The dual Voronoi diagrams with respect to representational Bregman divergences

Frank Nielsen and Richard Nock

frank.nielsen@polytechnique.edu

École Polytechnique, LIX, France

Sony Computer Science Laboratories Inc, FRL, Japan

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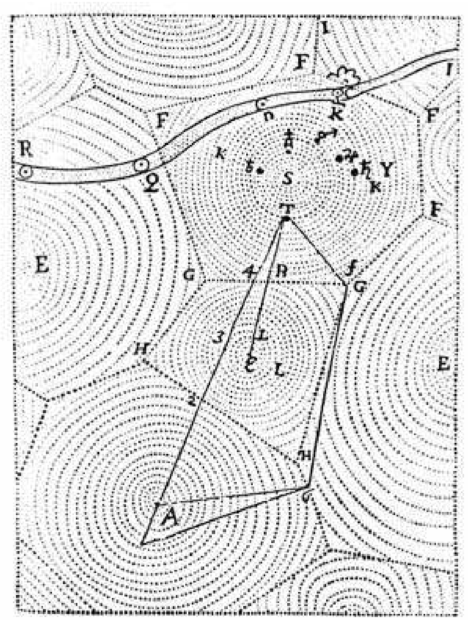
Ordinary Voronoi diagram

$\mathcal{P} = \{P_1, ..., P_n\} \in \mathcal{X}$: point set with vector coordinates $\mathbf{p}_1, ..., \mathbf{p}_n \in \mathbb{R}^d$.
Voronoi diagram: partition in *proximal regions* $\mathrm{vor}(P_i)$ of $\mathcal{X}$ wrt. a distance:

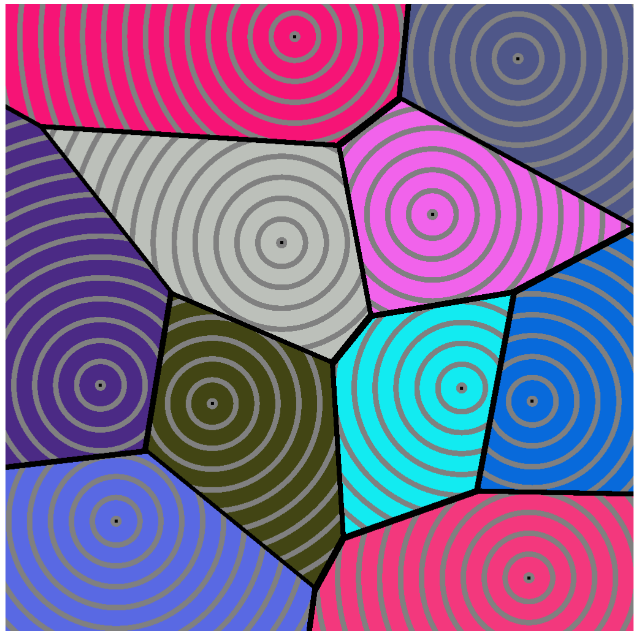
$$\mathrm{vor}(P_i) = \{X \in \mathcal{X} \mid D(X, P_i) \leq D(X, P_j)\ \forall j \in \{1, ..., n\}\}.$$

Ordinary Voronoi diagram in Euclidean geometry defined for

$$D(X, Y) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2}.$$



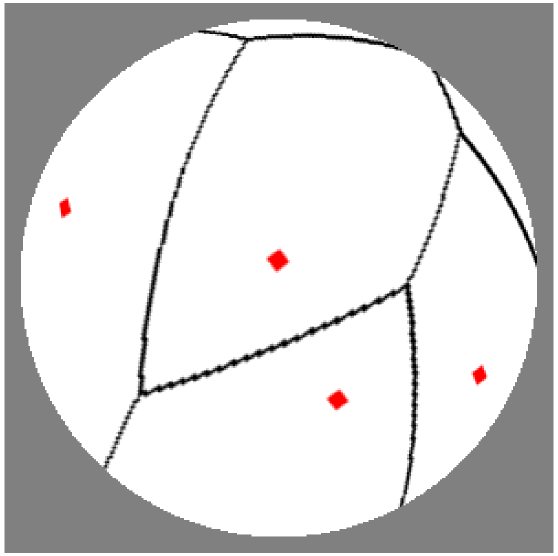
René Descartes' manual rendering (17th C.)



computer rendering

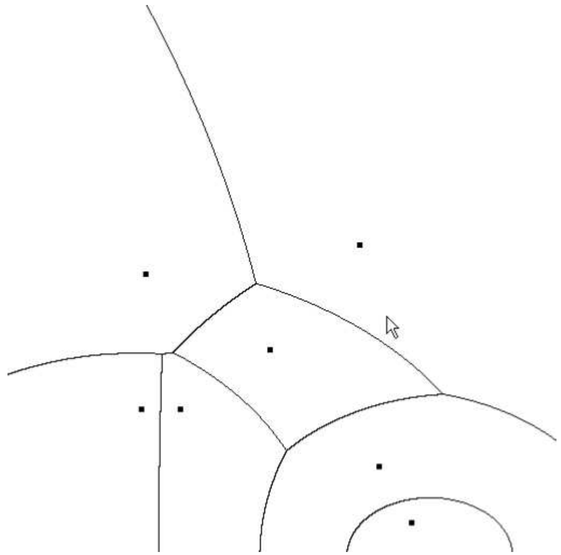
Voronoi diagram in abstract geometries

Birth of non-Euclidean geometries (accepted in 19th century)
Spherical (elliptical) and hyperbolic (Lobachevsky) imaginary geometries



Spherical Voronoi

$$D(p,q) = \arccos\langle p, q\rangle$$



Hyperbolic Voronoi (Poincaré upper plane)

$$D(p,q) = \mathrm{arccosh} 1 + \frac{\|p-q\|^2}{2 p_y q_y}$$ with

$$\mathrm{arccosh} x = \log(x + \sqrt{x^2 - 1})$$

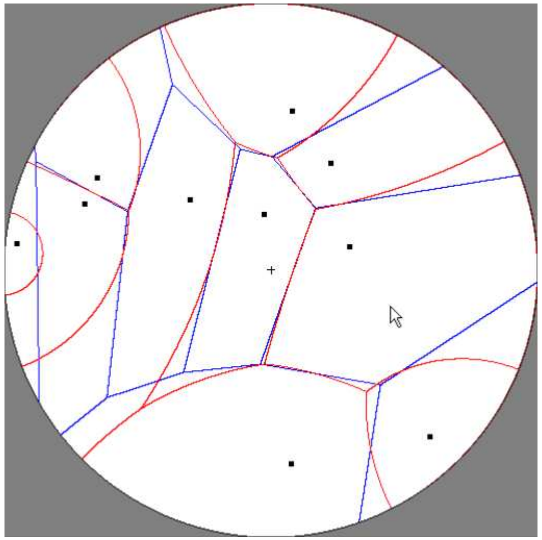
$$D(p,q) = \log \frac{p_y}{q_y}$$ (vertical line)

