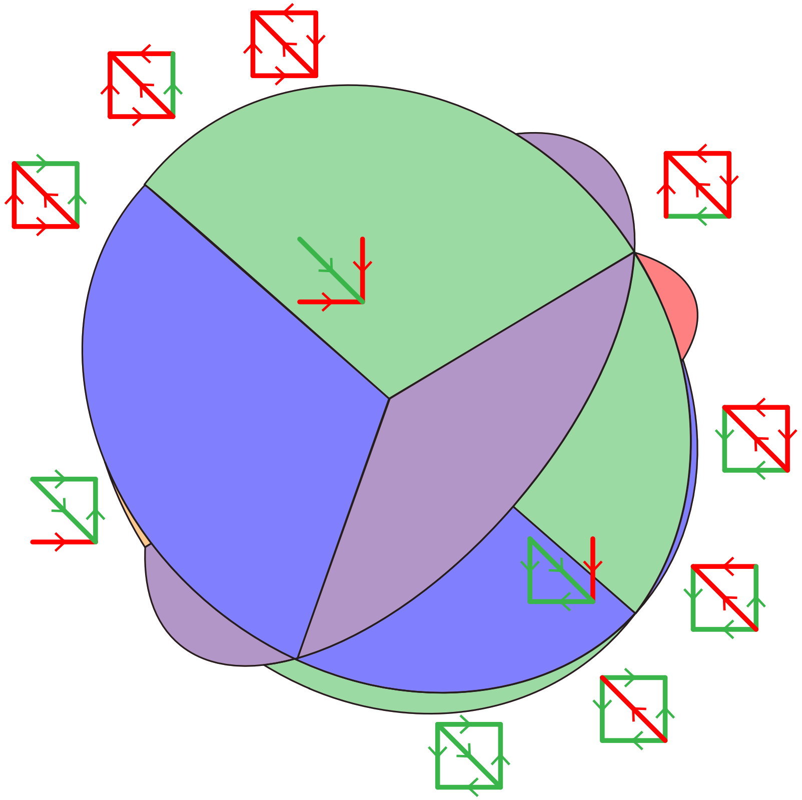


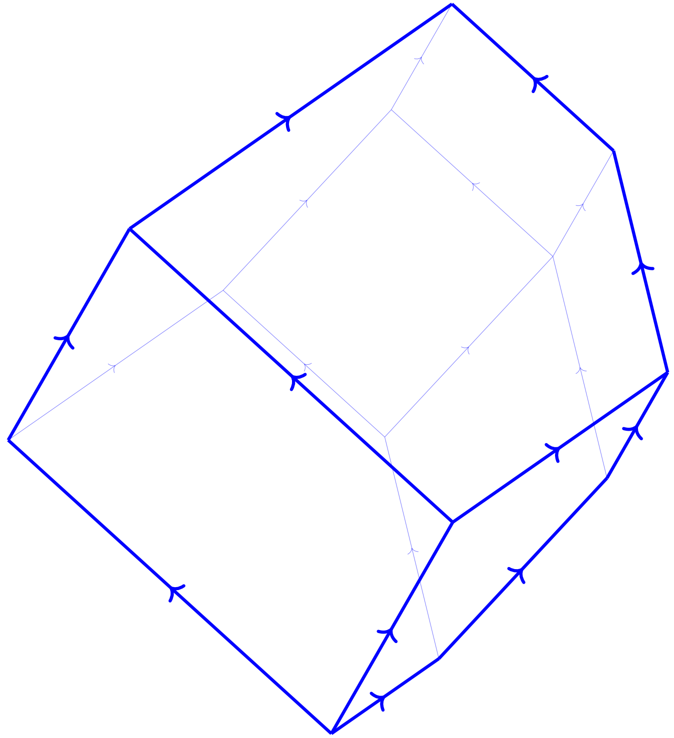


QUOTIENTOPES

THM. The quotient fan $\mathcal{F}_{\equiv}$ of any lattice congruence $\equiv$ of $\mathcal{AR}_D$ is the normal fan of

- a Minkowski sum of associahedra of Hohlweg – Lange, and
- a Minkowski sum of shard polytopes of Padrol – P. – Ritter





ρ-alternating matching = pair $(M_\triangledown, M_\vartriangle)$ with $M_\triangledown \subseteq \{u\} \cup \triangledown$ and $M_\vartriangle \subseteq \vartriangle \cup \{v\}$ s.t. $M_\triangledown$ and $M_\vartriangle$ are alternating along the transitive reduction of D

shard polytope of ρ = convex hull of signed charact. vectors of ρ-alternating matchings

