INF555

# Fundamentals of 3D

Lecture 1 Follow-up: Connected Components

Lecture 2: Convolutions and Filters
Matrix decompositions

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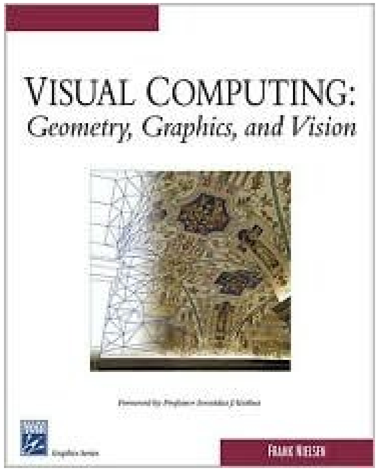
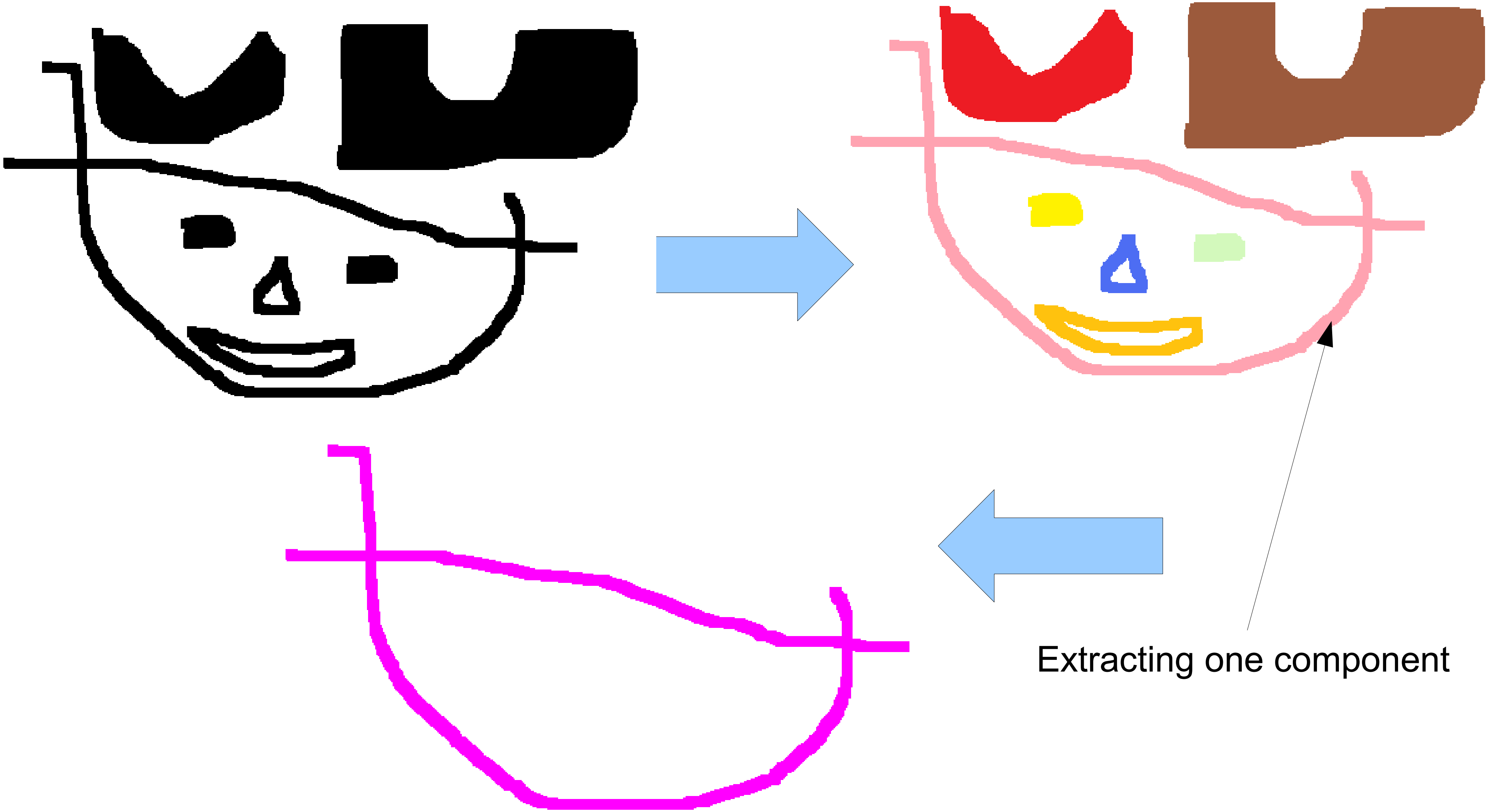
21 Septembre 2011

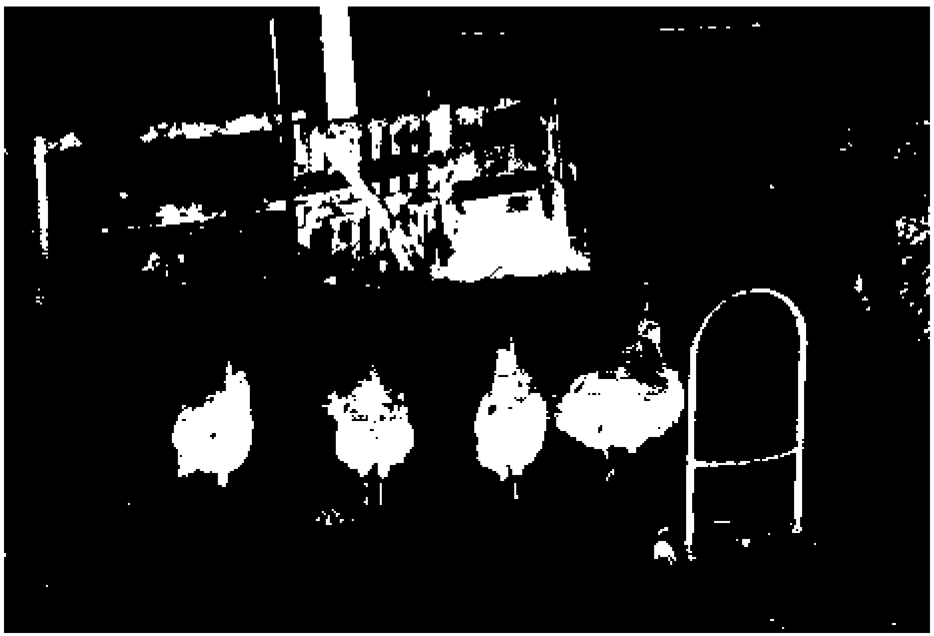


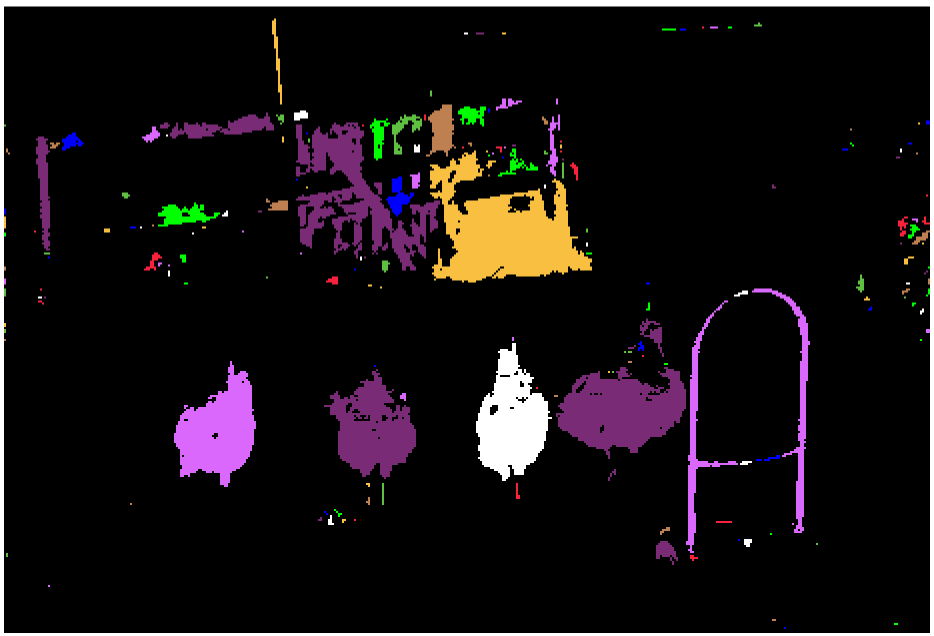
# Labelling connected components (Union-Find)




Input: binary image Foreground/Background pixels
Output: Each connected component

# Useful in computer vision...
# → Image segmentation





How many (large) objects?

# A simple algorithm

Initially each pixel defines its own region
 (labelled by say the pixel index: x+y*width)

Scan the image from left to right and top to bottom:

If current pixel is background AND East pixel is background:
Merge their regions into one = « connect » them

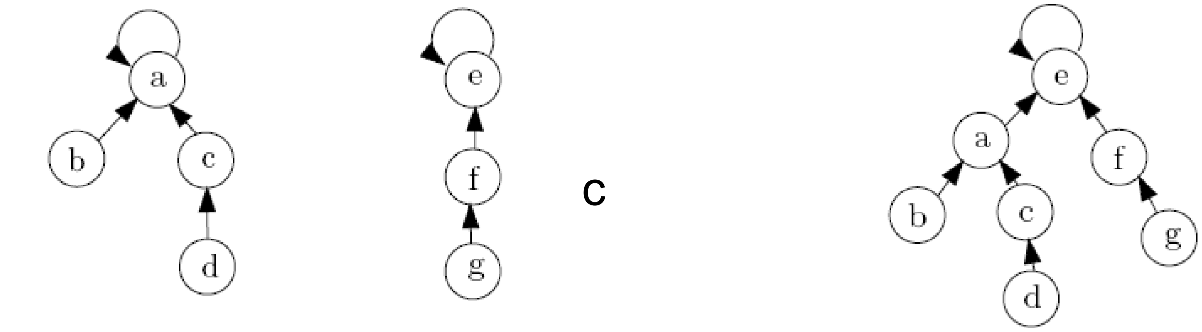
If current pixel is background AND South pixel is background:
Merge their regions into one = « connect » them

Extract regions (floodfilling or single pass algorithm)

How do we merge quickly **disjoint sets**?