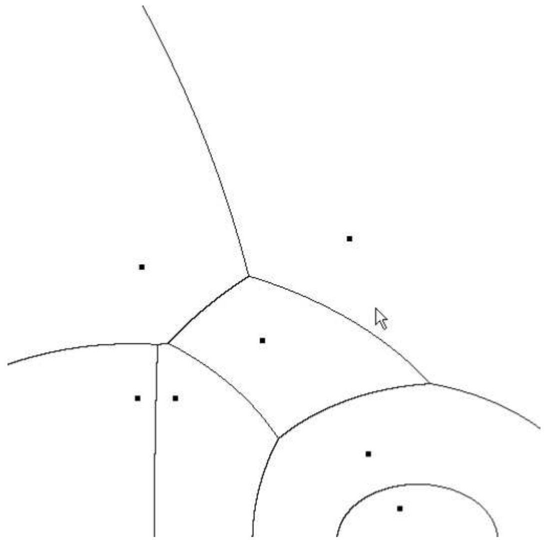
$$D(p,q) = \frac{|p_x - q_x|}{p_y}$$ (horizontal line)

Voronoi diagram in embedded geometries

Imaginary geometry can be realized in **many different** ways.
For example, hyperbolic geometry:

- Conformal Poincaré upper half-space,
- Conformal Poincaré disk,
- Non-conformal Klein disk,
- Pseudo-sphere in Euclidean geometry, etc.





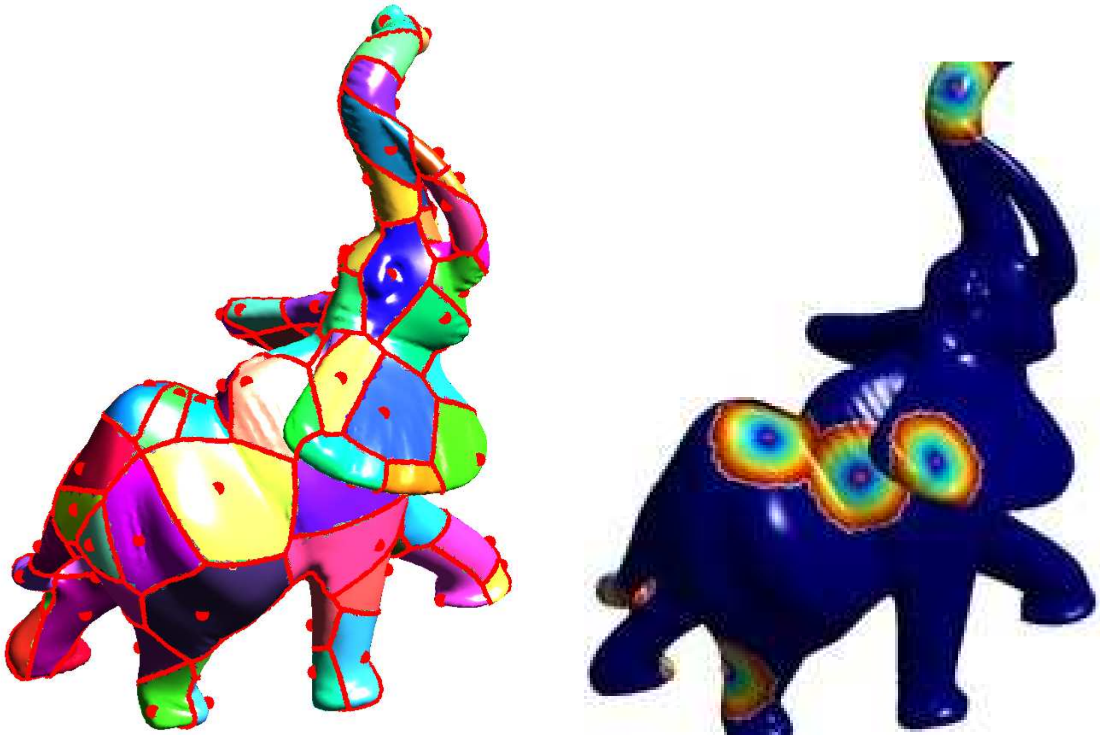
Hyperbolic Voronoi diagrams made easy, arXiv:0903.3287, 2009.

Distance between two corresponding points in *any* isometric embedding is the same.

Voronoi diagrams in Riemannian geometries

Riemannian geometry ($\longrightarrow \infty$ many abstract geometries).
Metric tensor g_{ij} (Euclidean $g_{ij}(p) = \mathrm{Id}$)
Geodesic: minimum length path (non-uniqueness, cut-loci)

Geodesic Voronoi diagram

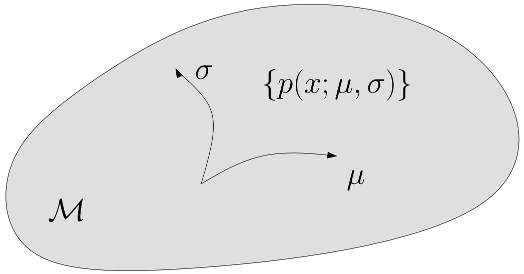


Nash embedding theorem: Every Riemannian manifold can be isometrically embedded in a Euclidean space $\mathbb{R}^d$.

Voronoi diagram in information geometries

Information geometry: Study of manifolds of probability (density) families.
$\longrightarrow$ Relying on differential geometry.

For example, $\mathcal{M} = \{p(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp -\frac{(x-\mu)^2}{2\sigma^2}\}$



Riemannian setting: Fisher information and induced Riemannian metric:

$$I(\theta) = \mathbb{E}\left[\frac{\partial}{\partial \theta_i} \log p(x;\theta) \frac{\partial}{\partial \theta_j} \log p(x;\theta) | \theta\right] = g_{ij}(\theta)$$

Distance is geodesic length (Rao, 1945)

$$D(P,Q) = \int_{t=0}^{t=1} \sqrt{g_{ij}(t(\theta))} \mathrm{d}t, \qquad t(\theta_0) = \theta(P), t(\theta_1) = \theta(Q)$$

Voronoi diagram in information geometries

Non-metric oriented divergences: $D(P,Q) \neq D(P,Q)$
Fundamental statistical distance is the Kullback-Leibler divergence:

$$\mathrm{KL}(P||Q) = \mathrm{KL}(p(x)||q(x)) = \int_x p(x) \log \frac{p(x)}{q(x)} \mathrm{d}x.$$

$$\mathrm{KL}(P||Q) = \mathrm{KL}(p(x)||q(x)) = \sum_{i=1}^{m} p_i \log \frac{p_i}{q_i}$$