QUOTIENTOPES

THM. The quotient fan $\mathcal{F}_{\equiv}$ of any lattice congruence $\equiv$ of $\mathcal{AR}_D$ is the normal fan of
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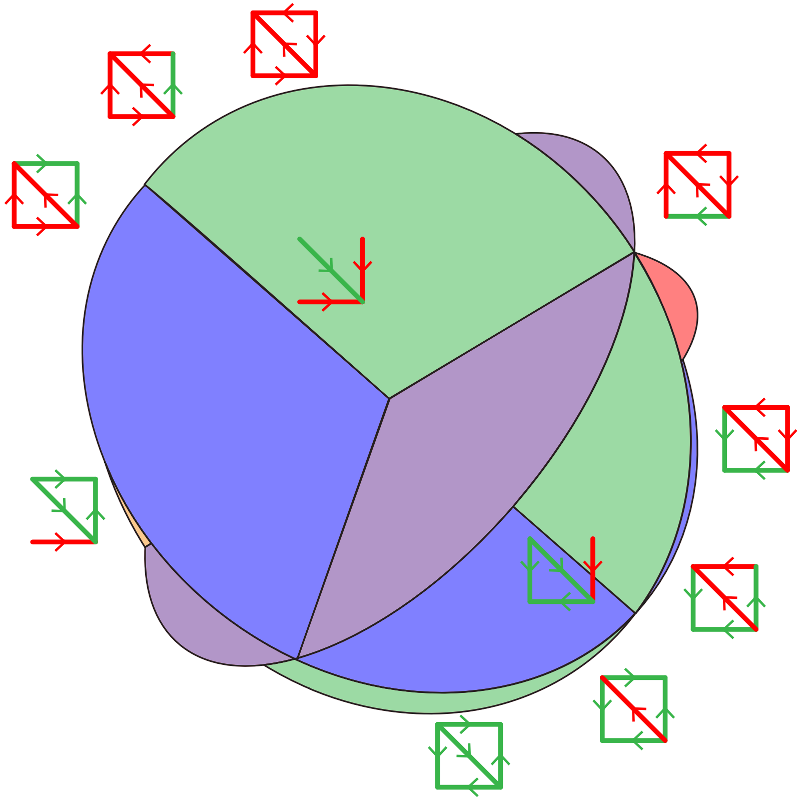
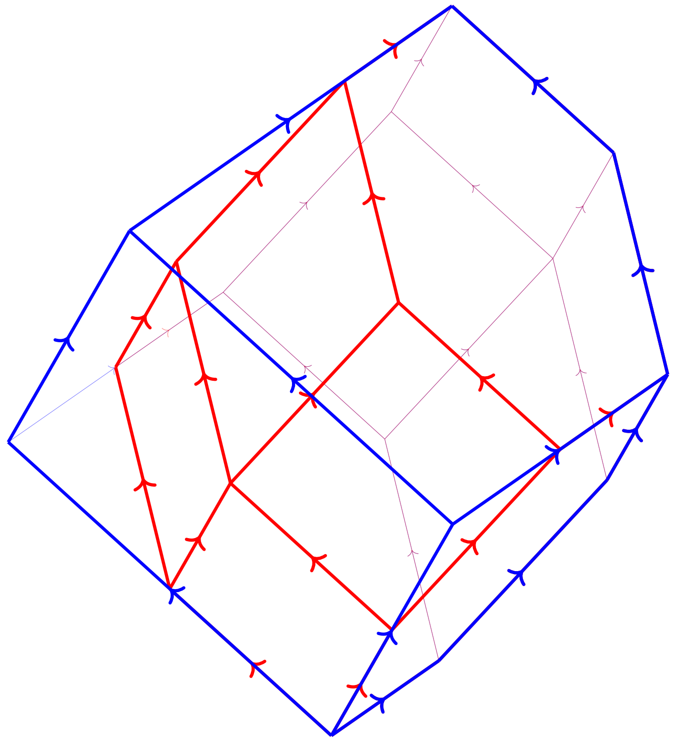
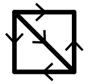
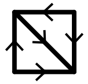


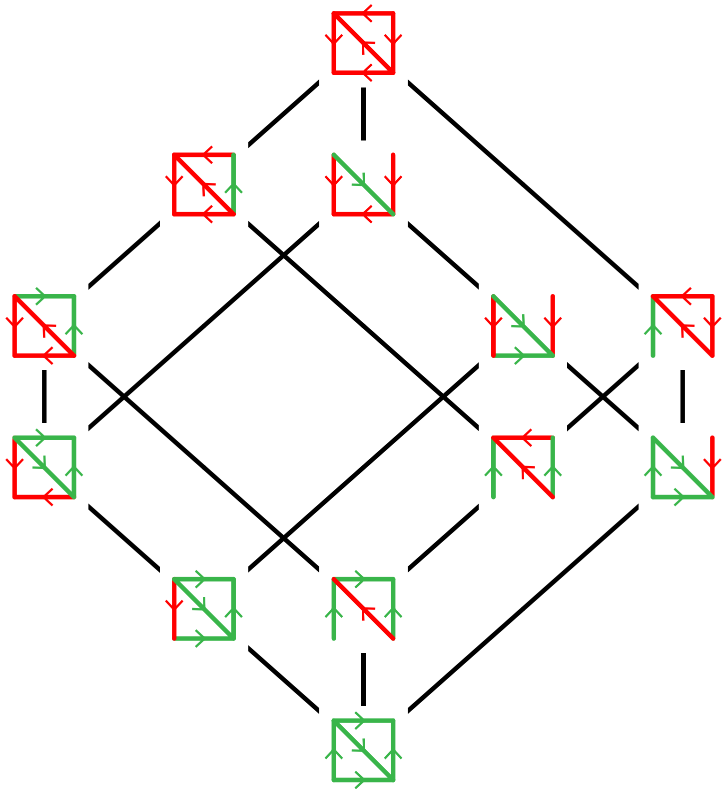


PROP. For the sylvester congruence, all facets defining inequalities of the associahedron of D are facet defining inequalities of the graphical zonotope of D