# Union-Find abstract data-structures

$\textsc{MakeSet}(x)$
1. $\text{parent}(x) \leftarrow x$
2. $\text{rank}(x) \leftarrow 0$

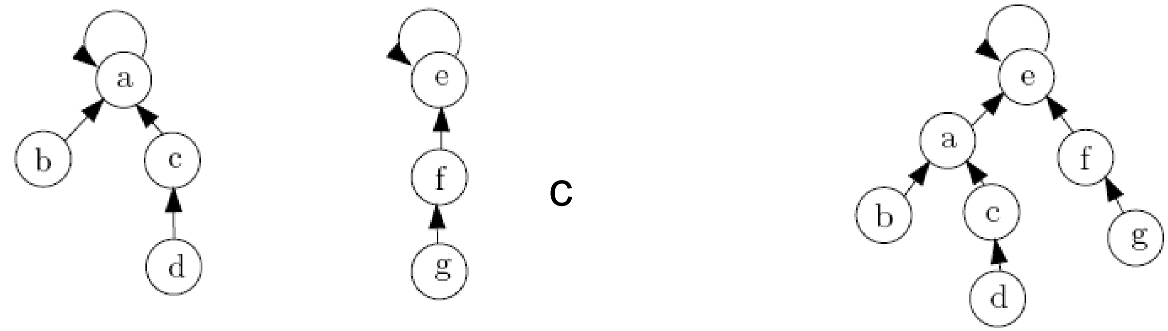



$\mathcal{S}_1 = \{a, b, c, d\}$ $\mathcal{S}_2 = \{e, f, g\}$ $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$

**RANK=DEPTH**

**Visual Depiction**

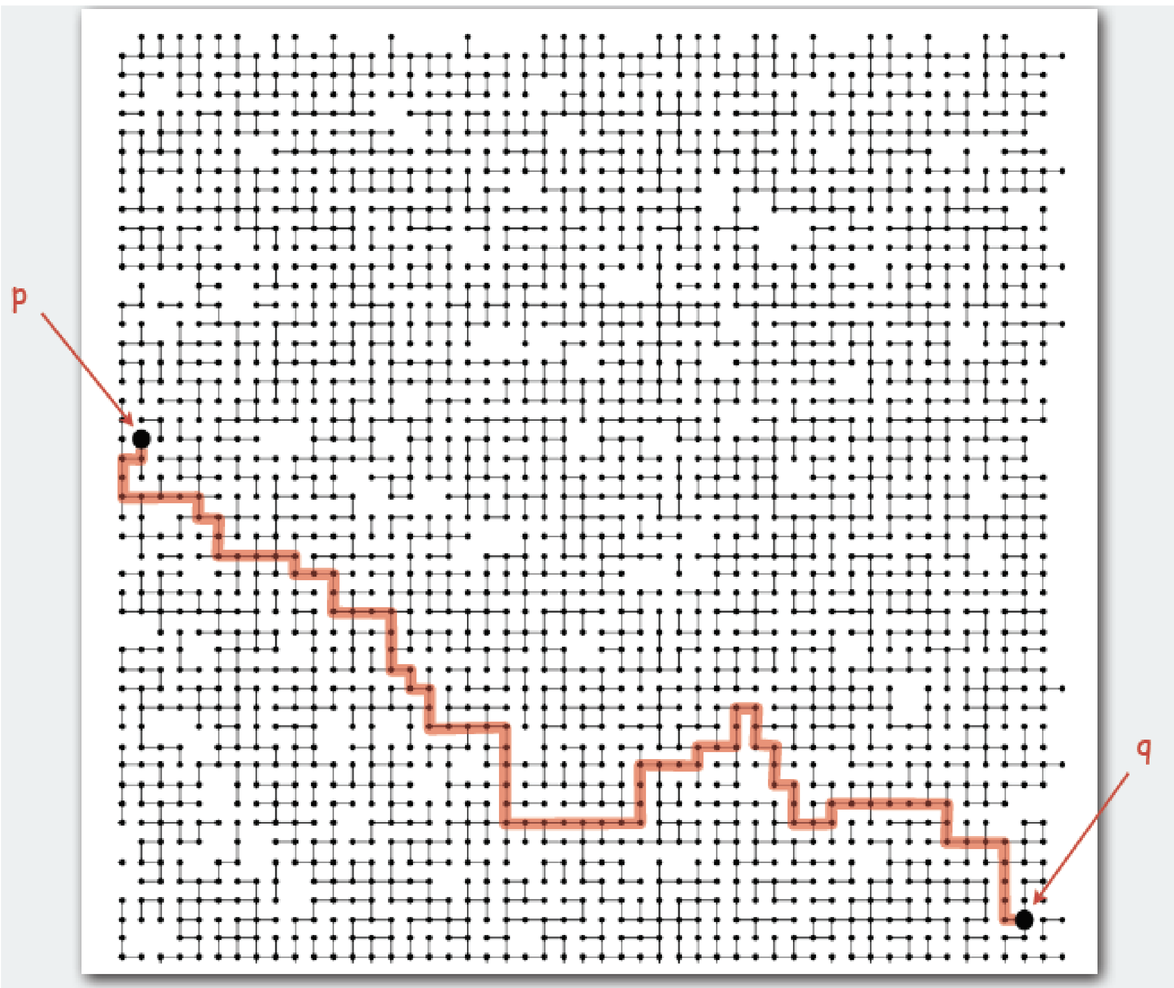
```
class UnionFind{
int  [ ] rank; int [ ] parent;

UnionFind(int n)
{int k;
parent=new int[n];
rank=new int[n];

for (k = 0; k < n; k++)
     {parent[k]   = k; rank[k] = 0;     }
}

int Find(int k)
{while (parent[k]!=k ) k=parent[k]; return k;}


int Union(int x, int y)
{
x=Find(x);y=Find(y);
if ( x == y ) return -1; // Do not perform union of same set

if (rank[x] > rank[y])
     {parent[y]=x;
       return x;}
     else
     { parent[x]=y;
      if (rank[x]==rank[y]) rank[y]++;return y;}
}
}
```




$\mathcal{S}_1 = \{a, b, c, d\}$ $\mathcal{S}_2 = \{e, f, g\}$ $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$

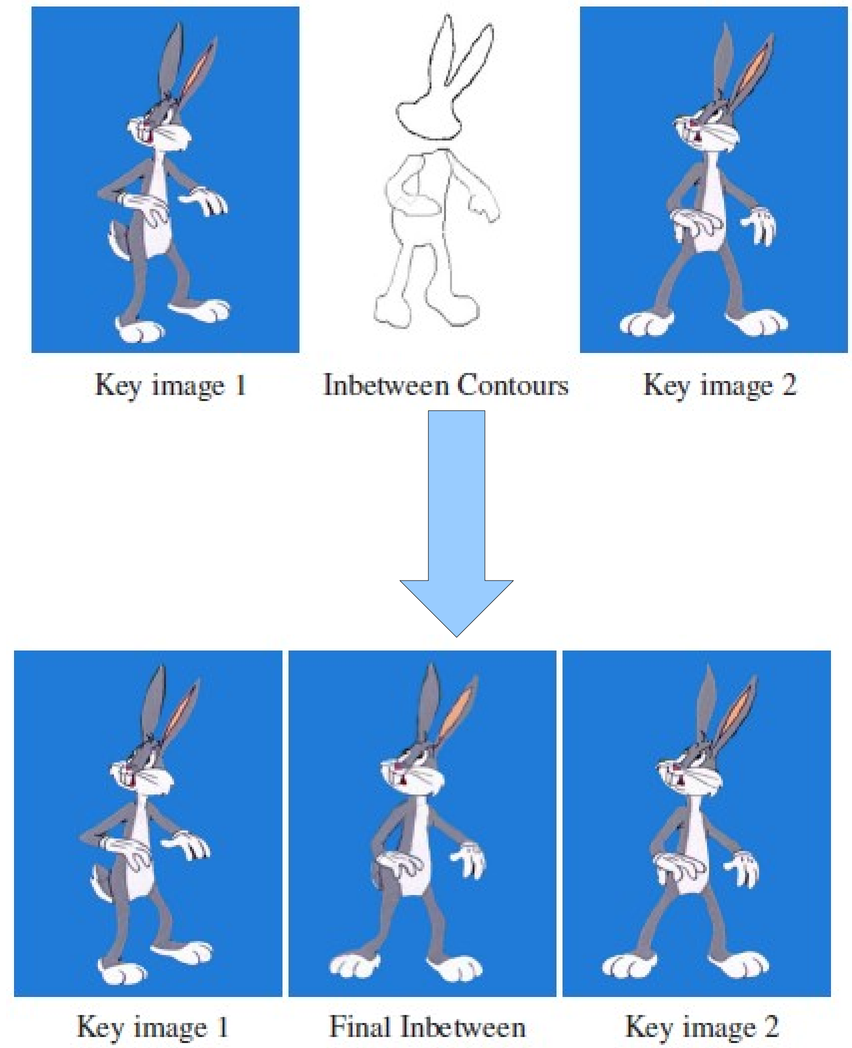
```
// Attach tree(y) to tree(x)
// else
// Attach tree(x) to tree(y)
     and if same depth, then increase depth of y by one
```

# Many applications of the union-find data-structure



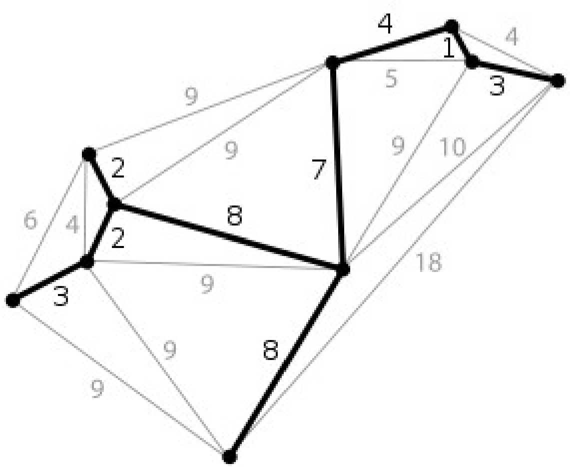

- Variable name aliases.
- Pixels in a digital photo.
- Computers in a network.
- Web pages on the Internet.
- Transistors in a computer chip.

