The α-representations of the Fisher Information Matrix
— On equivalent expressions of the FIM —

Frank Nielsen

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The Fisher Information Matrix [1] (FIM) for a family of *parametric* probability models $\{p(x;\theta)\}_\theta$ (densities $p(x;\theta)$ expressed with respect to a base measure ν) indexed by a D-dimensional parameter vector $\theta := (\theta^1, \ldots, \theta^D)$ is originally defined by

$$I(\theta) := (I_{ij}(\theta)), \quad I_{ij}(\theta) := E_{p(x;\theta)}[\partial_i l(x;\theta) \partial_j l(x;\theta)], \tag{1}$$

where $l(x;\theta) := \log p(x;\theta)$ is the log-likelihood function, and $\partial_i :=: \frac{\partial}{\partial \theta^i}$ (by notational convention). The FIM is a $D \times D$ positive semi-definite matrix for a D-order family.

The FIM is a cornerstone in statistics and occurs in many places, like for example the celebrated Cramér-Rao lower bound [2] for an unbiased estimator $\hat{\theta}$

$$\mathrm{Var}_{p(x;\theta)}[\hat{\theta}] \succeq I^{-1}(\theta),$$

where $\succeq$ denotes the Löwner ordering of positive semi-definite matrices: $A \succeq B$ iff. $A - B \succ 0$ is positive semi-definite. Another use of the FIM is in gradient descent method for deep learning using the natural gradient [4].

Yet, it is common to encounter another equivalent expression of the FIM in the litterature [2, 1]

$$I'_{ij}(\theta) := 4 \int \partial_i \sqrt{p(x;\theta)} \partial_j \sqrt{p(x;\theta)} \mathrm{d}\nu(x) \tag{2}$$

This form of the FIM is well-suited to prove that the FIM is always positive semi-definite matrix [1]: $I(\theta) \succeq 0$.

It turns out that one can define a family of equivalent representations of the FIM using the α-embedding of the parametric family. We define the α-representation of densities $l^{(\alpha)}(x;\theta) := k_\alpha(p(x;\theta))$ with

$$k_\alpha(u) := \begin{cases} \frac{2}{1-\alpha} u^{\frac{1-\alpha}{2}}, & \text{if } \alpha \neq 1 \\ \log u, & \text{if } \alpha = 1. \end{cases} \tag{3}$$

The function $l^{(\alpha)}(x;\theta)$ is called the α-likelihood function.

The α-representation of the FIM is

$$\boxed{I_{ij}^{(\alpha)}(\theta) := \int \partial_i l^{(\alpha)}(x;\theta) \partial_j l^{(-\alpha)}(x;\theta) \mathrm{d}\nu(x)} \tag{4}$$

In compact notation, we have $I_{ij}^{(\alpha)}(\theta) = \int \partial_i l^{(\alpha)} \partial_j l^{(-\alpha)} \mathrm{d}\nu(x)$ (this is the α-FIM). We can expand the α-FIM expressions as follows

$$I_{ij}^{(\alpha)}(\theta) = \begin{cases} \frac{1}{1-\alpha^2} \int \partial_i p(x;\theta)^{\frac{1-\alpha}{2}} \partial_j p(x;\theta)^{\frac{1+\alpha}{2}} \mathrm{d}\nu(x) & \text{for } \alpha \neq \pm 1 \\ \int \partial_i \log p(x;\theta) \partial_j p(x;\theta) \mathrm{d}\nu(x) & \text{for } \alpha \in \{-1, 1\} \end{cases}$$

The proof that $I_{ij}^{(\alpha)}(\theta) = I_{ij}(\theta)$ follows from the fact that

$$\partial_i l^\alpha = p^{-\frac{\alpha+1}{2}} \partial_i p = p^{\frac{1-\alpha}{2}} \partial_i l,$$

since $\partial_i l = \frac{\partial_i p}{p}$.

Therefore we get

$$\partial_i l^{(\alpha)} \partial_j l^{(-\alpha)} = p \partial_i l \partial_j l,$$

and $I_{ij}^{(\alpha)}(\theta) = E[\partial_i l \partial_j l] = I_{ij}(\theta)$.

Thus Eq. 1 and Eq. 2 where two examples of the α-representation, namely the 1-representation and the 0-representation, respectively. The 1-representation of Eq. 1 is called the logarithmic representation, and the 0-representation of Eq. 2 is called the square root representation.

Note that $I_{ij}(\theta) = E[\partial_i l \partial_j l] = \int p \partial_i l \partial_j l \mathrm{d}\nu(x) = \int \partial_i p \partial_j l \mathrm{d}\nu(x) = I_{ij}^{(1)}(\theta)$ since $\partial_i l = \frac{\partial_i p}{p}$

In information geometry [1], $\{\partial_i l^{(\alpha)}\}_i$ plays the role of tangent vectors, the α-scores. Geometrically speaking, the tangent plane $T_{p(x;\theta)}$ can be described using any α-base. The statistical manifold $M = \{p(x;\theta)\}_\theta$ is imbedded into the function space $\mathbb{R}^{\mathcal{X}}$, where $\mathcal{X}$ denotes the support of the densities.

Under regular conditions [2, 1], the α-representation of the FIM can further be rewritten as

$$I_{ij}^{(\alpha)}(\theta) = -\frac{2}{1+\alpha} \int p(x;\theta)^{\frac{1+\alpha}{2}} \partial_i \partial_j l^{(\alpha)}(x;\theta) \mathrm{d}\nu(x).$$

Since we have

$$\partial_i \partial_j l^{(\alpha)}(x;\theta) = p^{\frac{1-\alpha}{2}} \left(\partial_i \partial_j l + \frac{1-\alpha}{2} \partial_i l \partial_j l \right),$$

it follows that

$$I_{ij}^{(\alpha)}(\theta) = -\frac{2}{1+\alpha} \left(-I_{ij}^{(\alpha)}(\theta) + \frac{1-\alpha}{2} I_{ij}^{(\alpha)} \right) = I_{ij}(\theta).$$

Notice that when $\alpha = 1$, we recover the equivalent expression of the FIM (under mild conditions)

$$I_{ij}^{(1)}(\theta) = -E[\nabla^2 \log p(x;\theta)].$$

In particular, when the family is an exponential family [3] with cumulant function $F(\theta)$, we have

$$I(\theta) = \nabla^2 F(\theta).$$

Similarly, the coefficients of the α-connection can be expressed using the α-representation as

$$\Gamma_{ij,k}^{(\alpha)} = \int \partial_i \partial_j l^{(\alpha)} \partial_k^{(-\alpha)} \mathrm{d}\nu(x).$$

The Riemannian metric tensor g_{ij} (a geometric object) can be expressed in matrix form $I_{ij}^{(\alpha)}(\theta)$ using the α-base, and this tensor is called the Fisher metric tensor.

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References

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[4] Ke Sun and Frank Nielsen. Relative Fisher information and natural gradient for learning large modular models. In Doina Precup and Yee Whye Teh, editors, *Proceedings of the 34th International Conference on Machine Learning*, volume 70 of *Proceedings of Machine Learning Research*, pages 3289–3298, International Convention Centre, Sydney, Australia, 06–11 Aug 2017. PMLR.