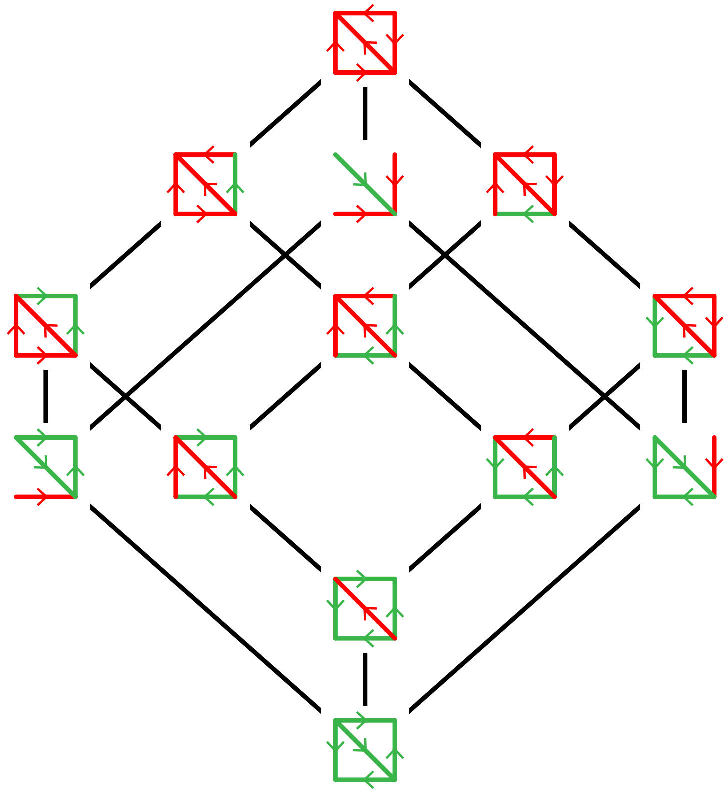
SOME OPEN PROBLEMS

SIMPLE ASSOCIAHEDRA

CONJ. D has no induced subgraph isomorphic to  or 
$\iff$ the Hasse diagram of the D-Tamari lattice is regular
$\iff$ the D-associahedron is a simple polytope