# Yet another example of UF/segmentation: Inbetweening for cell animation




http://www.vuse.vanderbilt.edu/~bobbyb/pubs/sca06.html

# Minimum Spanning Tree (Kruskal's algorithm)




Implemented using Union-Find data structure

```
function Kruskal(G)
  for each vertex v in G do
    Define an elementary cluster C(v) ← {v}.
  Initialize a priority queue Q to contain all edges in G, using the weights as keys.
  Define a tree T ← Ø       //T will ultimately contain the edges of the MST
   // n is total number of vertices
  while T has fewer than n-1 edges do
    // edge u,v is the minimum weighted route from/to v
    (u,v) ← Q.removeMin()
    // prevent cycles in T. add u,v only if T does not already contain a path between u and v.
    // Note that the cluster contains more than one vertex only if an edge containing a pair of
    // the vertices has been added to the tree.
    Let C(v) be the cluster containing v, and let C(u) be the cluster containing u.
    if C(v) ≠ C(u) then
      Add edge (v,u) to T.
      Merge C(v) and C(u) into one cluster, that is, union C(v) and C(u).
  return tree T
```

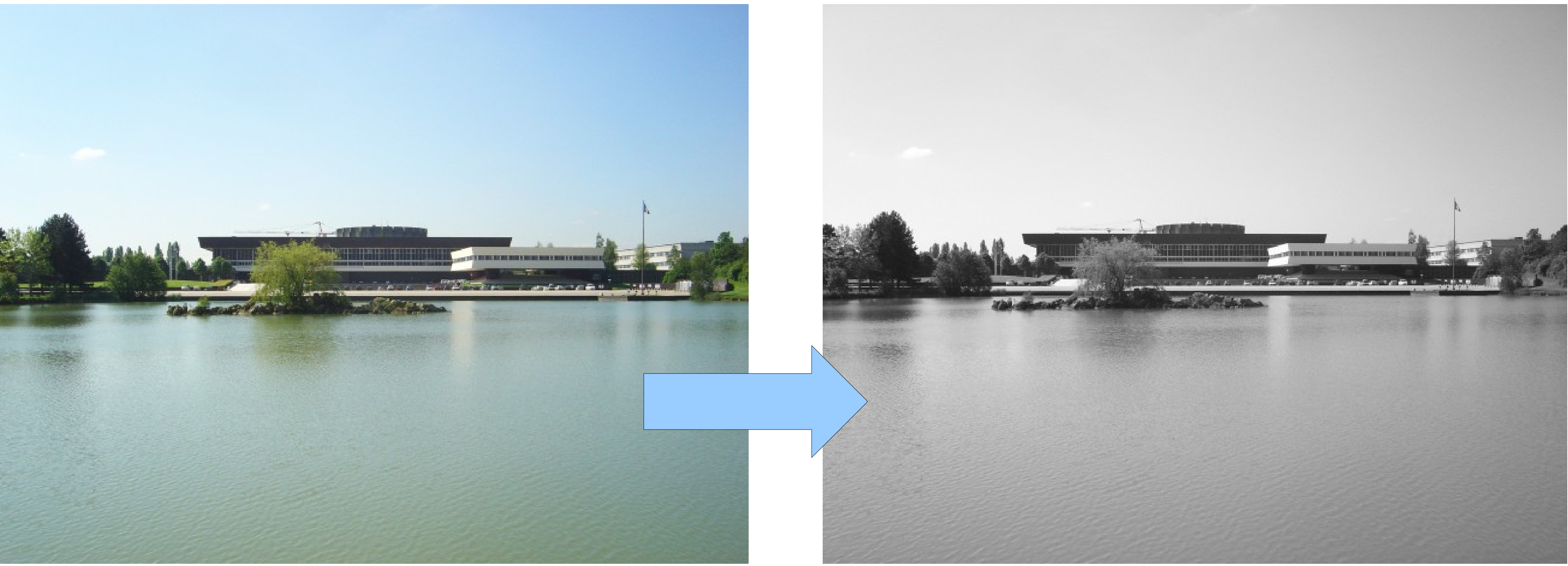
http://en.wikipedia.org/wiki/Minimum_spanning_tree

http://en.wikipedia.org/wiki/Kruskal's_algorithm

# Convolutions et filtres

Color image

Grey image



Intensity=0.3red+0.59green+0.11blue

# Image Convolution

Roberts Cross (edge) detection

Discrete gradient (maximal response for edge at 45 degrees)

| | |
|---|---|
| +1 | 0 |
| 0 | -1 |

Gx

| | |
|---|---|
| 0 | +1 |
| -1 | 0 |

Gy

The two filters are 90 degrees apart (inner product is zero)

$$|G| = \sqrt{Gx^2 + Gy^2}$$

$$\theta = \arctan(Gy/Gx) - 3\pi/4$$

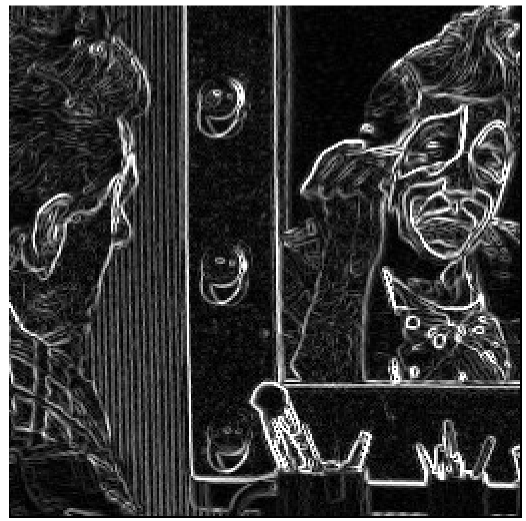
| | |
|---|---|
| $P_1$ | $P_2$ |
| $P_3$ | $P_4$ |

Approximated by

$$|G| = |P_1 - P_4| + |P_2 - P_3|$$

# Roberts Cross edge detector

Input



Filter response (x5)

Thresholded

| | |
|---|---|
| $P_1$ | $P_2$ |
| $P_3$ | $P_4$ |

Approximated by

$$|G| = |P_1 - P_4| + |P_2 - P_3|$$

# Sobel edge detector

| | | |
|---|---|---|
| -1 | 0 | +1 |
| -2 | 0 | +2 |
| -1 | 0 | +1 |

Gx

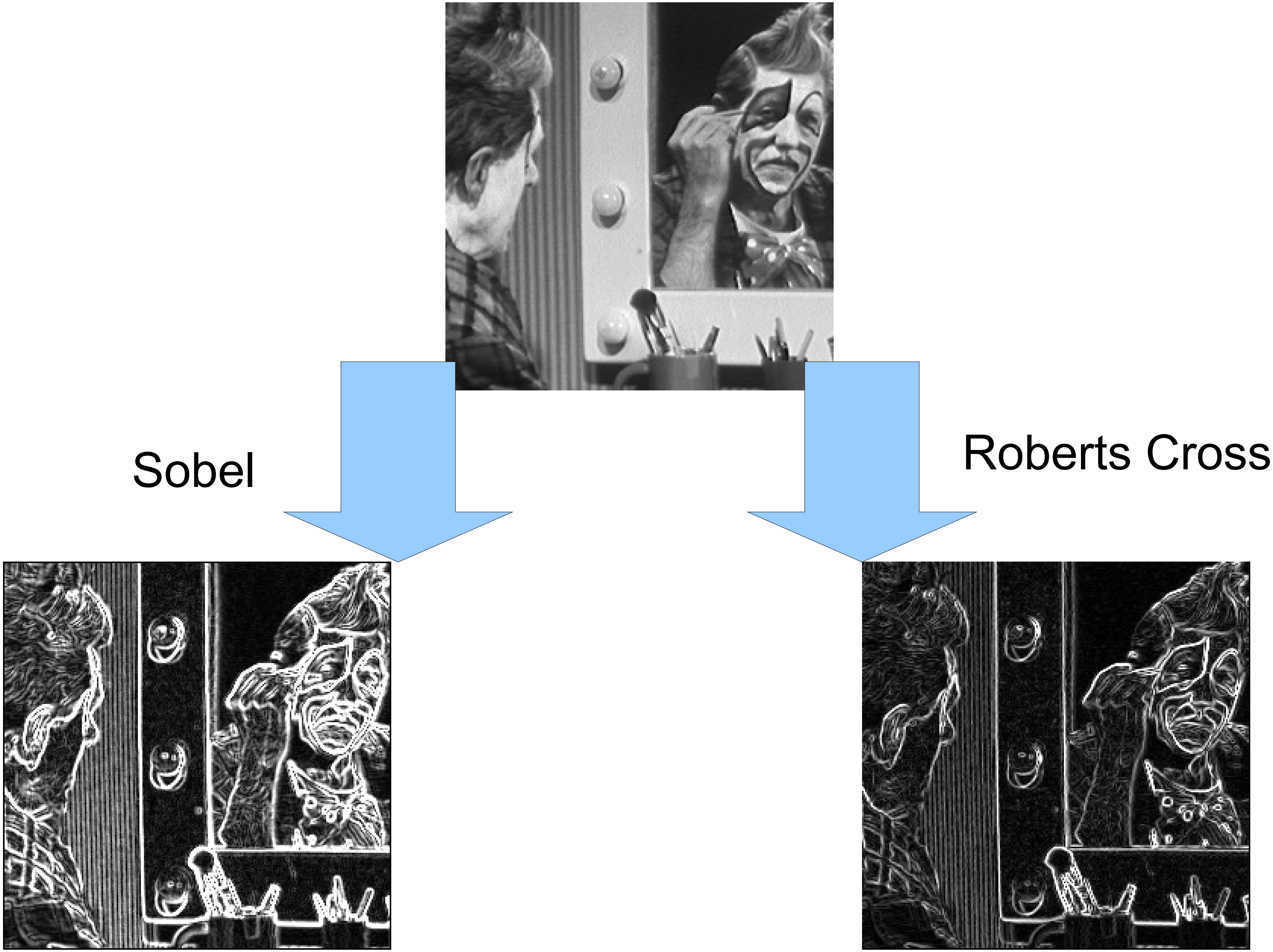
| | | |
|---|---|---|
| +1 | +2 | +1 |
| 0 | 0 | 0 |
| -1 | -2 | -1 |

