2nd MAGC Workshop

November 17–19, 2000

Building robust (hyper)elliptic curve cryptosystems

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I. Introduction and motivations

Goal: build an effective group of cryptographic strength, resisting all known attacks.

Dream: find **Nechaev** groups G, in which the best attack will be $O(\sqrt{\#G})$ (existence?)

Best groups so far: hyperelliptic curves of genus g, with size $\approx q^g$ over some finite field $\mathbb{F}_q$. Typical size $q^g \approx 2^{160--200} \approx 10^{50--60}$.

- Miller, Koblitz (1986): elliptic curves are suggested for use, following the breakthrough of Lenstra in integer factorization (1985).
- Koblitz (1988): hyperelliptic cryptosystems.

In this series of talks:

- Put the emphasis on elliptic curves, but take a more general view from time to time; $g > 1$ is the **next case**; sometimes, hec's yield info on ec's.
- Consider **any base field**, with some preference for large prime fields, or $\mathbb{F}_{2^n}$; few places where it really matters.

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General overview of the three lectures

I. Point counting algorithms: basic approaches.

II. Point counting algorithms: elaborate methods.

III. Discrete log algorithms.

Bibliography and links

- *A course in algorithmic algebraic number theory* (Cohen);
- *The arithmetic of elliptic curves* (Silverman);
- *Elliptic curve public key cryptosystems* (Menezes);
- *Elliptic curves in cryptography* (Blake, Seroussi, Smart);
- Lercier's thesis.
- *Algebraic aspects of cryptography* (Koblitz, appendix on hec by Menezes, Wu, Zuccherato).


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I. Point counting algorithms: basic approaches

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Plan

I. Elements of theory.

II. Generic methods.

III. Particular curves.

IV. Schoof's algorithm.

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I. Elements of theory

Let C be a plane smooth projective curve of genus g with equation $F(X,Y)=0$ with coefficients in $\mathbb{K}$, $\mathrm{char}(\mathbb{K})=p$.

Conic: (genus 0) $x^2+y^2=1$.

Elliptic curve: (genus 1) $y^2=x^3+x+1$.

Hyperelliptic curve: (genus g) $y^2=x^{2g+1}+\cdots$ (or in some cases $y^2=x^{2g+2}+\cdots$).

Rem. To simplify things, we assume that C is "at most" hyperelliptic (no C_{ab} or $X_0(N)$).

Def. $C(\mathbb{K})=\{P=(x,y)\in\mathbb{K}^2, F(x,y)=0\}$.

Thm. When $g\leq 1$, there is a group law on $C(\mathbb{K})$. When $g>1$, there is a group law on the *jacobian* of the curve.

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Elliptic curves

$$E: Y^2+a_1XY+a_3Y=X^3+a_2X^2+a_4X+a_6$$

$$b_2=a_1^2+4a_2, b_4=2a_4+a_1a_3, b_6=a_3^2+4a_6,$$
$$b_8=a_1^2a_6+4a_2a_6-a_1a_3a_4+a_2a_3^2-a_4^2,$$
$$c_4=b_2^2-24b_4, c_6=b_2^3+36b_2b_4-216b_6,$$

$$\Delta=-b_2^2b_8-8b_4^3-27b_6^2+9b_2b_4b_6\neq 0$$
$$j(E)=\frac{c_4^3}{\Delta}$$

When $p=2$**:** $Y^2+XY=X^3+a_2X^2+a_6$, $j=1/a_6$.

When $p>3$**:** $Y^2=X^3+AX+B$, $\Delta=-16(4A^3+27B^2)$.

$E(\mathbb{K})$, tangent-and-chord ($\oplus$, O_E), multiplication by n noted $[n]P$.