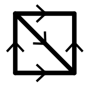


regular



non regular

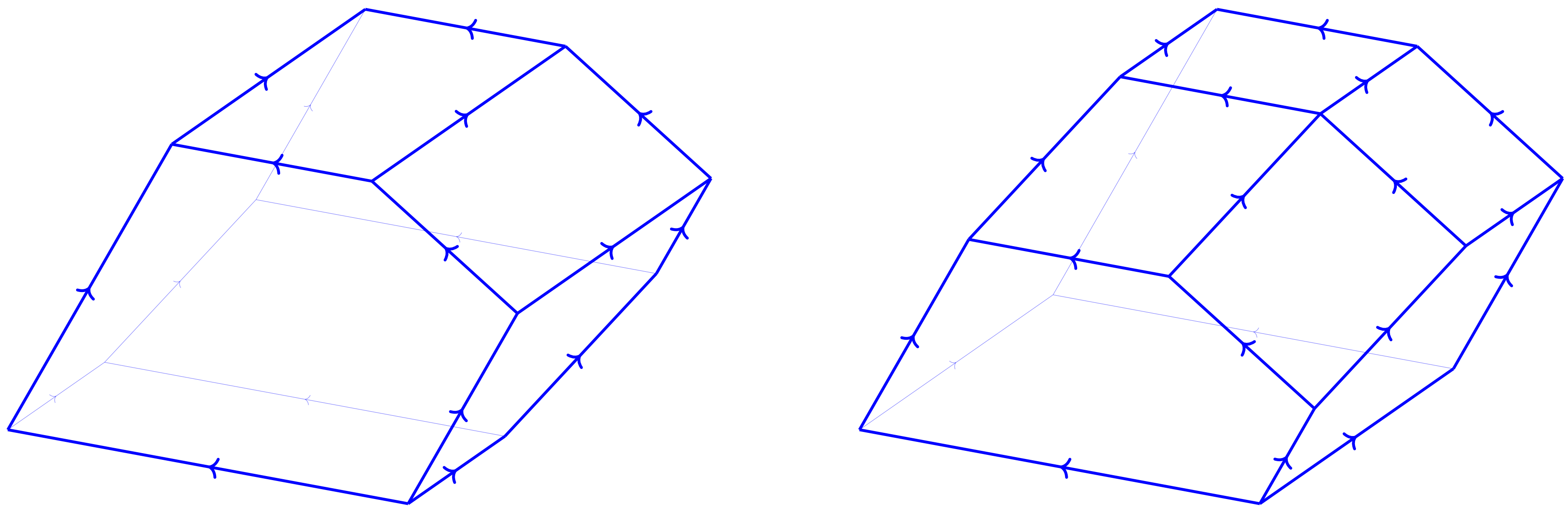
ISOMORPHIC CAMBRIAN ASSOCIAHEDRA

CONJ. D has no induced subgraph isomorphic to 

$\iff$ all Cambrian associahedra of D have the same number of vertices

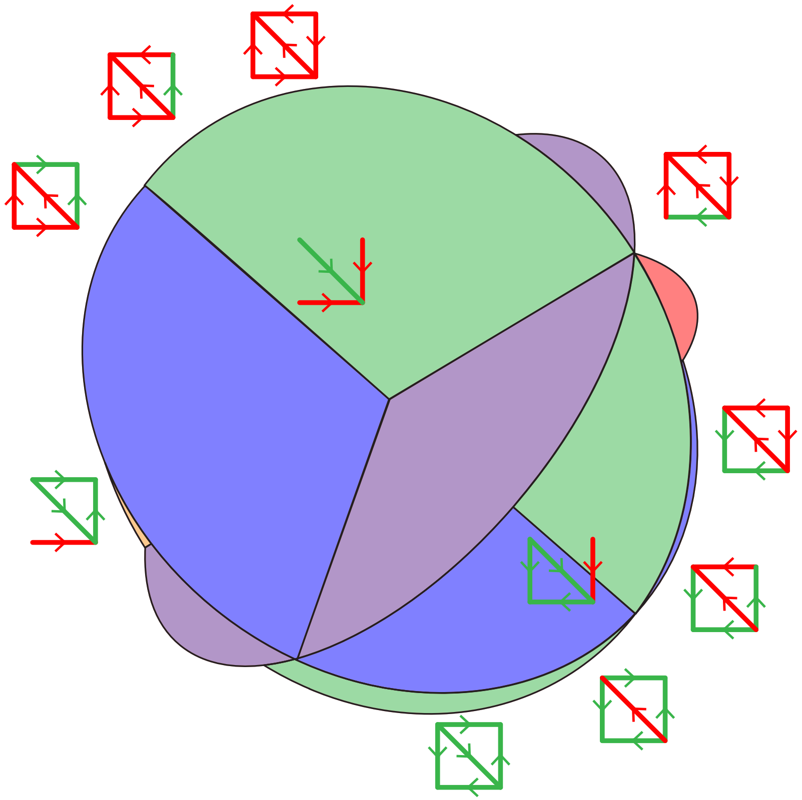
$\iff$ all Cambrian associahedra of D have isomorphic 1-skeleta

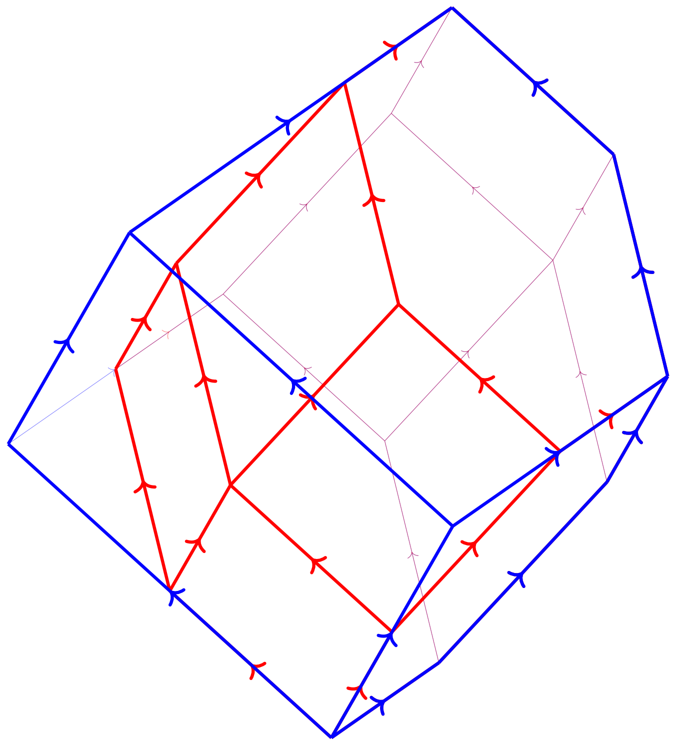
$\iff$ all Cambrian associahedra of D have isomorphic face lattices