Gy

Approximated by

| | | |
|---|---|---|
| $P_1$ | $P_2$ | $P_3$ |
| $P_4$ | $P_5$ | $P_6$ |
| $P_7$ | $P_8$ | $P_9$ |

$$|G| = |(P_1 + 2 \times P_2 + P_3) - (P_7 + 2 \times P_8 + P_9)| + |(P_3 + 2 \times P_6 + P_9) - (P_1 + 2 \times P_4 + P_7)|$$

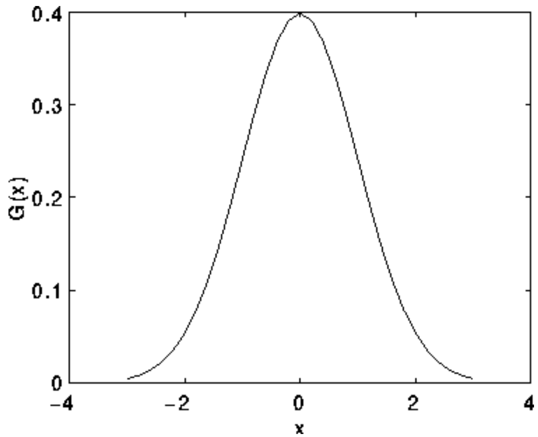


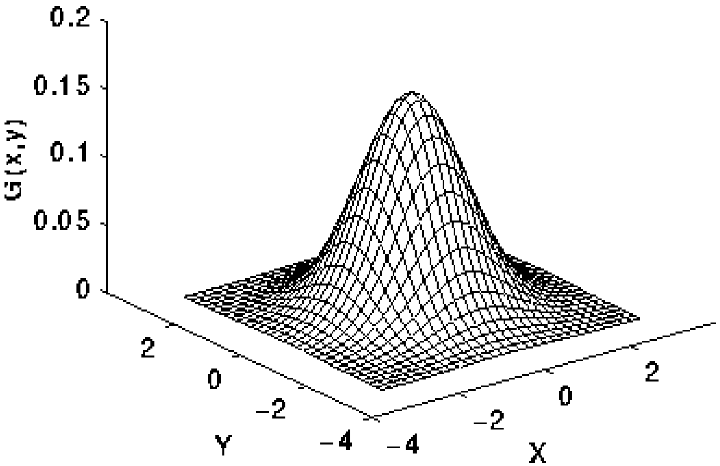

http://homepages.inf.ed.ac.uk/rbf/HIPR2/sobeldemo.htm

# Gaussian smoothing

$$G(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$




Normalize discrete Gaussian kernel to 1

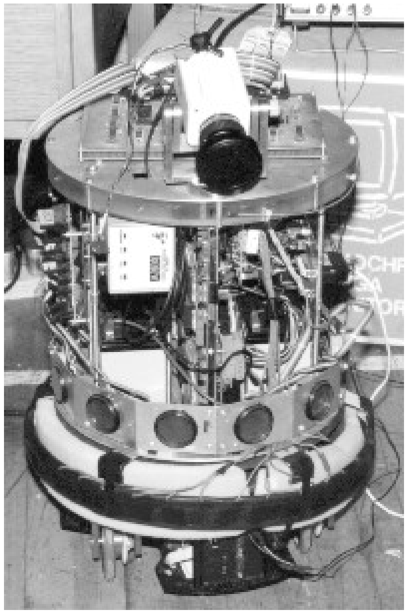
$\frac{1}{273}$

| | | | | |
|---|---|---|---|---|
| 1 | 4 | 7 | 4 | 1 |
| 4 | 16 | 26 | 16 | 4 |
| 7 | 26 | 41 | 26 | 7 |
| 4 | 16 | 26 | 16 | 4 |
| 1 | 4 | 7 | 4 | 1 |

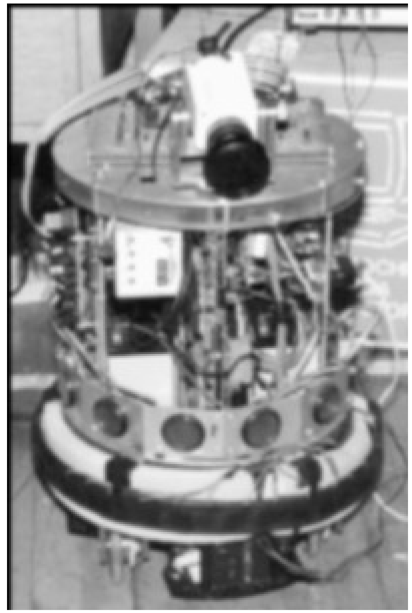
Two parameters to tune:

- K, the dimension of the matric
- Sigma, the smoothing width...

# Blurring, low-pass filter eliminates edges...



Source



Sigma=1, 5x5

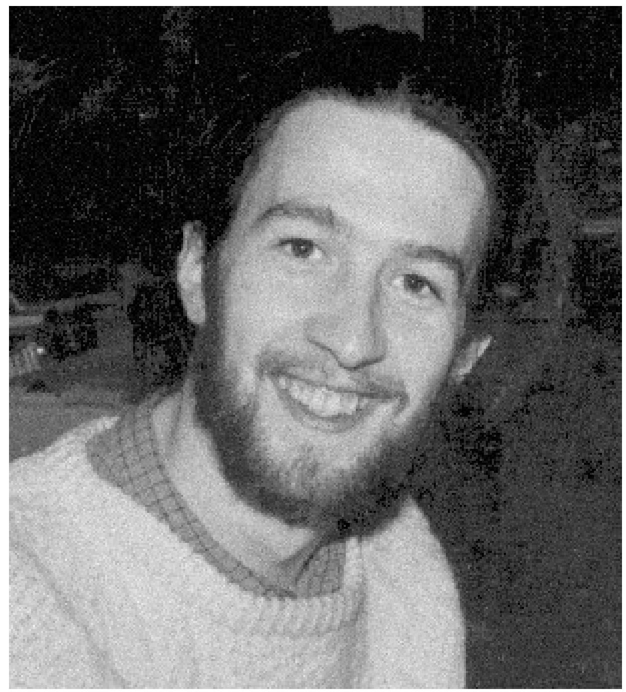
Sigma=2, 9x9

Sigma=4, 15x15

# Mean filter= Uniform average

$$\begin{array}{|c|c|c|}\hline \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \hline \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \hline \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \hline \end{array}$$

Source



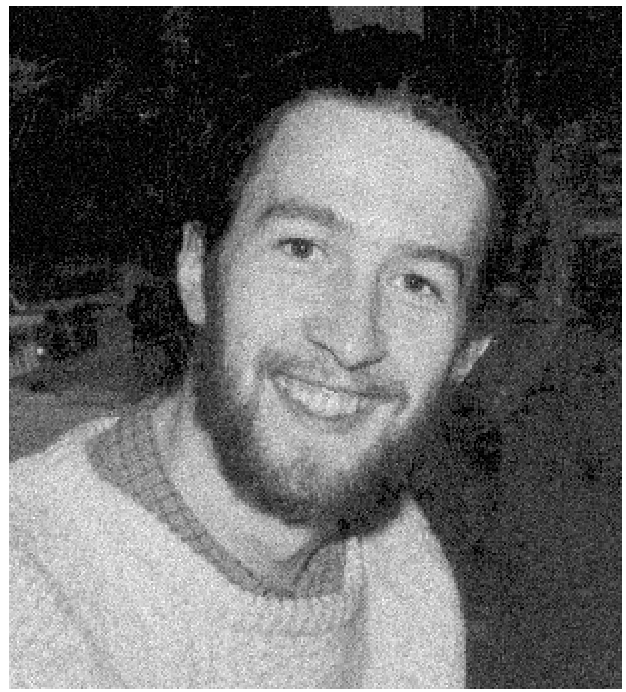
Corrupted, Gaussian noise (sigma=8)

Mean filter 3x3

# Mean filter= Uniform average

| | | |
|---|---|---|
| $\frac{1}{9}$ | $\frac{1}{9}$ | $\frac{1}{9}$ |
| $\frac{1}{9}$ | $\frac{1}{9}$ | $\frac{1}{9}$ |
| $\frac{1}{9}$ | $\frac{1}{9}$ | $\frac{1}{9}$ |

Source



Corrupted, Gaussian noise (sigma=13)

Mean filter 3x3

# Median filter

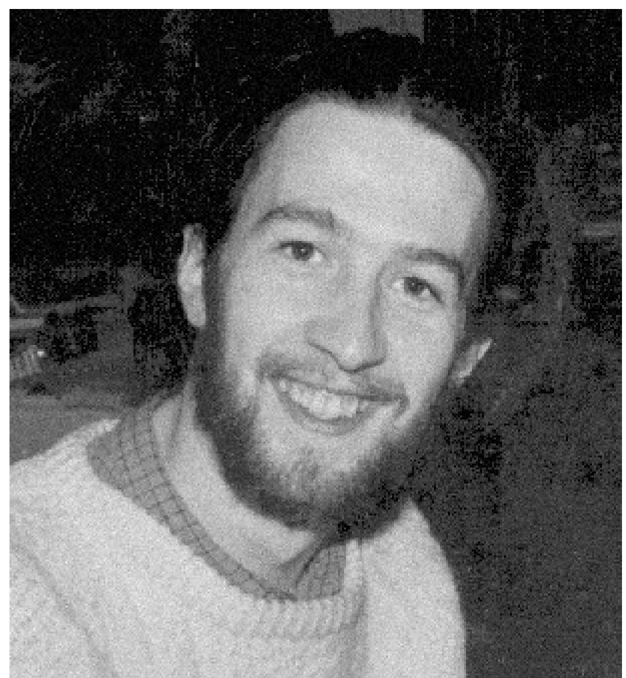
| | | | | |
|---|---|---|---|---|
| 123 | 125 | 126 | 130 | 140 |
| 122 | 124 | 126 | 127 | 135 |
| 118 | 120 | 150 | 125 | 134 |
| 119 | 115 | 119 | 123 | 133 |
| 111 | 116 | 110 | 120 | 130 |