Relative entropy, information divergence, discrimination measure, differential entropy.
Foothold in information/coding theory:

$$\mathrm{KL}(P||Q) = H^{\times}(P||Q) - H(P) \geq 0$$

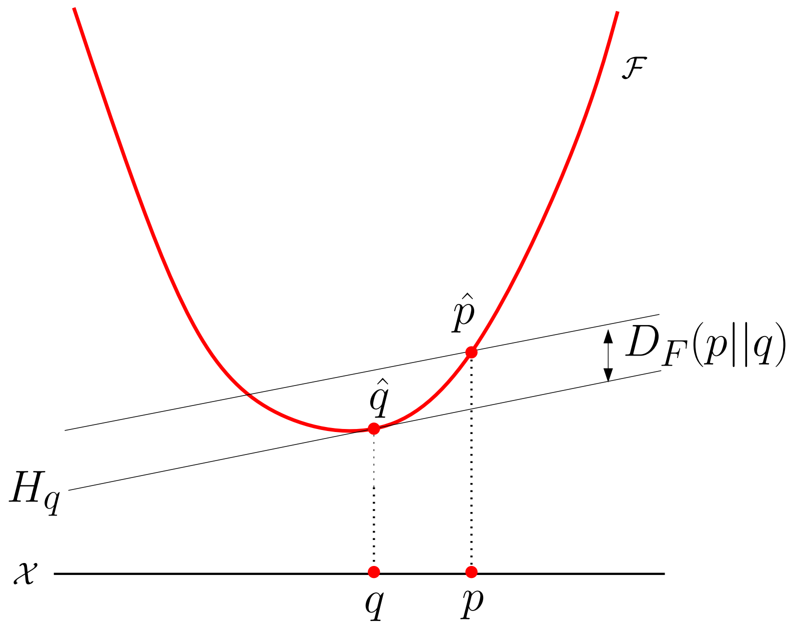
where $H(P) = -\int p(x) \log p(x) \mathrm{d}x$ and $H^{\times}(P||Q) = -\int p(x) \log q(x) \mathrm{d}x$ (cross-entropy).
$\rightarrow$ Dual connections & non-Riemannian geodesics.

Dually flat spaces: Canonical Bregman divergences

Strictly convex and differentiable generator $F : \mathbb{R}^d \rightarrow \mathbb{R}$.
Bregman divergence between any two vector points $\mathbf{p}$ and $\mathbf{q}$:

$$D_F(\mathbf{p}||\mathbf{q}) = F(\mathbf{p}) - F(\mathbf{q}) - \langle \mathbf{p}-\mathbf{q}, \nabla F(\mathbf{q})\rangle,$$

where $\nabla F(\mathbf{x})$ denote the gradient of F at $\mathbf{x} = [x_1 \ldots x_d]^T$.



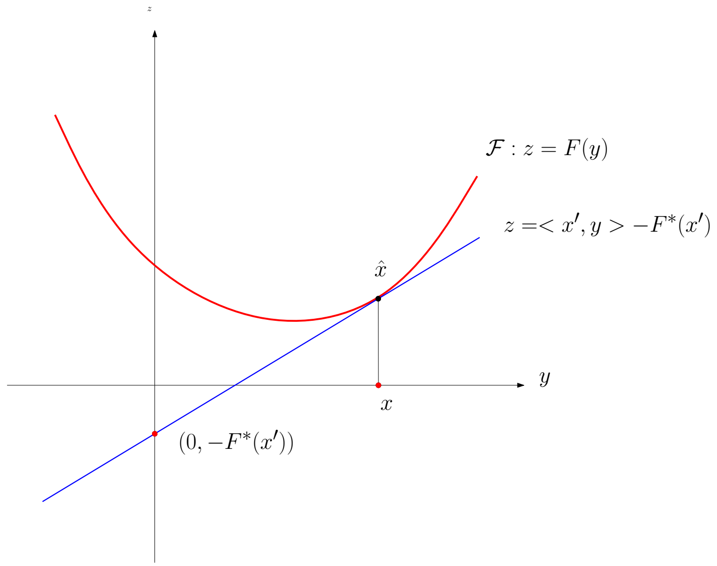
$F(\mathbf{x}) = \mathbf{x}^T\mathbf{x} = \sum_{i=1}^d x_i^2 \longrightarrow$ squared Euclidean distance: $\|\mathbf{p}-\mathbf{q}\|^2$.
$F(\mathbf{x}) = \sum_{i=1}^d x_i \log x_i$ (Shannon's negative entropy) $\longrightarrow$ Kullback-Leibler divergence: $\sum_i p_i \log \frac{p_i}{q_i}$

Legendre transformation & convex conjugates

Divergence D_F written in dual form using **Legendre transformation**:

$$F^*(\mathbf{x}^*) = \max_{\mathbf{x} \in \mathbb{R}^d} \{ \langle \mathbf{x}, \mathbf{x}^* \rangle - F(\mathbf{x}) \}$$

is convex in $\mathbf{x}^*$.
Legendre convex conjugates $F, F^* \rightarrow$ dual Bregman generators.
$\mathbf{x}^* = \nabla F(\mathbf{x})$: one-to-one mapping defining a dual coordinate system.



$$F^{**} = F, \qquad \nabla F^* = (\nabla F)^{-1}$$

Bregman Voronoi Diagrams: Properties, Algorithms and Applications, arXiv:0709.2196, 2007.

Canonical divergences (contrast functions)

Convex conjugates F and F^* (with $\mathbf{x}^* = \nabla F(\mathbf{x})$ and $\mathbf{x} = \nabla F^*(\mathbf{x}^*)$):

$$B_F(\mathbf{p}||\mathbf{q}) = F(\mathbf{p}) + F^*(\mathbf{q}^*) - \langle \mathbf{p}, \mathbf{q}^* \rangle.$$

Dual Bregman divergence B_{F^*}:

$$B_F(\mathbf{p}||\mathbf{q}) = B_{F^*}(\mathbf{q}^*||\mathbf{p}^*).$$

Two coordinate systems $\mathbf{x}$ and $\mathbf{x}^*$ define a **dually flat structure** in $\mathbb{R}^d$:

$$\begin{aligned} \mathbf{c}(\lambda) &= (1-\lambda)\mathbf{p} + \lambda\mathbf{q} && F\text{-geodesic passing through } P \text{ to } Q \\ \mathbf{c}^*(\lambda) &= (1-\lambda)\mathbf{p}^* + \lambda\mathbf{q}^* && F^*\text{-geodesic (dual)} \end{aligned}$$

$\longrightarrow$ Two “straight” lines with respect to the dual coordinate systems $\mathbf{x}/\mathbf{x}^*$.
Non-Riemannian geodesics.

Separable Bregman divergences & representation functions

- **Separable** Bregman divergence:

$$B_F(\mathbf{p}||\mathbf{q}) = \sum_{i=1}^{d} B_F(p_i||q_i),$$

where $B_F(p||q)$ is a 1D Bregman divergence acting on scalars.

$$F(\mathbf{x}) = \sum_{i=1}^{d} F(x_i)$$

for a **decomposable** generator F.