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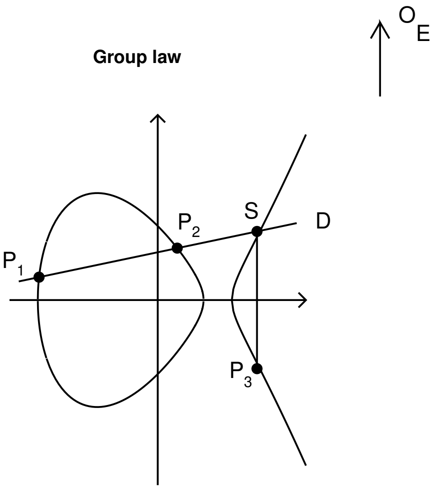
Group law



$$P_3=P_1\oplus P_2$$
$$[k]P=\underbrace{P\oplus\cdots\oplus P}_{k\text{ times}}$$

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Hyperelliptic curves

$$y^2 + h(x)y = f(x) = x^{2g+1} + \cdots$$

Representing $\mathrm{Jac}(C)$: (Mumford)

An element (= a divisor) of $\mathrm{Jac}(C)$ is

$$D = < u(z), v(z) >, \deg(u) \leq g, \deg(v) < \deg(u),$$

defined by (if $P_i = (x_i, y_i)$),

$$u(z) = \prod_{i=1}^{g} (z - x_i), \text{ and } v(x_i) = y_i, \ \forall i.$$

Rem. If $D = < u(z), v(z) >$, then $-D = < u(z), -v(z) >$.

Group law: Cantor's algorithm (or special formulae for fixed g *à la* Spallek, Harley, Nagao).

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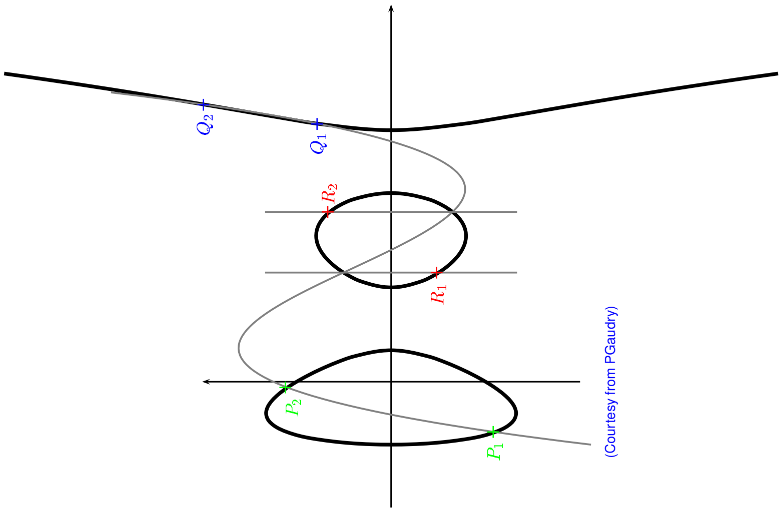

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Cardinality

$\mathbb{K} = \mathbb{F}_q = \mathbb{F}_{p^n}$; $N_r = \#C(\mathbb{K}_r)$ where $[\mathbb{K}_r : \mathbb{K}] = r$:

$$Z(T) = \exp\left(\sum_{r \geq 1} N_r \frac{T^r}{r}\right).$$

Ex. $\mathbb{P}^1(\mathbb{F}_{q^r}) = \{(x_0, x_1) \neq (0,0) \in \mathbb{F}_{q^r}^2\}/\sim$.

$$\#\mathbb{P}^1(\mathbb{F}_{q^r}) = 1 + q^r$$

$$Z(T) = \frac{1}{(1-T)(1-qT)}.$$

Thm. (Weil) $Z(T) \in \mathbb{Q}[T]$

$$Z(T) = \frac{L(T)}{(1-T)(1-qT)}$$

(i) $L(T) = 1 + a_1 T + \cdots + q^g T^{2g}$, $a_i \in \mathbb{Z}$;

(ii) $a_{2g-i} = q^{g-i} a_i$ for $0 \leq i \leq g$;

(iii) if $L(T) = \prod(1 - \alpha_i T)$, then $\alpha_i \alpha_{g+i} = q$ and $|\alpha_i| = \sqrt{q}$.