Neighbourhood values:

115, 119, 120, 123, 124,
125, 126, 127, 150

Median value: 124

Source



Corrupted, Gaussian noise (sigma=8)

Median filter 3x3

# Sharpening: Identity+Laplacian kernel

| | | |
|---|---|---|
| -1 | -1 | -1 |
| -1 | 8 | -1 |
| -1 | -1 | -1 |

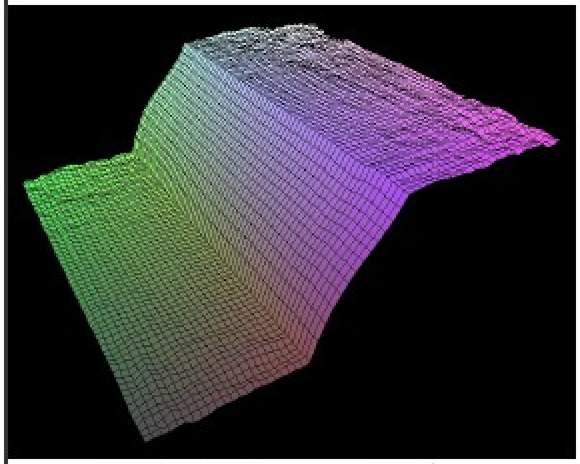
$$\mathbf{S} = \begin{bmatrix} 0 & -\lambda & 0 \\ -\lambda & 1+4\lambda & -\lambda \\ 0 & -\lambda & 0 \end{bmatrix}$$

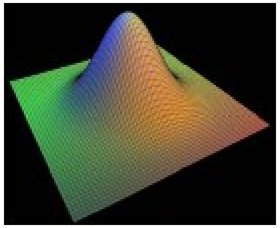
Source image

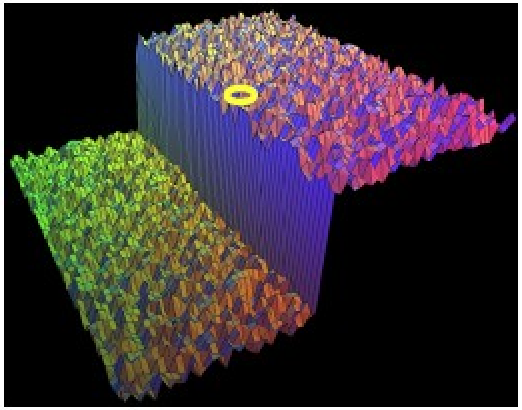
# Bilateral filtering

## Traditional spatial gaussian filtering

$$J(x) = \sum_{\xi} f(x,\xi) \, I(\xi)$$





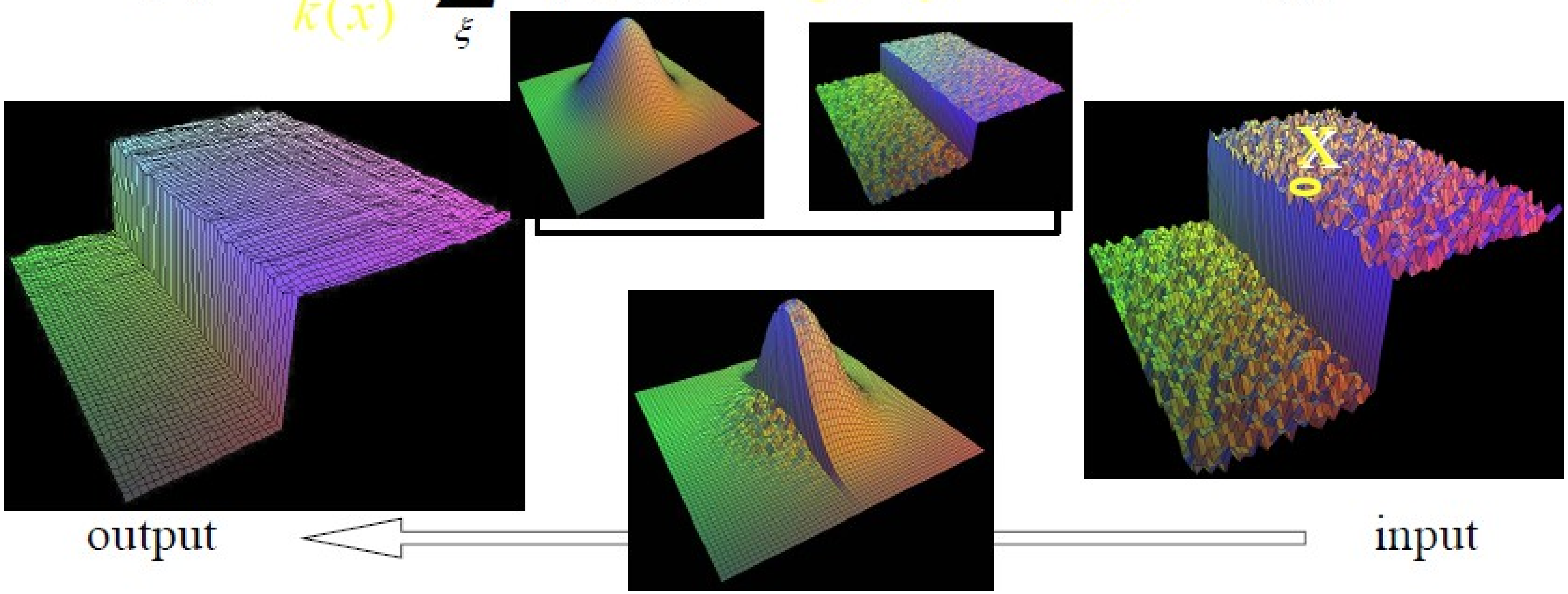


output ⟵ input

# Bilateral filtering

New! gaussian on the intensity difference filtering

$$J(x) = \frac{1}{k(x)} \sum_{\xi} f(x,\xi) \quad g(I(\xi) - I(x)) \quad I(\xi)$$