- Strictly monotonous **representation** function $k(\cdot)$
$\rightarrow$ a non-linear coordinate system $x_i = k(s_i)$ (and $\mathbf{x} = k(\mathbf{s})$).
Mapping is bijective

$$\mathbf{s} = k^{-1}(\mathbf{x}).$$

Representational Bregman divergences

- Bregman generator

$$U(\mathbf{x}) = \sum_{i=1}^{d} U(x_i) = \sum_{i=1}^{d} U(k(s_i)) = F(\mathbf{s})$$

  with $F = U \circ k$.

- Dual 1D generator $U^*(x^*) = \max_x \{xx^* - U(x)\}$ induces dual coordinate system $x_i^* = U'(x_i)$, where U' denotes the derivative of U. $\nabla U(\mathbf{x}) = [U'(x_1) \ldots U'(x_d)]^T$.

Canonical **separable representational Bregman divergence**:

$$B_{U,k}(\mathbf{p}||\mathbf{q}) = U(k(\mathbf{p})) + U^*(k^*(\mathbf{q}^*)) - \langle k(\mathbf{p}), k^*(\mathbf{q}^*)\rangle,$$

with $k^*(\mathbf{x}^*) = U'(k(\mathbf{x}))$.

Often, a Bregman by setting $F = U \circ k$. But although U is a strictly convex and differentiable function and k a strictly monotonous function, $F = U \circ k$ may not be strictly convex.

Dual representational Bregman divergences

$$B_{U,k}(\mathbf{p}||\mathbf{q}) = U(k(\mathbf{p})) - U(k(\mathbf{q})) - \langle k(\mathbf{p}) - k(\mathbf{q}), \nabla U(k(\mathbf{q})) \rangle .$$

This is the Bregman divergence acting on the k-representation:

$$B_{U,k}(\mathbf{p}||\mathbf{q}) = B_U(k(\mathbf{p}), k(\mathbf{p})).$$

$$k^*(\mathbf{x}^*) = \nabla F(\mathbf{x})$$

$$B_{U^*,k^*}(\mathbf{p}^*||\mathbf{q}^*) = B_{U,k}(\mathbf{q}||\mathbf{p}).$$

Amari's α-divergences

α-divergences on positive arrays (unnormalized discrete probabilities), $\alpha \in \mathbb{R}$:

$$D_\alpha(\mathbf{p}||\mathbf{q}) = \begin{cases} \sum_{i=1}^d \frac{4}{1-\alpha^2}\left(\frac{1-\alpha}{2}p_i + \frac{1+\alpha}{2}q_i - p_i^{\frac{1-\alpha}{2}} q_i^{\frac{1+\alpha}{2}}\right) & \alpha \neq \pm 1 \\ \sum_{i=1}^d p_i \log \frac{p_i}{q_i} + q_i - p_i = \mathrm{KL}(\mathbf{p}||\mathbf{q}) & \alpha = -1 \\ \sum_{i=1}^d q_i \log \frac{q_i}{p_i} + p_i - q_i = \mathrm{KL}(\mathbf{q}||\mathbf{p}) & \alpha = 1 \end{cases}$$

Duality

$$\boxed{D_\alpha(\mathbf{p}||\mathbf{q}) = D_{-\alpha}(\mathbf{q}||\mathbf{p}).}$$

α-divergences: Special cases of Csiszár f-divergences

Special case of Csiszár f-divergences associated with any convex function f satisfying $f(1) = f'(1) = 0$:

$$C_f(p||q) = \sum_{i=1}^{d} p_i f\left(\frac{q_i}{p_i}\right).$$

For statistical measures, $C_f(p||q) = \mathbb{E}_P[f(Q/P)]$, function of the 'likelihood ratio'.
For $\alpha \neq 0$, take

$$f_\alpha(x) = \frac{4}{1-\alpha^2}\left(\frac{1-\alpha}{2} + \frac{1+\alpha}{2}x - x^{\frac{1+\alpha}{2}}\right)$$

$$D_\alpha(p||q) = C_{f_\alpha}(p||q)$$

α-divergences are canonical divergences of constant-curvature geometries.

α-divergences are representational Bregman divergences in disguise.

β-divergences

Introduced by Copas and Eguchi.
Applications in statistics: Robust blind source separation, etc.

$$D_\beta(\mathbf{p}||\mathbf{q}) = \begin{cases} \sum_{i=1}^d q_i \log \frac{q_i}{p_i} + p_i - q_i = \mathrm{KL}(\mathbf{q}||\mathbf{p}) & \beta = 0 \\ \sum_{i=1}^d \frac{1}{\beta+1}(p_i^{\beta+1} - q_i^{\beta+1}) - \frac{1}{\beta} q_i (p_i^\beta - q_i^\beta) & \beta > 0 \end{cases}$$

β-divergences are also representational Bregman divergences (with $U_0(x) = \exp x$).

β-divergences are representational Bregman divergences in disguise.

Note that $F_\beta(x) = \frac{1}{\beta+1} x^{\beta+1}$ and $F_\beta^*(x) = \frac{x^{\beta+1} - x}{\beta(\beta+1)}$ are degenerated to linear functions for $\beta = 0$, and that k_β is a strictly monotonous increasing function.

Representational Bregman divergences of α-/β-divergences

Divergence	Convex conjugate functions	Representation functions
Bregman divergences B_F, B_{F^*}	U $U' = (U^{*\prime})^{-1}$ U^*	$k(x) = x$ $k^*(x) = U'(k(x))$
α-divergences$(\alpha \neq \pm 1)$ $F_\alpha(x) = \frac{2}{1+\alpha} x$ $F^*_\alpha(x) = \frac{2}{1-\alpha} x$	$U_\alpha(x) = \frac{2}{1+\alpha}(\frac{1-\alpha}{2}x)^{\frac{2}{1-\alpha}}$ $U'_\alpha(x) = \frac{2}{1+\alpha}(\frac{1-\alpha}{2}x)^{\frac{1+\alpha}{1-\alpha}}$ $U^*_\alpha(x) = \frac{2}{1-\alpha}(\frac{1+\alpha}{2}x)^{\frac{2}{1+\alpha}} = U_{-\alpha}(x)$	$k_\alpha(x) = \frac{2}{1-\alpha} x^{\frac{1-\alpha}{2}}$ $k^*_\alpha(x) = \frac{2}{1+\alpha} x^{\frac{1+\alpha}{2}} = k_{-\alpha}(x)$
β-divergences$(\beta > 0)$ $F_\beta(x) = \frac{1}{\beta+1} x^{\beta+1}$ $F^*_\beta(x) = \frac{x^{\beta+1}-x}{\beta(\beta+1)}$	$U_\beta(x) = \frac{1}{\beta+1}(1+\beta x)^{\frac{1+\beta}{\beta}}$ $U'_\beta(x) = (1+\beta x)^{\frac{1}{\beta}}$ $\quad U^{*\prime}_\beta(x) = \frac{x^\beta - 1}{\beta}$ $U^*_\beta(x) = \frac{x^{\beta+1}-x}{\beta(\beta+1)}$	$k_\beta(x) = \frac{x^\beta - 1}{\beta}$ $k^*_\beta(x) = x$

α- and β-divergences are representational Bregman divergences in disguise.