REMOVAHEDRA

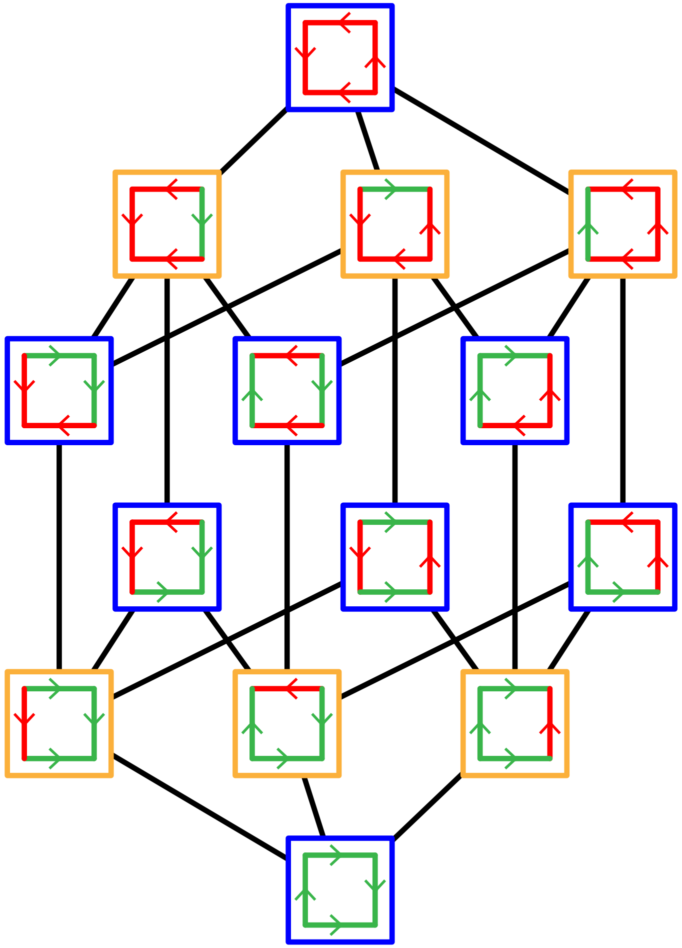
PROP. For the sylvester congruence, all facets defining inequalities of the associahedron of D are facet defining inequalities of the graphical zonotope of D

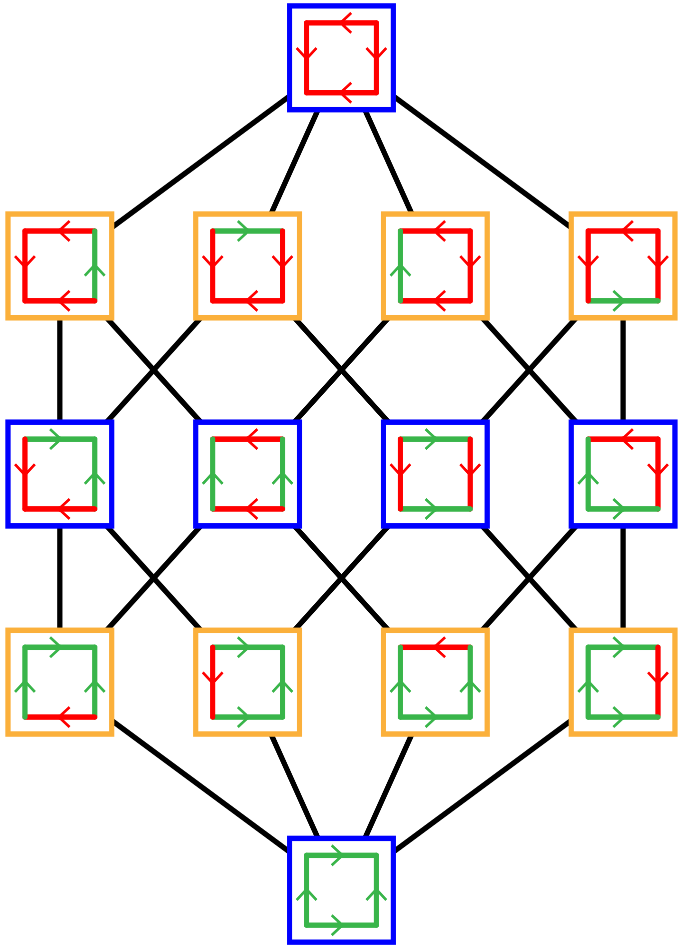




CONJ. For any $\mho, \Omega \subseteq V$, the quotient fan $\mathcal{F}_{(\mho,\Omega)}$ is the normal fan of the polytope obtained by deleting inequalities of the graphical zonotope of D

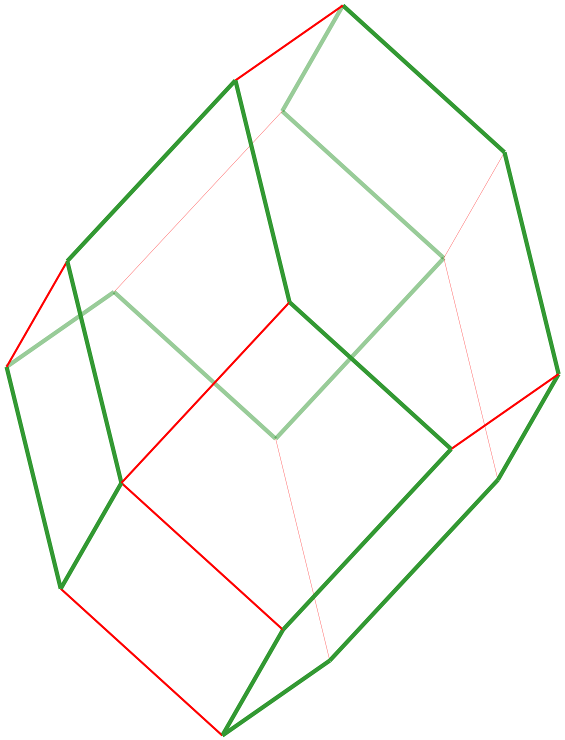
Not all acyclic reorientation flip graphs admit a Hamiltonian cycle