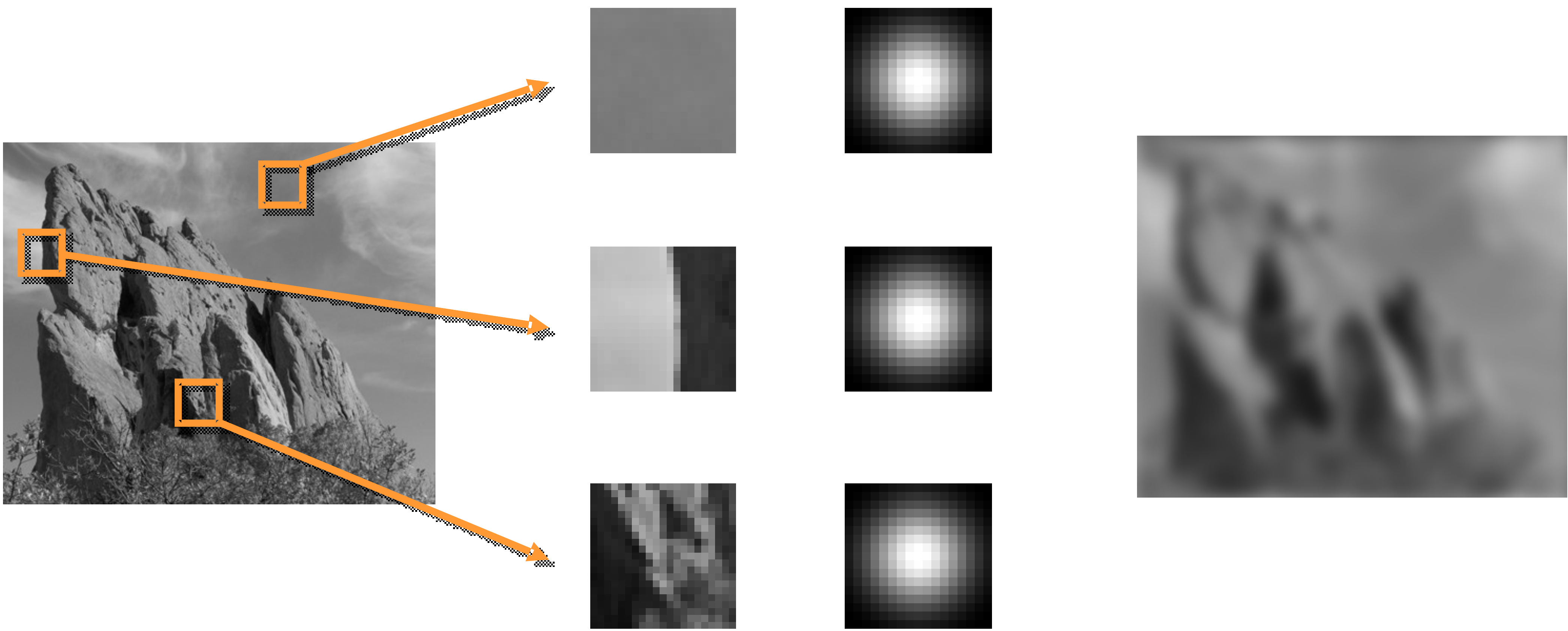


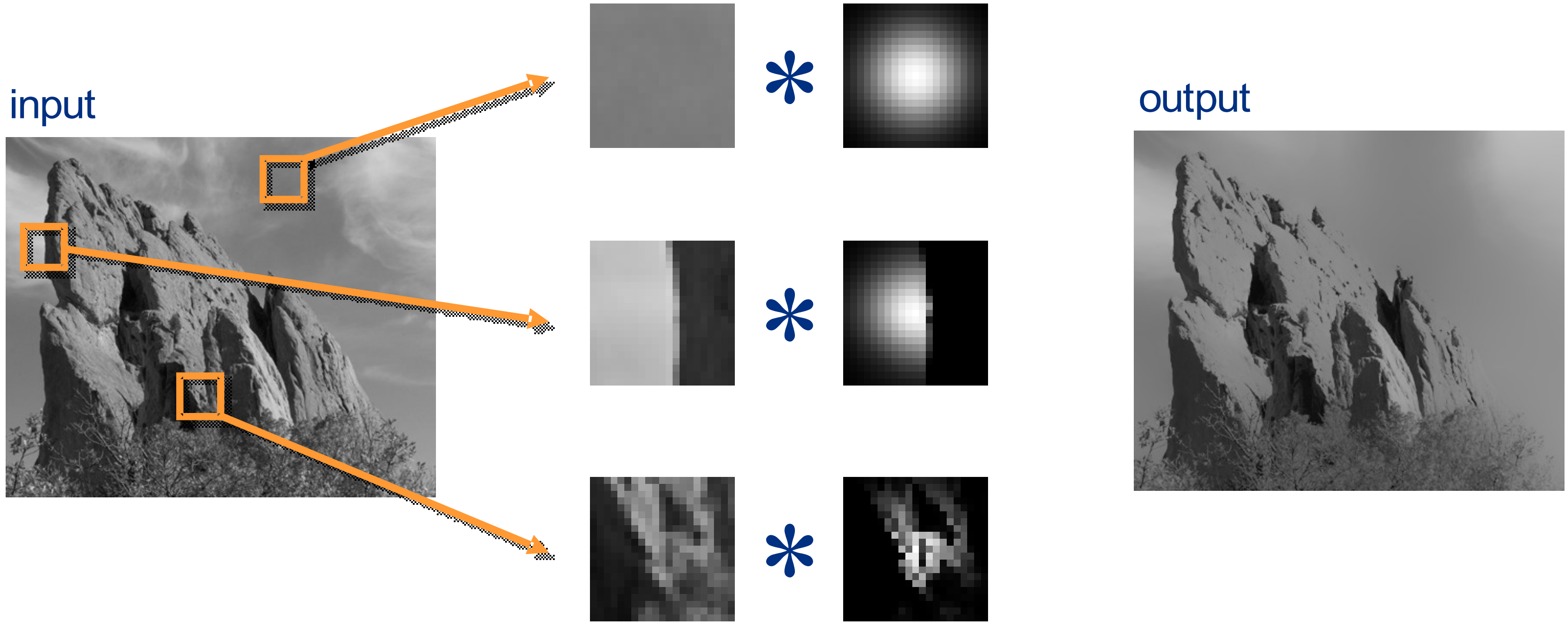

**Bilateral Filtering for Gray and Color Images, Tomasi and Manduchi 1998**

**…. SUSAN feature extractor...**

# Bilateral filtering

## Traditional spatial gaussian filtering





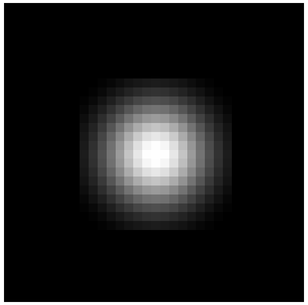

The kernel shape depends on the image content.

new — not new — new

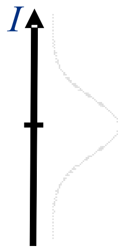
$$BF[I]_{\mathbf{p}} = \frac{1}{W_{\mathbf{p}}} \sum_{\mathbf{q}\in S} G_{\sigma_s}(\|\mathbf{p}-\mathbf{q}\|)\, G_{\sigma_r}(|I_{\mathbf{p}} - I_{\mathbf{q}}|)\, I_{\mathbf{q}}$$

normalization factor

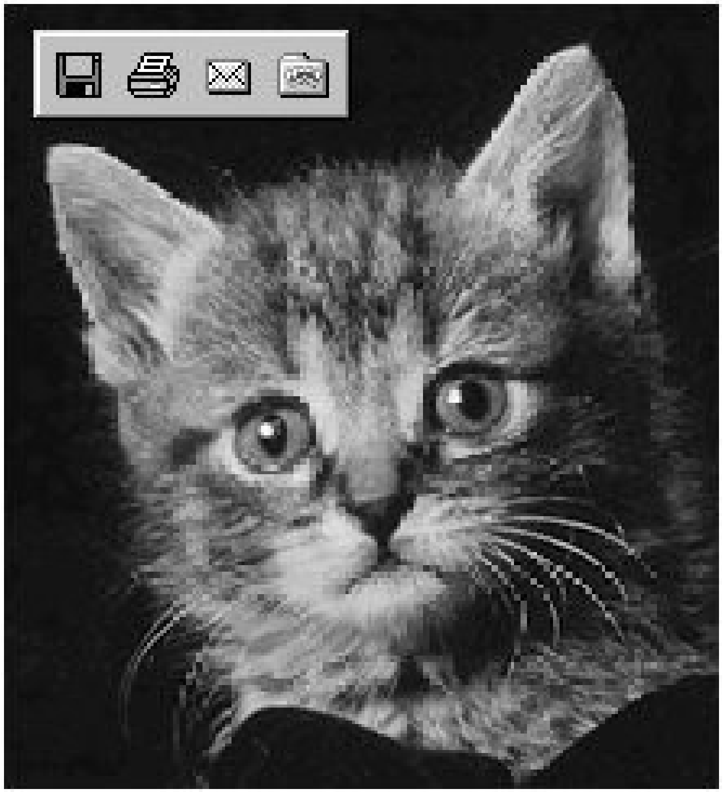
***space*** weight



***range*** weight



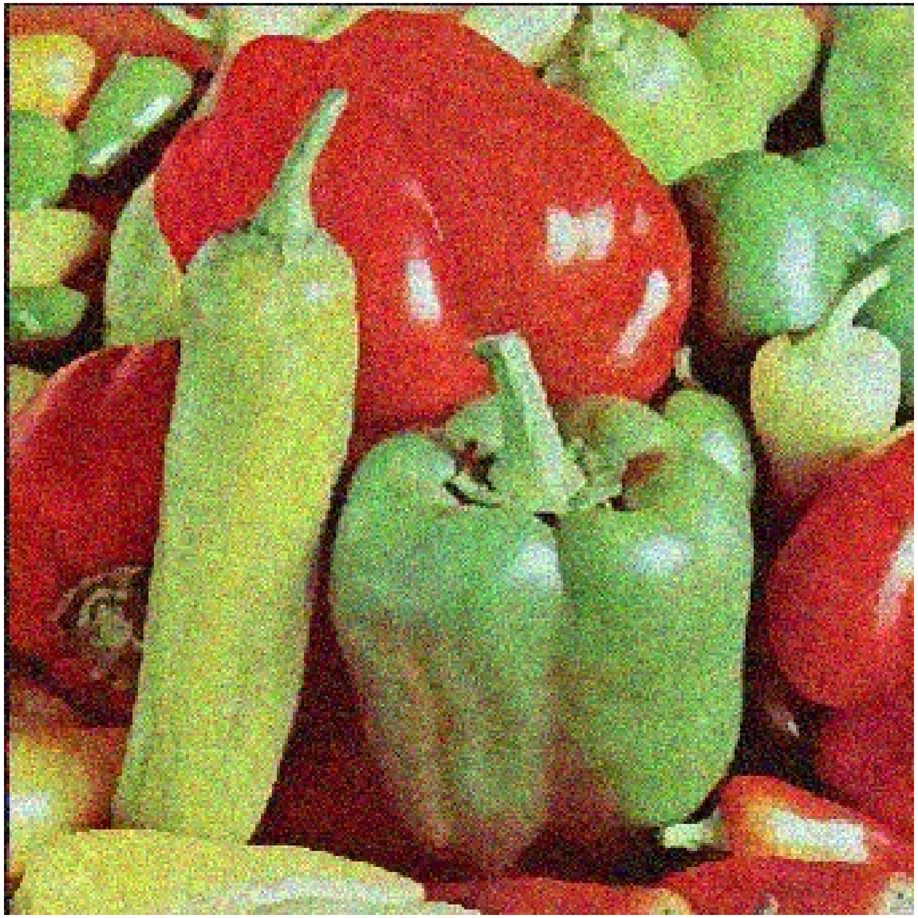
# Bilateral filtering



- Example of Bilateral filtering
- Low contrast texture has been removed

- Brute-force implementation is slow > 10min (but real-time with GPU and better algorithms)

# Bilateral filtering



Origin image

Bilateral (3)

# Results

Origin Image

One iteration

five iterations

**Bootstrapping**

http://people.csail.mit.edu/sparis/siggraph07_course/

# Bilateral filtering extends to meshes

Source

Bilateral mesh denoising

# Harris-Stephens edge detector: Feature points

Aim at finding good  feature

$$M = \sum_{x,y} w(x,y) \begin{bmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{bmatrix}$$

Gradient with respect to x, times gradient with respect to y

Sum over image region – area we are checking for corner

# Harris-Stephens edge detector

Measure the corner response as

$$\boxed{R = \det M - k\left(\operatorname{trace} M\right)^2}$$

$$\boxed{\begin{aligned} \det M &= \lambda_1 \lambda_2 \\ \operatorname{trace} M &= \lambda_1 + \lambda_2 \end{aligned}}$$

Avoid computing eigenvalues themselves.