Thm. $\#\mathrm{Jac}(C) = L(1)$.

Coro. $|\#C - (q+1)| \leq 2g\sqrt{q}$;
$(\sqrt{q} - 1)^{2g} \leq \#\mathrm{Jac}(C) \leq (\sqrt{q} + 1)^{2g}$.

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ℓ-torsion

Def. $\mathrm{Jac}[n] = \{P \in \mathrm{Jac}(\overline{\mathbb{K}}), [n]P = O_J\}$.

Thm. If $(n, \mathrm{char}(\mathbb{K})) = 1, \mathrm{Jac}[n] \sim (\mathbb{Z}/n\mathbb{Z})^{2g}$;
$\mathrm{Jac}[p^r] = (\mathbb{Z}/p\mathbb{Z}^r)^k, 0 \leq k \leq g$.

Rem. In general $k = g$ (ordinary curves); when $g = 1$, the case $k = 0$ corresponds to *supersingular* curves.

Coro. $\mathrm{Jac}(C)/\mathbb{K}$ is at most $C_1 \times C_2 \times \cdots \times C_{2g}$.

Division polynomials for elliptic curves:

Take $E : y^2 = x^3 + Ax + B$:

$$[n](X, Y) = \left(\frac{\phi_n(X,Y)}{\psi_n(X,Y)^2}, \frac{\omega_n(X,Y)}{\psi_n(X,Y)^3}\right)$$

$$\phi_n = X\psi_n^2 - \psi_{n+1}\psi_{n-1}$$

$$4Y\omega_n = \psi_{n+2}\psi_{n-1}^2 - \psi_{n-2}\psi_{n+1}^2$$

$$\phi_n, \psi_{2n+1}, \psi_{2n}/(2Y), \omega_{2n+1}/Y, \omega_{2n} \in \mathbb{Z}[A, B, X]$$

Rem. When $g > 1$, one can define analogous division polynomials (cf. Cantor).

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$$f_n(X) = \begin{cases} \psi_n(X,Y) & \text{for } n \text{ odd} \\ \psi_n(X,Y)/(2Y) & \text{for } n \text{ even} \end{cases}$$

$$f_{-1} = -1, \quad f_0 = 0, \quad f_1 = 1, \quad f_2 = 1$$

$$f_3(X,Y) = 3X^4 + 6AX^2 + 12BX - A^2$$

$$f_4(X,Y) = X^6 + 5AX^4 + 20BX^3 - 5A^2X^2 - 4ABX - 8B^2 - A^3$$

$$f_{2n} = f_n(f_{n+2}f_{n-1}^2 - f_{n-2}f_{n+1}^2)$$

$$f_{2n+1} = \begin{cases} f_{n+2}f_n^3 - f_{n+1}^3 f_{n-1}(16Y^4) & \text{if } n \text{ is odd} \\ (16Y^4)f_{n+2}f_n^3 - f_{n+1}^3 f_{n-1} & \text{otherwise.} \end{cases}$$

$$\deg(f_n(X)) = \begin{cases} (n^2-1)/2 & \text{if } n \text{ is odd} \\ (n^2-4)/2 & \text{otherwise.} \end{cases}$$

Thm. $P = (x, y)$ point of order ℓ in $E(\overline{\mathbb{K}})$ $\iff [2]P = O_E$ or $f_\ell(x) = 0$.

Rem. $f_3(X) = -A^2$ when $\mathrm{char}(\mathbb{K}) = 3$, so that all $Y^2 = X^3 + AX + B$ are supersingular in this case.

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II. Generic methods

Input: a finite abelian group $(G, +)$ with $\#G \leq B$.

Output: $\#G$.

Enumeration: $O(\#G)$.

Use Lagrange's theorem: for random $x \in G$, find $\omega =$ order of x. Deduce from this the order of G (take care to small orders, group structure with SNF, etc.; see Cohen).