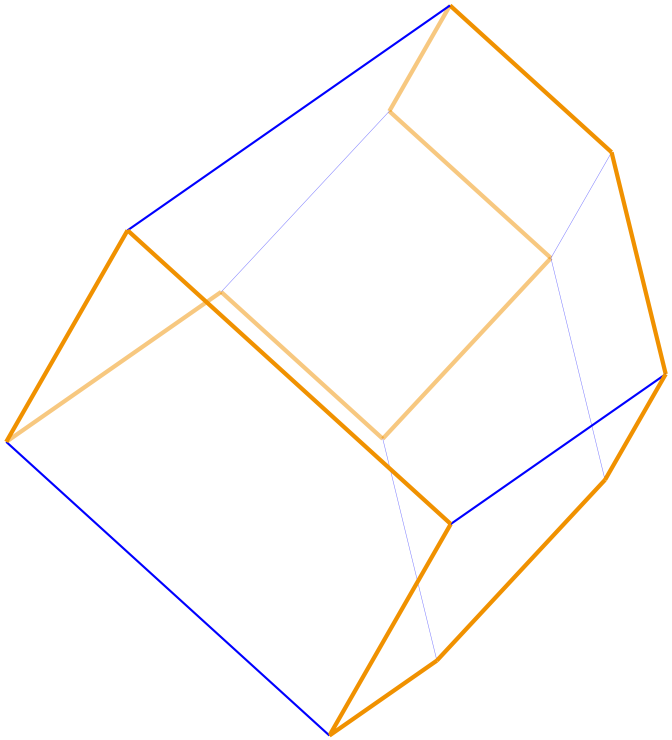




HAMILTONIAN CYCLES

THM [SSW'93]. For D chordal, the acyclic reorientation flip graph is Hamiltonian





CONJ. When D is skeletal, all quotientopes admit a Hamiltonian cycle

... checked for all quotients, for all skeletal acyclic directed graphs up to 5 vertices ...

LATTICE OF REGIONS OF HYPERPLANE ARRANGEMENTS

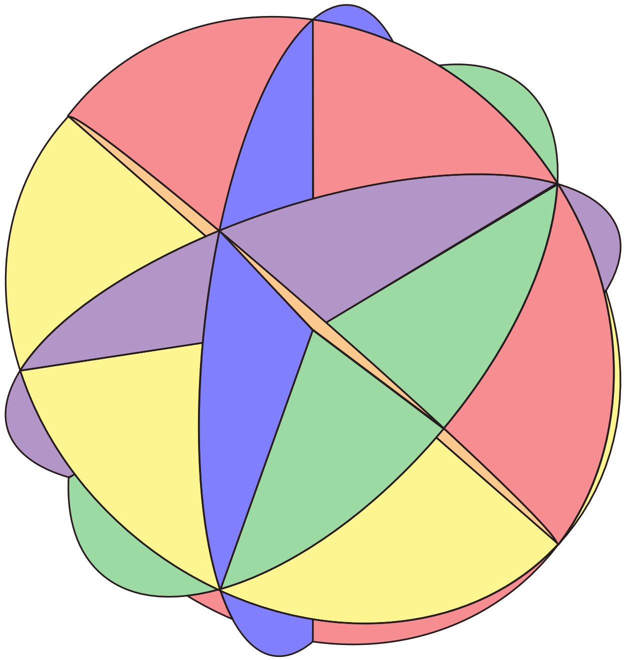
$\mathcal{H}$ hyperplane arrangement in $\mathbb{R}^n$

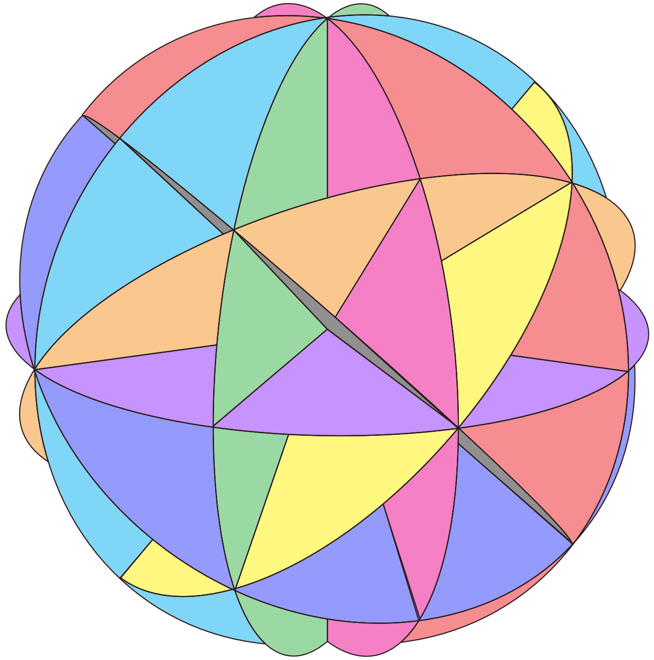
base region B = distinguished region of $\mathbb{R}^n \smallsetminus \mathcal{H}$

inversion set of a region C = set of hyperplanes of $\mathcal{H}$ that separate B and C

poset of regions $\mathrm{PR}(\mathcal{H}, B)$ = regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ ordered by inclusion of inversion sets

QU. For which $(\mathcal{H}, B)$ is the poset of regions PR a lattice?