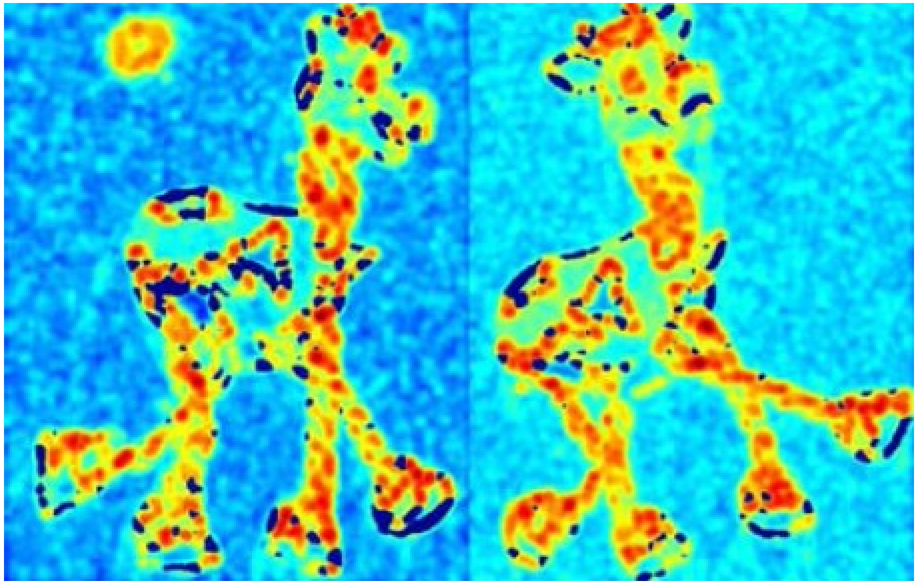
($k$ – empirical constant, $k$ = 0.04-0.06)

Algorithm:

– Find points with large corner response function $R$ ($R$ > threshold)

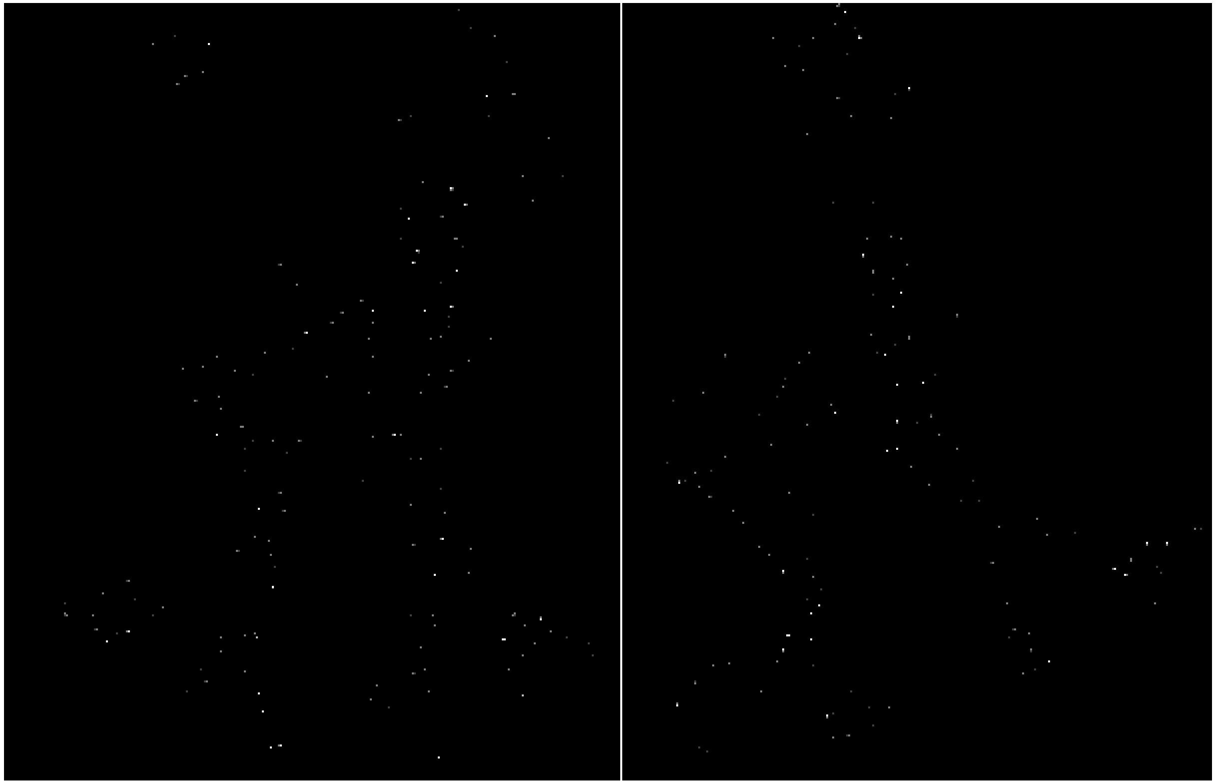
– Take the points of **local maxima** of $R$

# Harris-Stephens edge detector

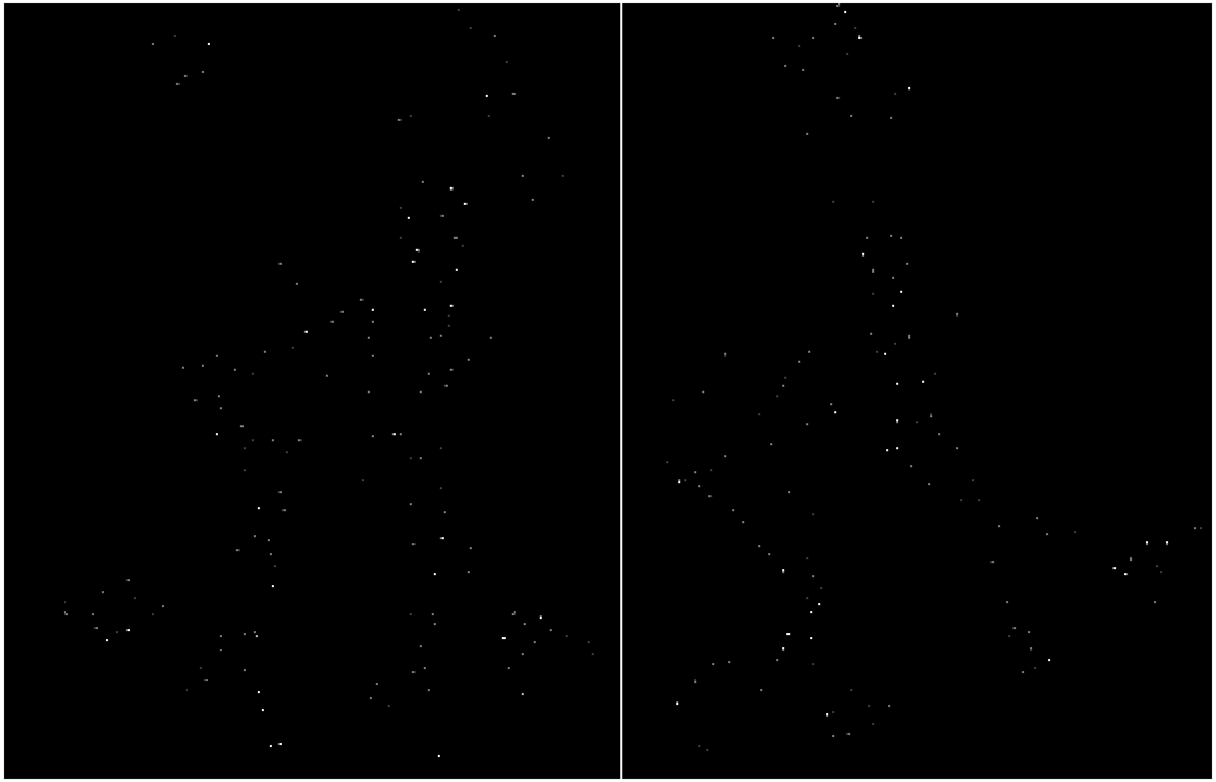


Corner response R

Thresholding R>cste



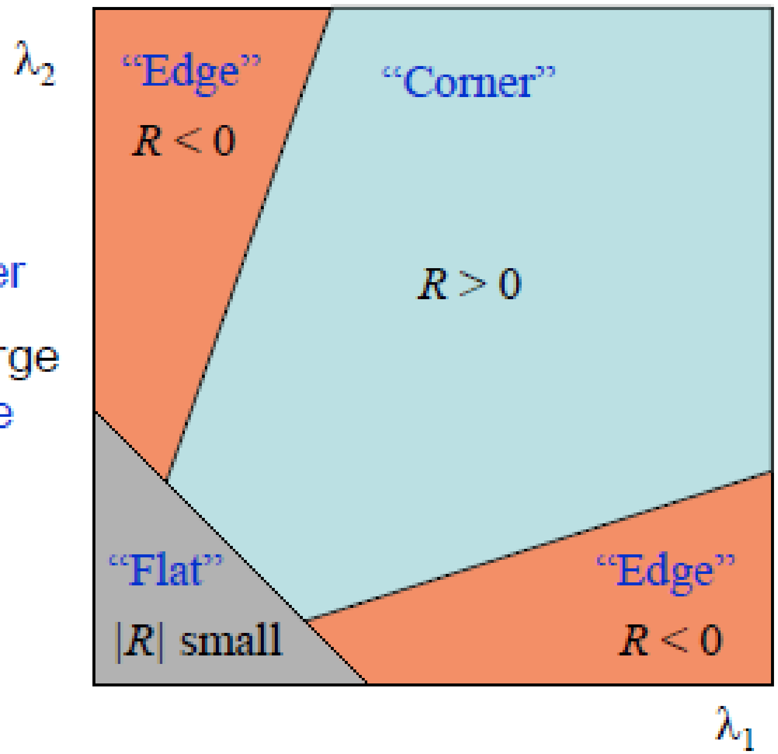
Local maxima of R

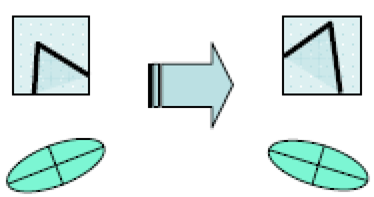


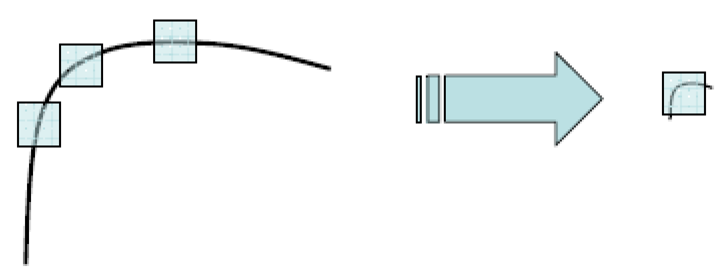
Superposing local maxima
on source images

# Invariant to orientation but
# Depends on the scaling of the image

- *R* depends only on eigenvalues of M
- *R* is large for a corner
- *R* is negative with large magnitude for an edge
- |*R*| is small for a flat region





All points will be classified as edges

Corner !