How to find ω:

- try increasing value of ω: $O(\omega) \leq O(B)$ (??).
- Shanks's *baby steps/giant steps method*: write $m = m_0 + m_1 b$ for some $b, 0 \leq m_0 < b$, $0 \leq m_1 \leq B/b$ and write

$$[m]x = 0 \iff [m_1]([-b]x) = [m_0]x.$$

1. **baby steps**: precompute $\mathcal{B} = \{[m_0]x, 0 \leq m_0 \leq b\}$;
2. **giant steps**: find all m_1 s.t. $[m_1]([-b]x) = [m_0]x$ for some m_0.

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Cost: $b + B/b$ minimized with $b = \sqrt{B}$. Time and space are $O(\sqrt{B})$ group operations, **deterministic**.

Rem. membership must be tested very fast (hashing).

Rem. can be modified when $A \leq \#G \leq B$, yielding a method in $O(\sqrt{B-A})$.

• Using kangaroos (Stein-Teske, Gaudry-Harley): **probabilistic** method in $O(\sqrt{B-A})$ time and $O(1)$ space (see discrete logs in part III).

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Application to elliptic curves

- **Enumeration:** find all $x \in \mathbb{F}_q$ s.t. $f(x)$ is a square.
- **Lagrange:** $[q+1]P = [\pm c]P$ for $0 \leq c \leq 2\sqrt{q}$.

  Rem. If $\mathrm{ord}(P)$ is large enough, then

  $$\#\{c \in [-2\sqrt{q}, 2\sqrt{q}], [q+1-c]P = O_E\} = 1$$

  and we can bypass the structure problem.
- **Kangaroos:** idem.
- **Shanks:**

  More precisely: $c = n_0 + n_1 W, 0 \leq n_0 < W$, $|n_1| \leq 2\sqrt{q}/W$. Write

  $$[q+1-n_0]P = [\pm n_1][W]P, 0 \leq n_1 \leq 2\sqrt{q}/W$$

  Cost: $W = \sqrt{2\sqrt{q}}$, so $O(2\sqrt{2\sqrt{q}})$.

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Application to hyperelliptic curves

$$L(1) = 1 - s_1 + \cdots + (-1)^g s_g$$
$$+(-1)^{g+1} q s_{g-1} + \cdots - q^{g-1} s_1 + q^g,$$
$$|s_i| \leq \binom{2g}{i} q^{i/2}.$$

A) Enumeration

$g = 2$**:** compute $N_1(C)$ and $N_2(C)$ and deduce
$s_1 = q + 1 - N_1(C)$,
$s_2 = (s_1^2 + N_2(C) - (q^2 + 1))/2$.

$g = 3$**:** $s_3 = (s_1^3 - 3s_1 s_2 - N_3 + q^3 + 1)/3$.

Prop. Method in $O(q^g)$.

B) Lagrange

Hasse-Weil gives $w = (\sqrt{q}+1)^{2g} - (\sqrt{q}-1)^{2g} = \mathbf{4gq^{g-1/2}} + O(q^{g-1})$ (for fixed g, $q \to +\infty$).

Prop. Method in $O(q^{(2g-1)/2})$ (for fixed g).

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C) Shanks/Kangaroos

Prop. Method in $O(q^{(2g-1)/4})$ (for fixed g).

Using partial information

Prop. (Elkies) If the $k-1$ first s_i's are known, then

$$w' = 2\binom{2g}{k} q^{g-k/2} + O(q^{g-(k+1)/2})$$

and with more work

$$w'' = \frac{4g}{k} q^{g-k/2} + O(q^{g-(k+1)/2}).$$

Cost: $O(q^{k-1}) + O(q^{(2g-k)/4})$ leading to $k = (2g+4)/5$ and $O(q^{(2g-1)/5})$.

Ex. $g = 3$ leads to a $O(q)$ method by computing s_1 instead of $O(q^{5/4})$.