Centroids wrt. representational Bregman divergences

In Euclidean geometry, the centroid is the minimizer of the sum of squared distances (a Bregman divergence for $F(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle$).
Right-sided and left-sided barycenters are respectively a k-mean, and a ∇F-mean (for stricly convex $F = U \circ k$) or the k-representation of a ∇U-mean (for degenerated $F = U \circ k$):

$$\mathbf{b}^R = k^{-1}\left(\sum_i w_i k(\mathbf{p}_i)\right)$$

$$\mathbf{b}^L = k^{-1}\left(\nabla U^*\left(\sum_i w_i \nabla U(k(\mathbf{p}_i))\right)\right)$$

Generalized mean:

$$M_f(x_1, ..., x_n) = f^{-1}\left(\sum_{i=1}^n f(x_i)\right)$$

include Pythagoras' means (arithmetic $f(x) = x$, geometric $f(x) = \log x$, harmonic $f(x) = \frac{1}{x}$)

Centroids (proof)

$$\min_{\mathbf{c}} \frac{1}{n} \sum_i B_{U,k}(\mathbf{p}_i || \mathbf{c})$$

$$\equiv \quad \min_{\mathbf{c}} \left(\frac{1}{n} \sum_i U(k(\mathbf{p}_i)) - U(k(\mathbf{c})) - \sum_i \langle k(\mathbf{p}_i) - k(\mathbf{c}), \nabla U(k(\mathbf{c})) \rangle \right)$$

$$\overset{\text{mod. constants}}{\equiv} \quad \min_{\mathbf{c}} -U(k(\mathbf{c})) - \left\langle \frac{1}{n} \sum_i k(\mathbf{p}_i) - k(\mathbf{c}), \nabla U(k(\mathbf{c})) \right\rangle$$

$$\overset{\text{Legendre}}{\equiv} \quad \min_{\mathbf{c}} B_{U,k} \left(\frac{1}{n} \sum_i k(\mathbf{p}_i) || k(\mathbf{c}) \right) \geq 0$$

It follows that this is minimized for $k(\mathbf{c}) = \frac{1}{n} \sum_i k(\mathbf{p}_i)$ since $B_{U,k}(\mathbf{p}||\mathbf{q}) = 0$ iff. $\mathbf{p} = \mathbf{q}$. Since k is strictly monotonous, we get $\mathbf{c} = k^{-1}(\frac{1}{n} \sum_i k(\mathbf{p}_i))$.
Note that $k^{-1} \circ U^{-1} = (U \circ k)^{-1}$ so that $\mathbf{b}^L$ is merely usually a $U \circ k$-mean.

α-centroids and β-centroids (barycenters)

Means	Left-sided	Right-sided
Generic	$k^{-1}\left(\nabla U^*\left(\sum_{i=1}^n \frac{1}{n}\nabla U\left(k(\mathbf{p}_i)\right)\right)\right)$	$k^{-1}\left(\frac{1}{n}\sum_{i=1}^n k(\mathbf{p}_i)\right)$
α-means ($\alpha \neq \pm 1$)	$n^{-\frac{2}{1+\alpha}}\left(\sum_{i=1}^n \mathbf{p}_i^{\frac{1+\alpha}{2}}\right)^{\frac{2}{1+\alpha}}$	$n^{-\frac{2}{1-\alpha}}\left(\sum_{i=1}^n \mathbf{p}_i^{\frac{1-\alpha}{2}}\right)^{\frac{2}{1-\alpha}}$
β-means ($\beta > 0$)	$\frac{1}{n}\sum_{i=1}^n \mathbf{p}_i$	$n^{-\frac{1}{\beta}}\left(\sum_{i=1}^n \mathbf{p}_i^\beta\right)^{\frac{1}{\beta}}$

Recover former result:

Amari, Integration of Stochastic Models by Minimizing α-Divergence, Neural Computation, 2007.

Generalized Bregman Voronoi diagrams as lower envelopes

Voronoi diagrams obtained as **minimization diagrams** of functions:

$$\min_{i \in \{1,\ldots,n\}} B_{U,k}(\mathbf{x}||\mathbf{p}_i).$$

Minimization diagram equivalent to

$$\min_i f_i(\mathbf{x})$$

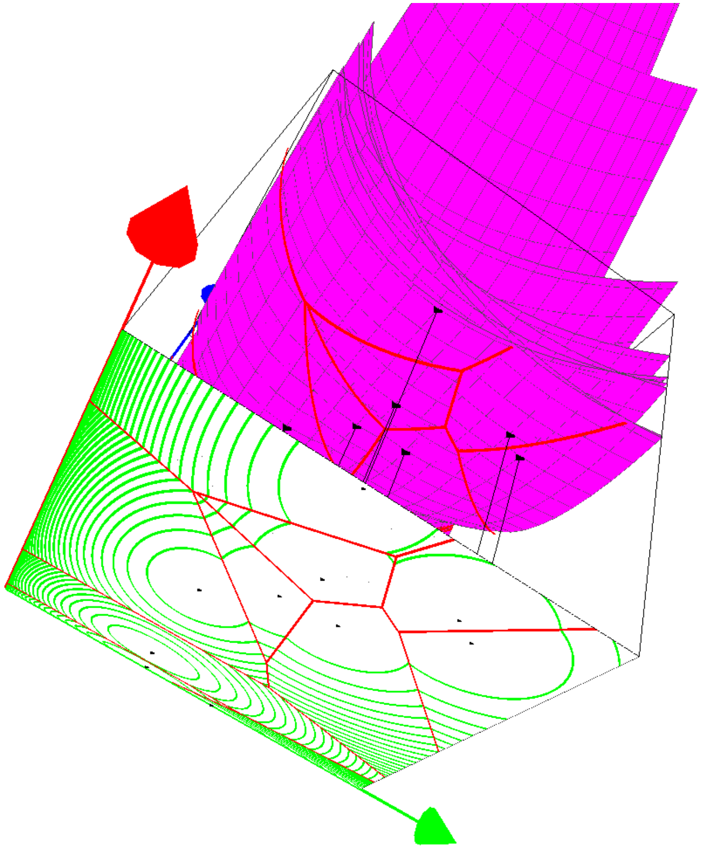
with

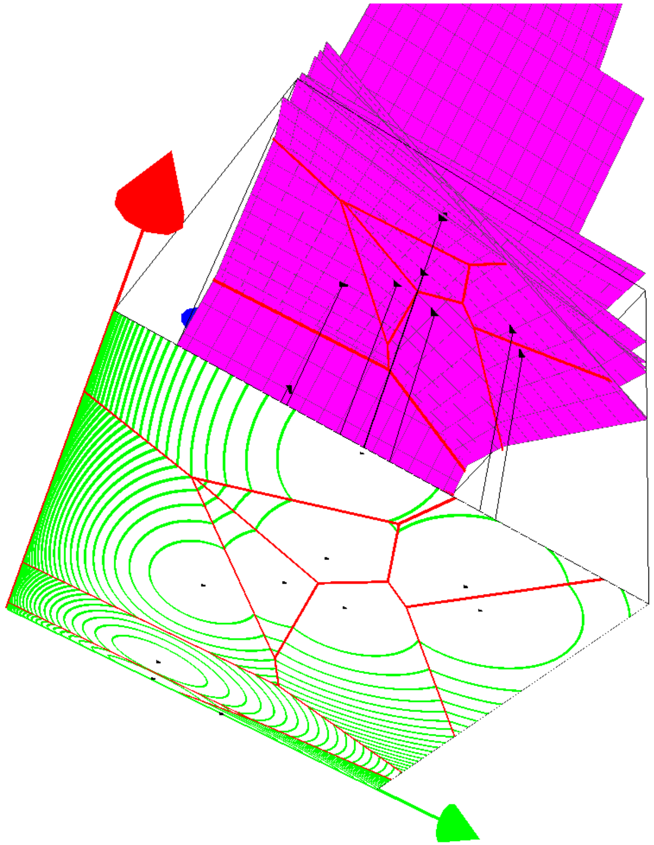
$$f_i(\mathbf{x}) = \langle k(\mathbf{p}_i) - k(\mathbf{x}), \nabla U(k(\mathbf{p}_i))\rangle - U(k(\mathbf{p}_i)).$$

Functions f_i's are linear in $k(\mathbf{x})$ and denote hyperplanes.
$\rightarrow$ Mapping the points $\mathcal{P}$ to the point set $\mathcal{P}_k$, obtain an **affine** minimization diagram.
$\rightarrow$ can be computed from Chazelle's optimal half-space intersection. algorithm of Chazelle.
Then pull back this diagram by the strictly monotonous k^{-1} function.

Voronoi diagrams as minimization diagrams

For example, for the Kullback-Leibler divergence (relative entropy):





Bregman Voronoi Diagrams: Properties, Algorithms and Applications, arXiv:0709.2196

Voronoi diagrams of representable Bregman divergences

Theorem.
The Voronoi diagram of n d-dimensional points with respect to a representational Bregman divergence has complexity $O(n^{\lceil \frac{d}{2} \rceil})$. It can be computed in $O(n \log n + n^{\lceil \frac{d}{2} \rceil})$ time.

Corollary.
The dual α-Voronoi and β-Voronoi diagrams have complexity $O(n^{\lceil \frac{d}{2} \rceil})$, and can be computed optimally in $O(n^{\lceil \frac{d}{2} \rceil})$ time.

α-Voronoi diagrams

Right-sided α-bisectors

$$H_\alpha(\mathbf{p},\mathbf{q}) : \{\mathbf{x} \in \mathcal{X} \; | D_\alpha(\mathbf{p}||\boxed{\mathbf{x}}) = D_\alpha(\mathbf{q}||\boxed{\mathbf{x}})\}$$

for $\alpha \neq \pm 1$.

$$\Longrightarrow H_\alpha(\mathbf{p},\mathbf{q}) : \sum_i \frac{1-\alpha}{2}(p_i - q_i) + x^{\frac{1+\alpha}{2}}(q^{\frac{1-\alpha}{2}} - p^{\frac{1-\alpha}{2}}) = 0.$$

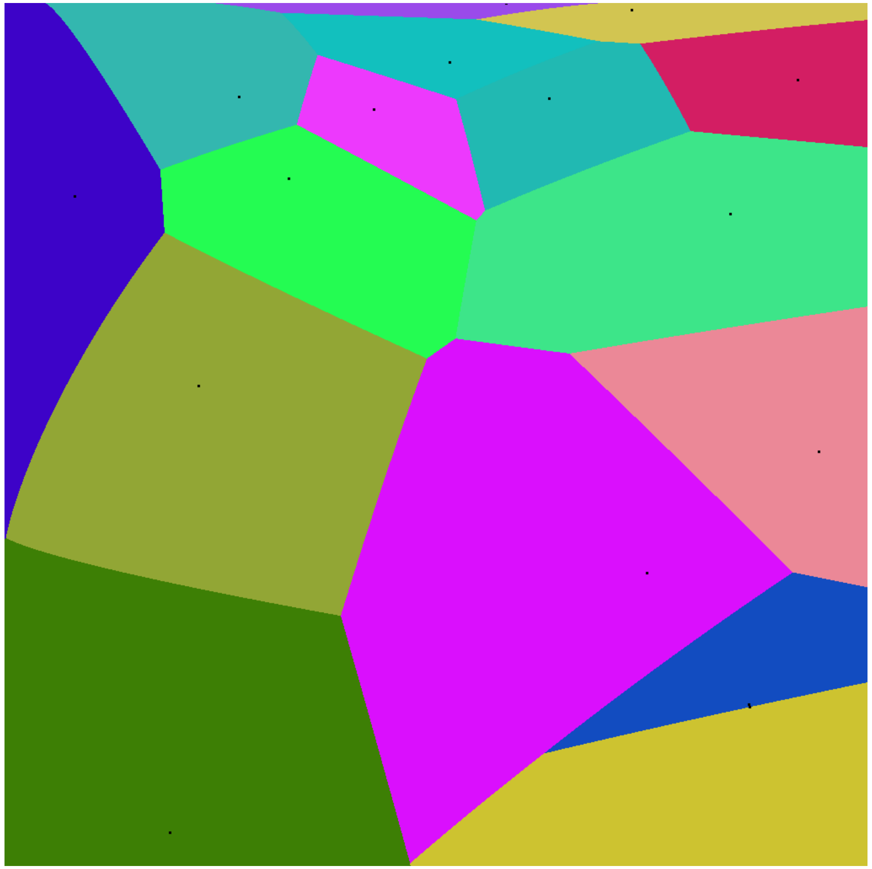
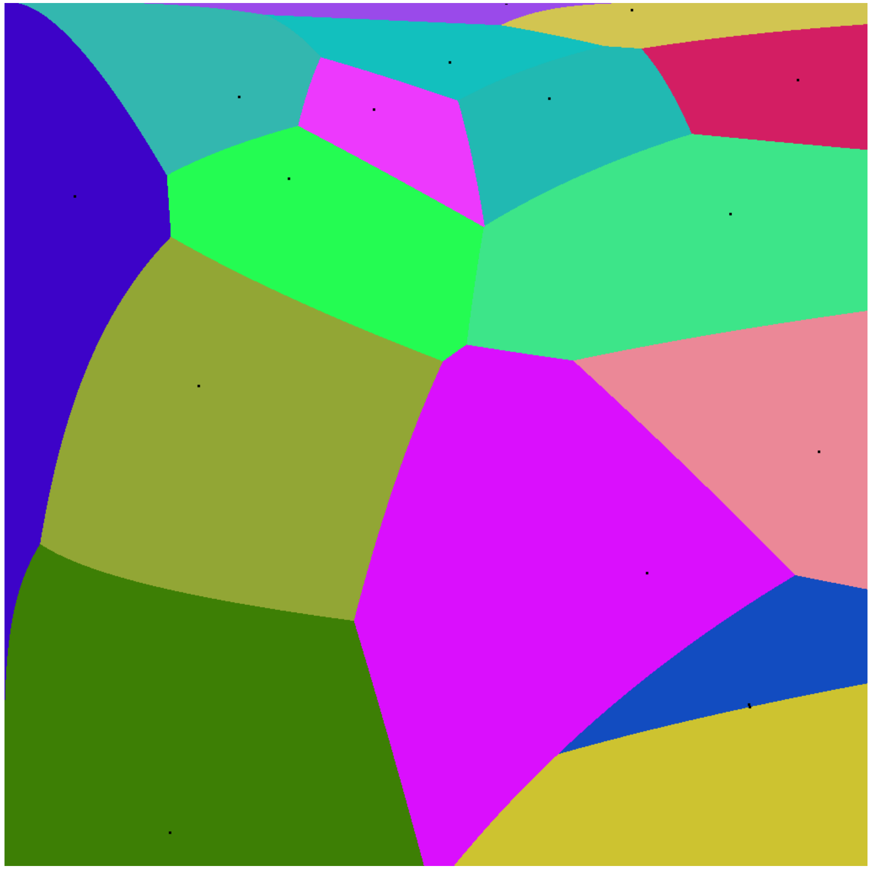
Letting $\mathbf{X} = [x_1^{\frac{1+\alpha}{2}} \; ... \; x_d^{\frac{1+\alpha}{2}}]^T$, we get hyperplane bisectors:

$$H_\alpha(\mathbf{p},\mathbf{q}) : \sum_i X_i(q^{\frac{1-\alpha}{2}} - p^{\frac{1-\alpha}{2}}) + \sum_i \frac{1-\alpha}{2}(p_i - q_i) = 0.$$

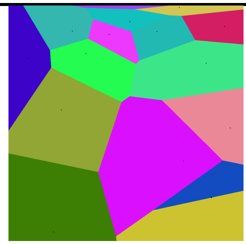
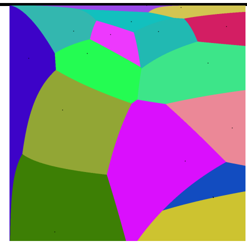
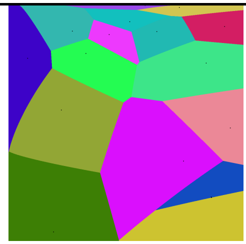
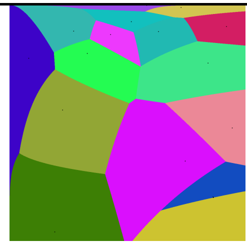
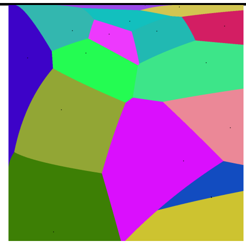
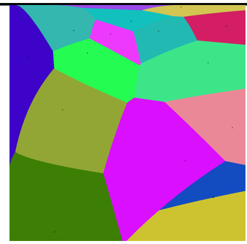
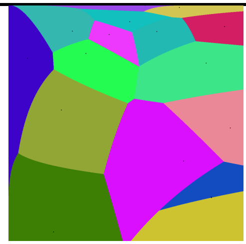
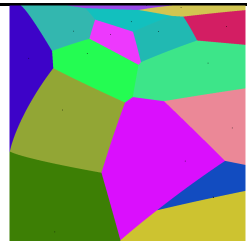
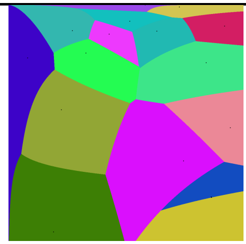
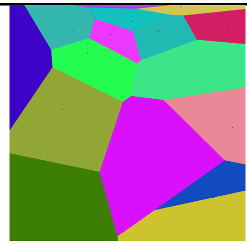
Right-sided α-Voronoi diagram is affine in the $k(x) = x^{\frac{1+\alpha}{2}}$-representation with complexity $O(n^{\lceil \frac{d}{2} \rceil})$.

Indeed, $D(X||P_i) = B_{U,k}(\mathbf{x}||\mathbf{p}_i) \leq D(X||P_j) = B_{U,k}(\mathbf{x}||\mathbf{p}_j) \Longleftrightarrow B_U(k(\mathbf{x})||k(\mathbf{p}_i)) \leq B_U(k(\mathbf{x})||k(\mathbf{p}_j))$.

Dual α-Voronoi diagrams ($\alpha = -\frac{1}{2}$)

$$D_\alpha(\mathbf{p}||\mathbf{q}) = D_{-\alpha}(\mathbf{q}||\mathbf{p}).$$

α	Left-sided	Right-sided
$\alpha = -1$ (KL)		
$\alpha = -\frac{1}{2}$		
$\alpha = 0$ (squared Hellinger)		
$\alpha = \frac{1}{2}$		
$\alpha = 1$ (KL*)		