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QU. For which $(\mathcal{H}, B)$ is the poset of regions PR a lattice?

THM. The poset of regions $\mathrm{PR}(\mathcal{H}, B)$ — Björner–Edelman–Ziegler ('90)
- is never a lattice when B is not a simplicial region
- is always a lattice when $\mathcal{H}$ is a simplicial arrangement

THM. The poset of regions $\mathrm{PR}(\mathcal{H}, B)$ is a semidistributive lattice
$\iff$ $\mathcal{H}$ is tight with respect to B — Reading ('16)

QUOTIENTOPES FOR HYPERPLANE ARRANGEMENTS

$\mathcal{H}$ hyperplane arrangement in $\mathbb{R}^n$
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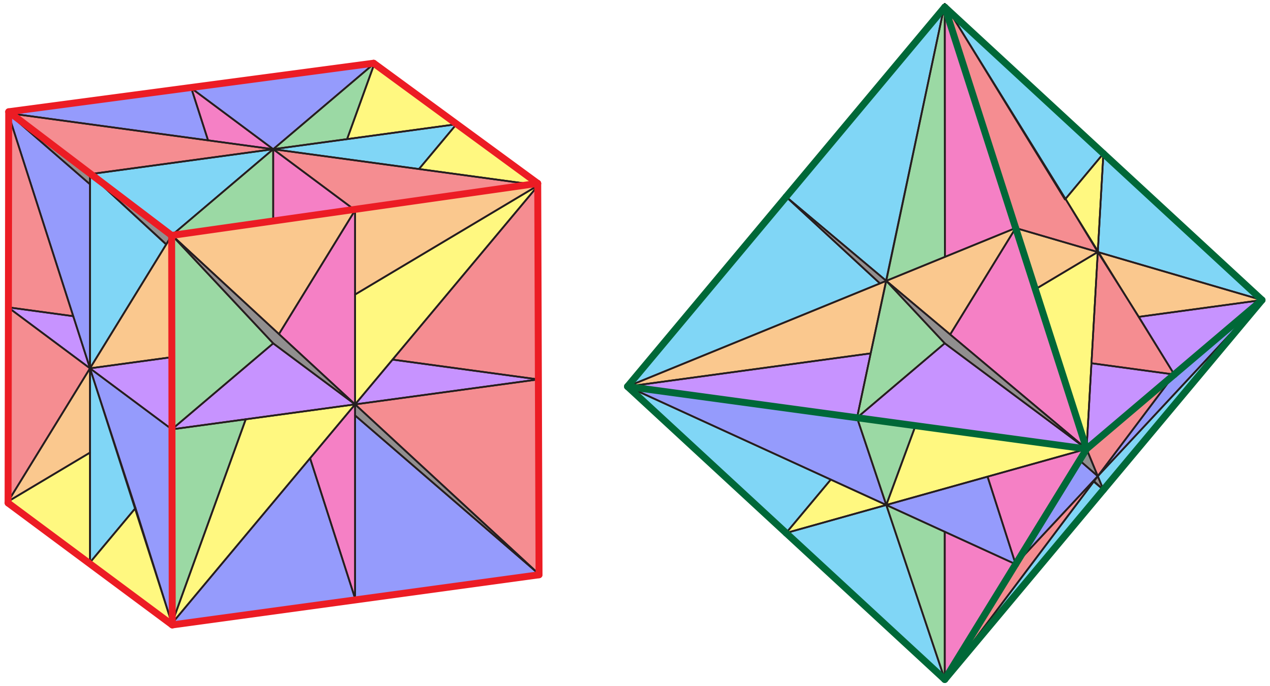
THM. If $\mathrm{PR}(\mathcal{H}, B)$ is a lattice, and $\equiv$ is a congruence of $\mathrm{PR}(\mathcal{H}, B)$, the cones obtained by glueing the regions of $\mathbb{R}^n \setminus \mathcal{H}$ in the same congruence class form a complete fan $\mathcal{F}_\equiv$

Reading ('05)

QU. Is the quotient fan $\mathcal{F}_\equiv$ always polytopal?

QUOTIENTOPES FOR HYPERPLANE ARRANGEMENTS

hyperoctahedral group = isometry group of the hypercube (or of its dual cross-polytope)

