MPRI – Cours 2-12-2

F. Morain

Lecture III: introduction to elliptic curves

2011/10/11

0. Conics.

I. Definition and group law.

II. Curves over finite fields.

III. ECDLP.

0. Conics

q odd prime power

$$C = \{(x,y) \in \mathbb{F}_q \times \mathbb{F}_q, x^2 + y^2 = 1\}.$$

$$(x_1,y_1) \oplus (x_2,y_2) = (x_1x_2 + y_1y_2, x_1y_2 + x_2y_1).$$

$(C, \oplus)$ is abelian, $O_C = (1,0)$, $\ominus(x,y) = (x,-y)$.

Thm. C is cyclic of order $q - \chi(-1)$, $\chi(a) = a^{(q-1)/2}$.
Proof:

Rem. Idem with $x^2 - Dy^2 = 1$.

I. Definition and group law

$\mathbf{K}$ field of characteristic $\neq 2, 3$. Elements of $\mathbf{K}^3 - \{(0,0,0)\}$ are equivalent iff

$$(x_1,y_1,z_1) \sim (x_1',y_1',z_1') \Longleftrightarrow \exists\, \lambda \neq 0, x_1 = \lambda x_1', y_1 = \lambda y_1', z_1 = \lambda z_1'.$$

Projective space: $\mathbf{P}^2(\mathbf{K})$ = equivalence classes of $\sim$.

Elliptic curve defined for points in $\mathbf{P}^2(\mathbf{K})$:

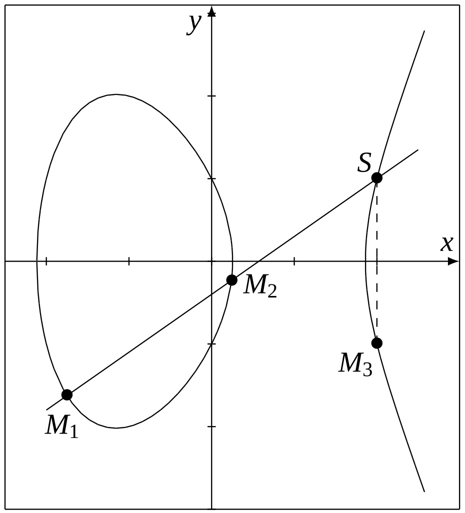
$$Y^2Z = X^3 + aXZ^2 + bZ^3 \quad (1)$$

with $4a^3 + 27b^2 \neq 0$ (discriminant of E).

Def. $E(\mathbf{K}) = \{(x : y : z)$ satisfying (1)$\}$.

Prop. $E(\mathbf{K}) = \{(0:1:0)\} \cup \{(x : y : 1)$ satisfying (1)$\}$ = point at infinity $\cup$ affine part.

The group law



$$M_3 = M_1 \oplus M_2$$

$$\lambda = \begin{cases} (y_1 - y_2)/(x_1 - x_2) \\ (3x_1^2 + a)/(2y_1) \end{cases}$$
$$x_3 = \lambda^2 - x_1 - x_2$$
$$y_3 = \lambda(x_1 - x_3) - y_1$$

$$[k]M = \underbrace{M \oplus \cdots \oplus M}_{k \text{ times}}$$

Rem. Standard equation and group law formulas for any field. Can be improved in many ways, see BS's part.

II. Curves over finite fields

Thm. (Hasse) $\#E(\mathbb{F}_p) = p + 1 - t$, $|t| \leq 2\sqrt{p}$.

Thm. (Deuring) given $|t|$, there exists E s.t. $\#E = p + 1 - t$.

Key advantage: enough groups of cardinality close to p (e.g., primality proving).

Caveat:

- no general formula for $\#E$ except in some special cases, e.g. $E : Y^2 = X^3 + X$ has $p + 1 - 2u$ points when $p = u^2 + v^2$.
- no efficient way for finding E given t except in some special cases (complex multiplication).

Rem. Generalizable to $q = p^n$.

Group structure

Thm. $E(\mathbb{F}_p) \simeq E_1 \times E_2$ of respective ordres m_1 and m_2 s.t. $m_2 \mid p - 1$ and $m_2 \mid m_1$.

Prop. (Murty; Vlăduţ) Almost always, $E(\mathbb{F}_p)$ is cyclic.

Consequence:

$$\sqrt{p} - 1 < \exp(E(\mathbb{F}_p)) < (\sqrt{p} + 1)^2.$$

Thm. (Schoof) For almost all curves $E/\mathbb{Q}$, there exists $C_E > 0$ s.t.

$$\frac{\exp(E(\mathbb{F}_p))}{\sqrt{p}} > C_E \frac{\log p}{(\log \log p)^2}.$$

Computing the cardinality

Invent a method in time:

- $O(p)$:
- $O(p^{1/2})$:
- $O(p^{1/4})$:

Algorithms:

- $g = 1$, p large: Schoof (1985). $\tilde{O}((\log p)^5)$, completely practical after improvements by Elkies, Atkin, and implementations by M., Lercier, etc. New recent record (2010/07) A. Sutherland, for $p = 16219299585 \cdot 2^{16612} - 1$ (5000dd), 1378 CPU days AMD Phenom II 3.0 GHz.
- $p = 2$: p-adic methods (Satoh, Fouquet/Gaudry/Harley; Mestre). Completely solved.

III. ECDLP

DLP in general resistant on an elliptic curve except

- supersingular curves ($t = 0$), due to the MOV reduction;
- anomalous curves ($t = 1$).

ECC112b: taken from `http://lacal.epfl.ch/page81774.html`, Bos/Kaihara/Kleinjung/Lenstra/Montgomery (EPFL/Alcatel-Lucent Bell Laboratories/MSR) $p = (2^{128} - 3)/(11 \times 6949)$, curve secp112r1

- 3.5 months on 200 PS3; 8.5×10^{16} ec additions ($\approx$ 14 full 56-bit DES key searches); started on January 13, 2009, and finished on July 8, 2009.
- half a billion distinguished points using 0.6 Terabyte of disk space.