Dual β-Voronoi diagrams

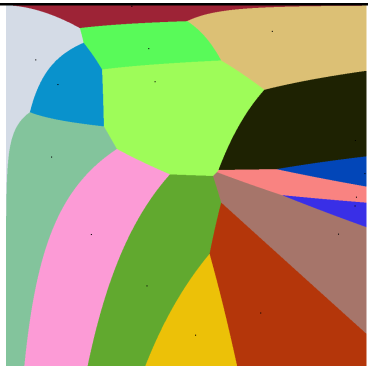
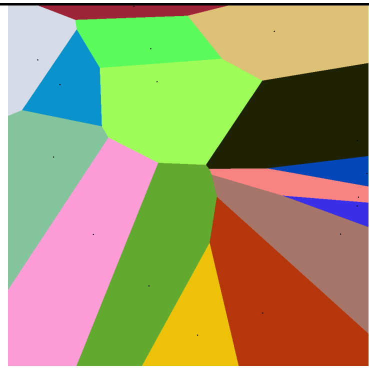
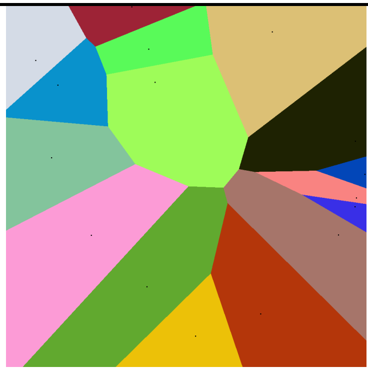
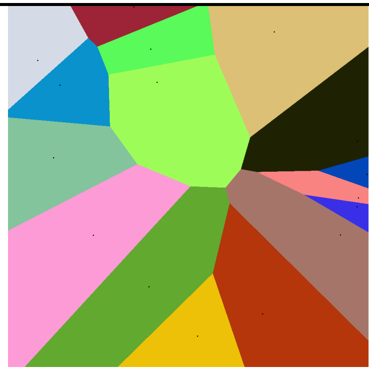
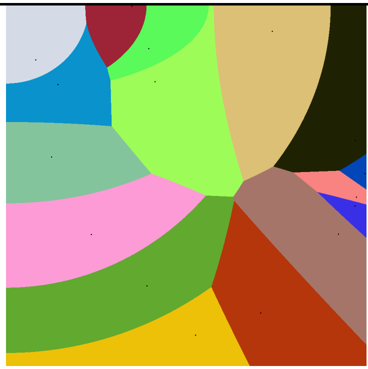
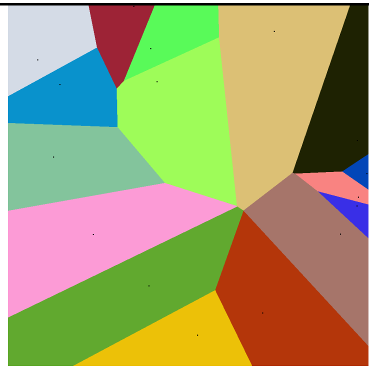
Right-sided β-Voronoi diagrams are affine for $\beta > 0$. Indeed, the β-bisector

$$H_\beta(\mathbf{p},\mathbf{q}) : \{\mathbf{x} \in \mathcal{X} \,|D_\beta(\mathbf{p}||\mathbf{x}) = D_\beta(\mathbf{q}||\mathbf{x})\}$$

yields a linear equation:

$$H_\beta(\mathbf{p},\mathbf{q}) : \sum_{i=1}^{d} \frac{1}{\beta+1}(p_i^{\beta+1} - q_i^{\beta+1}) - \frac{1}{\beta} x_i (p_i^\beta - q_i^\beta) = 0.$$

Dual β-Voronoi diagrams

β	Left-sided	Right-sided
$\beta = 0$ (KL)		
$\beta = 1$		
$\beta = 2$		

Power diagrams in Laguerre geometry

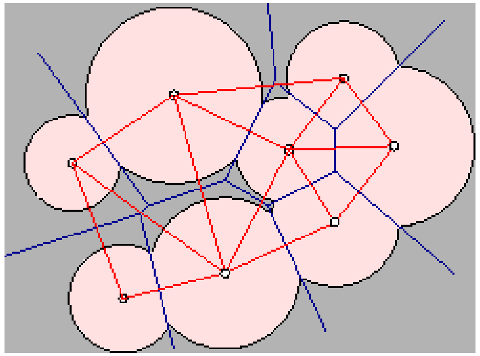
Power distance of $\mathbf{x}$ to a ball $B = B(\mathbf{p}, r)$:

$$D(\mathbf{x}, \mathrm{Ball}(\mathbf{p}, r)) = ||\mathbf{p} - \mathbf{x}||^2 - r^2$$

Radical hyperplane:

$$2\langle \mathbf{x}, \mathbf{p}_j - \mathbf{p}_i \rangle + ||\mathbf{p}_i||^2 - ||\mathbf{p}_j||^2 + r_j^2 - r_i^2 = 0$$

Power diagrams are affine diagrams



Universal construction theorem:
Any affine diagram is *identical* to the power diagram of a set of corresponding balls. (Aurenhammer'87)

Rep. Bregman Voronoi diagrams as power diagrams

Seek for transformations to match representable Bregman/power bisector equations:

$$2\langle \mathbf{x}, \mathbf{p}_j - \mathbf{p}_i \rangle + ||\mathbf{p}_i||^2 - ||\mathbf{p}_j||^2 + r_j^2 - r_i^2 = 0$$

$$f_i(\mathbf{x}) = f_j(\mathbf{x})$$

with

$$f_l(\mathbf{x}) = \langle k(\mathbf{p}_l) - k(\mathbf{x}), \nabla U(k(\mathbf{p}_l)) \rangle - U(k(\mathbf{p}_l)).$$

We get

$$\mathbf{p}_i \rightarrow \nabla U(k(\mathbf{p}_i))$$

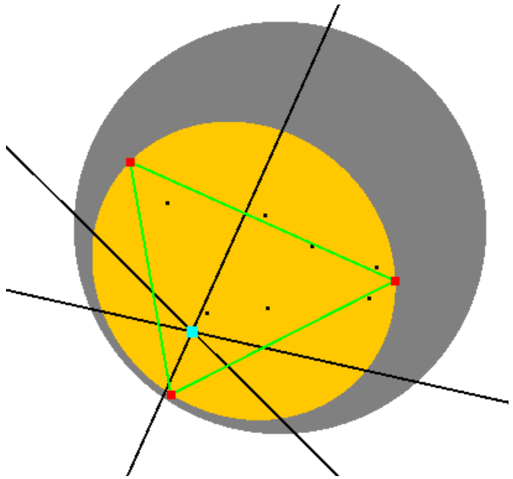
$$r_i = \langle U(k(\mathbf{p}_i)), U(k(\mathbf{p}_i)) \rangle + 2U(k(\mathbf{p}_i)) - \langle \mathbf{p}_i, U(k(\mathbf{p}_i)) \rangle \,.$$

Representational Bregman Voronoi diagrams can be built from power diagrans

Concluding remarks

- Geometries and embeddings
- Representable Bregman Voronoi diagrams
- Dual α- and β-Voronoi diagrams

Extensions:
Affine hyperplane in representation space for geometric computing.
Framework can be used for solving MINIBALL problems (L_∞-center, MINMAX center)



(For example, Hyperbolic geometry in Klein non-conformal disk.)
Hyperbolic Voronoi diagrams made easy, arXiv:0903.3287

Thank you very much

References:

co-authors: Jean-Daniel Boissonnat, Richard Nock.

- **On Bregman Voronoi diagrams.** *In Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms* (New Orleans, Louisiana, January 07 - 09, 2007). pp. 746-755.
- **Visualizing Bregman Voronoi diagrams.** *In Proceedings of the Twenty-Third Annual Symposium on Computational Geometry* (Gyeongju, South Korea, June 06 - 08, 2007). pp. 121-122.
- **Bregman Voronoi Diagrams: Properties, Algorithms and Applications,** *arXiv:0709.2196*