Stein-Teske-Williams: use truncated Euler product of the L-function to approximate $L(1)$, get same type of bound $O(q^{\lfloor (2g-1)/5 \rceil})$. Record (1999): $q = 10^8 + 7$, $g = 3$, $\#\mathrm{Jac} \approx 10^{24}$ in 10 hours on a Pentium Pro/200 with SIMATH.

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III. Particular curves

A) Supersingular curves

Supersingular elliptic curves: E s.t. $\#E = q + 1 - c, p \mid c$ (not every c, all is known).

For instance: when $n = 2m+1, q = 2^n$

E	c_n
$Y^2 + Y = X^3$	0
$Y^2 + Y = X^3 + X$	$-(2/n)\sqrt{2q}$
$Y^2 + Y = X^3 + X + 1$	$(2/n)\sqrt{2q}$

(See A. Menezes and S. Vanstone, *Utilitas Math.*, 38:135–153, 1990)

Pb: subject to the MOV reduction (see also Frey, Rück).

$g > 1$**:** can be generalized, but reductions still apply (see also Galbraith for security evaluation).

B) Weil-Koblitz

Build curves over $\mathbb{F}_q$ for q small and use $\mathrm{Jac}(C)/\mathbb{F}_{q^k}$.

Pb: ECDL might be a little easier.

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C) CM curves

(complex multiplication)

$g = 1$**:**

Thm. (Katre) If $p = x^2 + 4y^2$ with $x \equiv 1 \bmod 4$ and $a \not\equiv 0 \bmod p$, then $E : Y^2 = X^3 + aX$ has cardinality

$$p+1-\begin{cases} 2x & \text{if } (a/p)_4 = 1, \\ -2x & \text{if } (a/p)_4 = -1, \\ -4y & \text{otherwise with } y \text{ s.t. } 2y(a/p)_4 = x. \end{cases}$$

There are 13 cases of curves defined over $\mathbb{Q}$ having such properties; in general, $4p = A^2 + DB^2$, $\#E = p + 1 - A$: basis for primality proving with elliptic curves (ECPP, Atkin, M.).

$g > 1$**:**

Spallek, Weng ($g = 2$); Buhler-Koblitz; Duursma-Sakurai; Chao, Matsuda, Nakamura, Tsujii.

Pb: too much structure?

D) Weil descent

Start from ec's to build hec's (Smart *et al.*).

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IV. Schoof's algorithm

The Frobenius endomorphism

Ordinary:

$$\begin{array}{rccc} \varphi: & \overline{\mathbb{K}} & \to & \overline{\mathbb{K}} \\ & x & \mapsto & x^q \end{array}$$

Extension to C and $\mathrm{Jac}(C)$:

$$\begin{array}{rccc} \varphi: & C(\overline{\mathbb{K}}) & \to & C(\overline{\mathbb{K}}) \\ & (X, Y) & \mapsto & (X^q, Y^q) \end{array}$$

Fundamental thm. The minimal polynomial $\chi(T)$ of φ is the reciprocal of $L(T)$. Moreover $\#\mathrm{Jac}(C)/\mathbb{F}_q = \chi(1)$.

Consequence: computing $\#\mathrm{Jac}(C)/\mathbb{F}_q$ boils down to computing $\chi(T)$.

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$g = 1$: for E with $\chi(T) = T^2 - cT + q$, $|c| \leq 2\sqrt{q}$.

φ restricted to $E[\ell]$ satisfies:

$$\varphi^2 - c\varphi + q \equiv 0 \bmod \ell$$

so we can find $c_\ell \equiv c \bmod \ell$ such that

$$(X^{q^2}, Y^{q^2}) \oplus [q](X, Y) = [c_\ell](X^q, Y^q)$$

in $\mathbb{K}[X, Y]/(E, f_\ell(X))$ and use CRT once $\prod \ell > 4\sqrt{q}$. Yields a $O(\log^8 q)$ **deterministic algorithm**.

Pb. $\deg(f_\ell) = O(\ell^2)$.