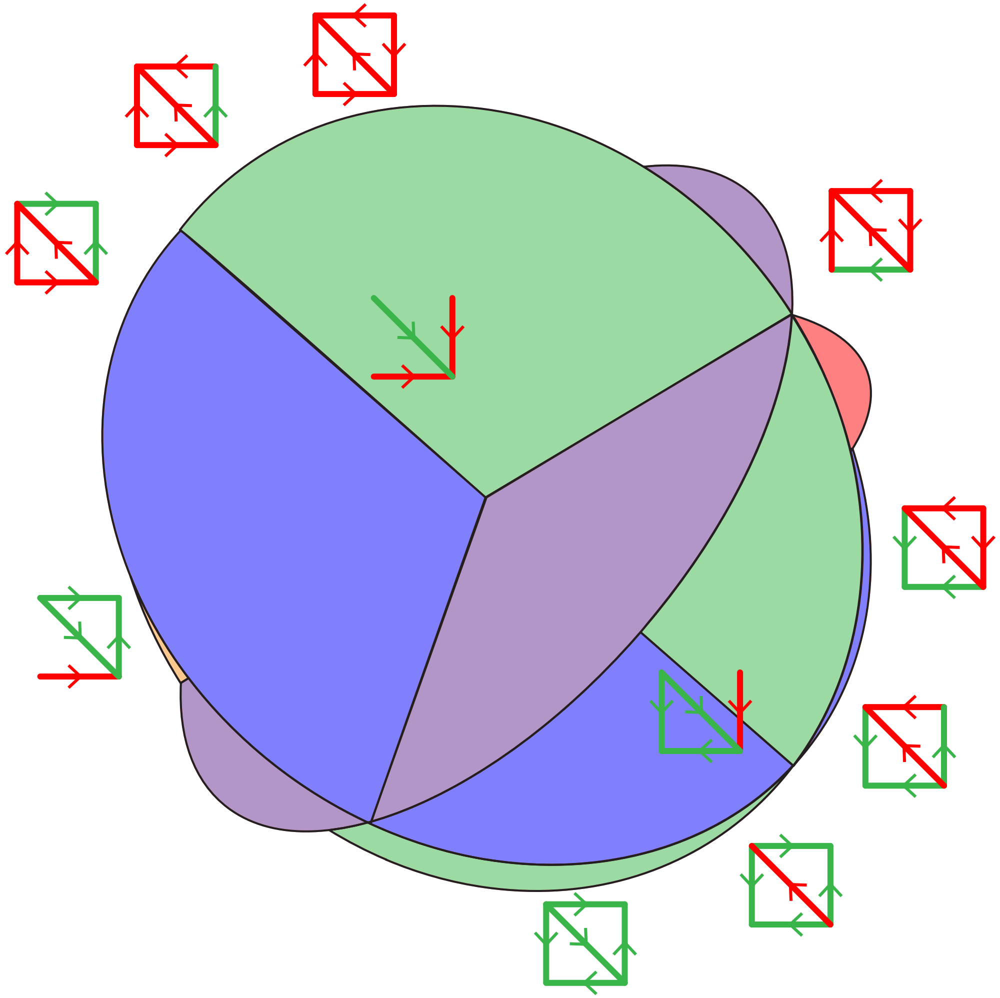


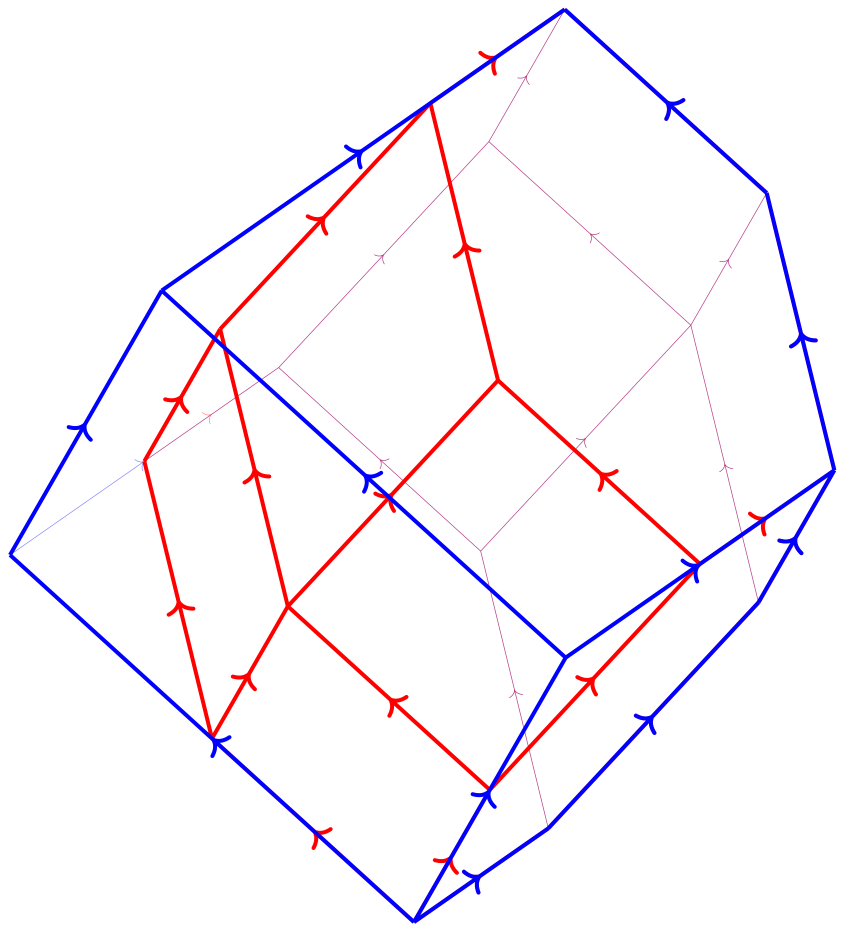
THM. The quotient fan of any lattice congruence of the type B weak order is polytopal

Padrol–P.–Ritter ('20$^+$)

Type B quotientopes are obtained

- not as removahedra,
- not as Minkowski sum of cyclohedra,
- but as Minkowski sum of shard polytopes (but this is another story...)





THANK YOU