# Application to feature matching for image panorama tools

http://www.ptgui.com/download.html

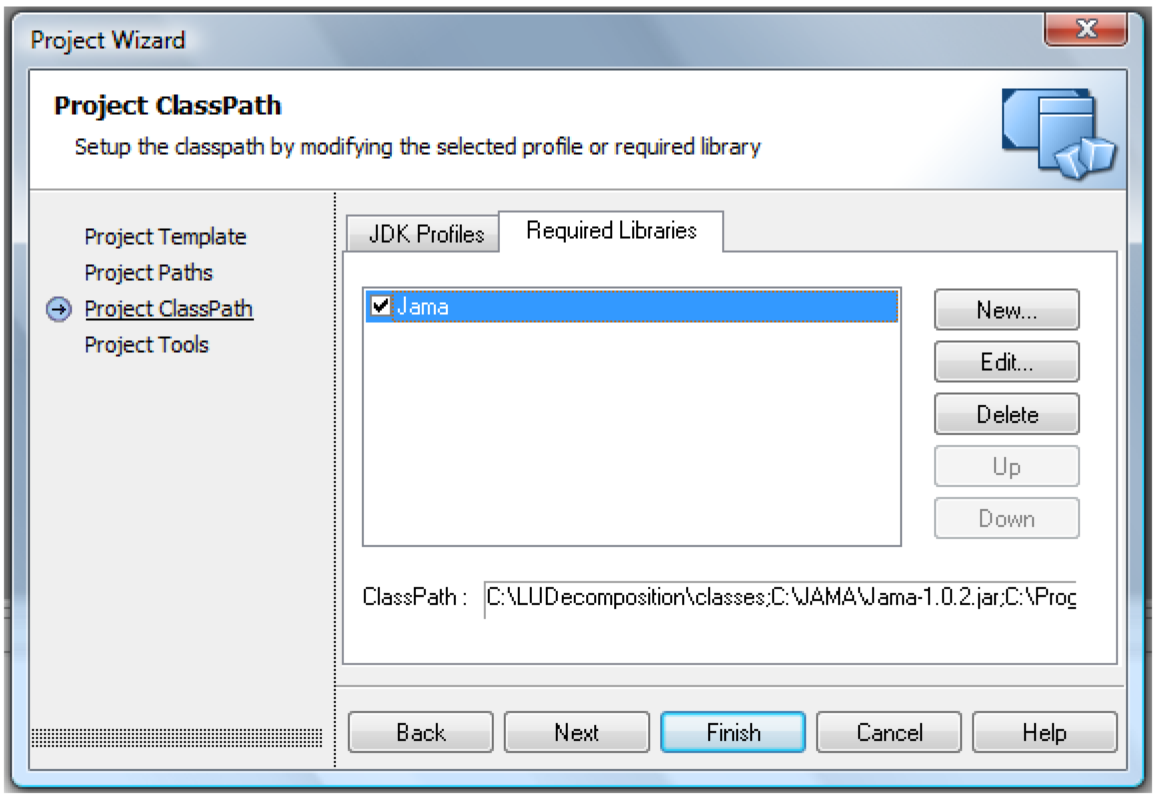
# Matrix operations using JAMA

# *JAMA* : A Java Matrix Package

Use JAMA library

javac -*classpath Jama-1.0.2.jar* filename.java

Using JCreator




***Summary of JAMA Capabilities***

| | |
|---|---|
| **Object Manipulation** | constructors<br>set elements<br>get elements<br>copy<br>clone |
| **Elementary Operations** | addition<br>subtraction<br>multiplication<br>scalar multiplication<br>element-wise multiplication<br>element-wise division<br>unary minus<br>transpose<br>norm |
| **Decompositions** | Cholesky<br>LU<br>QR<br>SVD<br>symmetric eigenvalue<br>nonsymmetric eigenvalue |
| **Equation Solution** | nonsingular systems<br>least squares |
| **Derived Quantities** | condition number<br>determinant<br>rank<br>inverse<br>pseudoinverse |

http://math.nist.gov/javanumerics/jama/

# *Write a class wrapper around Jama*

Essential operations are:

| | |
|---|---|
| CholeskyDecomposition | Cholesky Decomposition. |
| EigenvalueDecomposition | Eigenvalues and eigenvectors of a real matrix. |
| LUDecomposition | LU Decomposition. |
| Matrix | Jama = Java Matrix class. |
| QRDecomposition | QR Decomposition. |
| SingularValueDecomposition | Singular Value Decomposition. |

http://www.bluebit.gr/matrix-calculator/

# LU Decomposition of rectangular matrices

Product of a lower and upper triangular matrices

$$A = LU$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{pmatrix} . \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{pmatrix}$$

# LU Decomposition of rectangular matrices

$$P^{-1}A = LU$$

P permutation matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

LU Decomposition for solving linear systems:

$$A\{x\} = \{b\}$$

$$LU\{x\} = \{b\}$$

$$L\{U.x\} = \{b\}$$

$$\{U.x\} = \{y\}$$

$$L\{y\} = \{b\}$$

Trivial to solve since L is lower triangular matrix

```java
Matrix A=Matrix.random(3,3);
Matrix b = Matrix.random(3,1);
Matrix x= A.solve(b);

System.out.println("A="); A.print(6,3);
System.out.println("b="); b.print(6,3);
System.out.println("x="); x.print(6,3);

LUDecomposition luDecomp=A.lu();
Matrix L=luDecomp.getL();
Matrix U=luDecomp.getU();
int [ ] pivot= luDecomp.getPivot();

System.out.println("L="); L.print(6,3);
System.out.println("U="); U.print(6,3);
for(int i=0;i<pivot.length;i++)
    System.out.print(pivot[i]+" ");
System.out.println("");
```

```
A=

  0.345   0.090   0.599
  0.932   0.428   0.703
  0.420   0.089   0.439

b=

  0.342
  0.192
  0.806

x=

  4.432
 -7.913
 -0.791

L=

  1.000   0.000   0.000
  0.450   1.000   0.000
  0.370   0.660   1.000

U=

  0.932   0.428   0.703
  0.000  -0.103   0.122
  0.000   0.000   0.258

1 2 0
```

# QR Decomposition

$$A = QR,$$

Q: Orthogonal matrix
R: Upper triangular matrix

Useful for solving least squares problem

```
import Jama.*;

class QRDecompositionTest
{
public static void main(String args[])
{
Matrix A=Matrix.random(3,3);
QRDecomposition qr=new QRDecomposition(A);
System.out.println("A=");A.print(6,3);
Matrix Q=qr.getQ();
System.out.println("Q=");Q.print(6,3);
Matrix R=qr.getR();
System.out.println("R=");R.print(6,3);
}
}
```

```
A=

  0.821  0.098  0.374
  0.057  0.151  0.036
  0.994  0.150  0.703

Q=

 -0.637  0.131  0.760
 -0.044 -0.990  0.134
 -0.770 -0.052 -0.636

R=

 -1.290 -0.185 -0.781
  0.000 -0.145 -0.023
  0.000  0.000 -0.158
```

# Cholesky decomposition

$$\mathbf{A} = \mathbf{L}\mathbf{L}^*,$$

For a **symmetric positive definite matrix** A,
Cholesky decomposition yields:

L is a lower triangular matrix
L* is the conjugate transpose

```java
import Jama.*;

class CholeskyDecompositionTest
{
public static void main(String args[])
{
double [][] array=new double[3][3];
int i,j;

for(i=0;i<3;i++)
	for(j=0;j<=i;j++){
		array[i][j]=Math.random();
		array[j][i]=array[i][j];}

Matrix A=new Matrix(array);
CholeskyDecomposition cd=new CholeskyDecomposition(A);

System.out.println("A=");A.print(6,3);
Matrix L=cd.getL();
System.out.println("L=");L.print(6,3);

if (cd.isSPD())
	{L.times(L.transpose()).print(6,3);}



}
}
```

```
A=

  0.688   0.342   0.046
  0.342   0.411   0.345
  0.046   0.345   0.962

L=

  0.829   0.000   0.000
  0.412   0.491   0.000
  0.055   0.658   0.725


  0.688   0.342   0.046
  0.342   0.411   0.345
  0.046   0.345   0.962
```

# Application of Cholesky decomposition : A multivariate normal

Box-Muller transform for standard normal variate : $\sqrt{-2\ln U_1}\cos(2\pi U_2)$

U1,U2 are independent uniform distributions (`Math.random()`)

Multivariate normal density : $$f_X(x) = \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\Big(-\tfrac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\Big),$$

We obtain a random sample $z$ from $N(\mu,\Sigma)$ from its Cholesky decomposition

$$\Sigma = AA^T$$

by taking a vector of normal variates

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix}$$

with $x_i = \sqrt{-2\ln u_1}\cos(2\pi u_2) where u_1$ and $u_2$ are two uniformly drawn numbers (the Box-Müller transform), and scale and shift this vector as follows:

$$z = \mu + Ax$$

```java
import Jama.*;

class Gaussian
{
int dim; // dimension
Matrix mu; // mean
Matrix Sigma; // variance-covariance matrix
Matrix  A; // Auxiliary for Cholesky decomposition
Matrix Precision; // Auxiliary for inverse of the covariance matrix

// Constructor
Gaussian(Matrix m,Matrix S){
    this.Sigma=S;
    ...
    CholeskyDecomposition cd=new CholeskyDecomposition(Sigma);
    this.A=cd.getL(); // Sigma=L L'
}

// Draw a normal variate
Matrix draw(){
    Matrix x=new Matrix(dim,1); // column vector
    for(int i=0;i<dim;i++)
        {double u1=Math.random();
         double u2=Math.random();
         // Box-Muller transform
         double r=Math.sqrt(-2.0*Math.log(u1))*Math.sin(2.0*Math.PI*u2);
         x.set(i,0,r);}
    return mu.plus(A.times(x));
}
```