$g > 1$: general algorithm by Pila (1990), but impossible to implement; Kampkötter (1991) for any hyperelliptic, with precise equations for $g = 2$ (uses Gröbner bases).

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II. Point counting algorithms: elaborate methods

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Plan

I. What we saw yesterday.

II. Isogenies and point counting: Elkies, Atkin, Couveignes, Lercier.

III. Satoh's algorithm.

IV. Generalization to genus 2.

V. Generating cryptographically strong elliptic curves.

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I. What we saw yesterday

$$\begin{array}{rccc} \varphi: & C(\overline{\mathbb{K}}) & \to & C(\overline{\mathbb{K}}) \\ & (X,Y) & \mapsto & (X^q, Y^q) \end{array}$$

Fundamental thm. The minimal polynomial $\chi(T)$ of φ is the reciprocal of $L(T)$. Moreover $\#\mathrm{Jac}(C)/\mathbb{F}_q = \chi(1)$.

Consequence: computing $\#\mathrm{Jac}(C)/\mathbb{F}_q$ boils down to computing $\chi(T)$.

$g=1$: for E with $\chi(T) = T^2 - cT + q$, $|c| \leq 2\sqrt{q}$.

φ restricted to $E[\ell]$ satisfies:

$$\varphi^2 - c\varphi + q \equiv 0 \bmod \ell$$

so we can find $c_\ell \equiv c \bmod \ell$ such that

$$(X^{q^2}, Y^{q^2}) \oplus [q](X,Y) = [c_\ell](X^q, Y^q)$$

in $\mathbb{K}[X,Y]/(E, f_\ell(X))$ and use CRT once $\prod \ell > 4\sqrt{q}$. Yields a $O(\log^8 q)$ **deterministic algorithm**.

Pb. $\deg(f_\ell) = O(\ell^2)$.

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II. Isogenies and point counting

A) Elements of theory

Def. $\phi : E \to E^*$, $\phi(O_E) = O_{E^*}$; induces a morphism of groups.

First examples

1.
$$[k](X,Y) = \left(\frac{A_k}{\psi_k^2}, \frac{B_k}{\psi_k^3}\right)$$

2. $[i](X,Y) = (-X, iY)$ on $E : Y^2 = X^3 - X$.

3. $\varphi(X,Y) = (X^q, Y^q)$, $\mathbb{K} = \mathbb{F}_q$.

Thm. (dual isogeny) There is a unique $\hat{\phi} : E^* \to E$, $\hat{\phi} \circ \phi = [m]$, $m = \deg\phi$.

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[dr, "{[m]}"'] & E^* \arrow[d, "\hat{\phi}"] \\ & E \end{tikzcd}$$

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Isogenies and subgroups

Thm. If F is a finite subgroup of E, then there exists ϕ and E^* s.t.

$$\phi : E \to E^* = E/F, \quad \ker(\phi) = F.$$

Ex. $E : y^2 = x^3 + ax^2 + bx$, $F = <(0,0)>$;

$$E^* : Y^2 = X^3 - 2aX^2 + (a^2 - 4b)X,$$

$$\phi : (x,y) \mapsto \left(\frac{y^2}{x^2}, \frac{y(b-x^2)}{x^2}\right).$$

More generally: Vélu's formulas give

$$\phi(X,Y) = \left(\frac{G(X)}{H(X)^2}, \frac{J(X,Y)}{H(X)^3}\right).$$

(case $\deg\phi$ odd.)

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Application to point counting

Suppose F is a subgroup of order ℓ of E:

$$\begin{tikzcd} E \arrow[r, "I"] \arrow[dr, "{[\ell]}"'] & E^* \arrow[d, "\hat{I}"] \\ & E \end{tikzcd}$$

$$I(X,Y) = \left(\frac{G}{H^2}, \dots\right), \deg(H) = (\ell - 1)/2$$

$\ker(I) \subset E[\ell] \Rightarrow H(X) \mid f_\ell(X)$ in $\mathbb{K}[X]$.

Schoof's algorithm on a degree $O(\ell)$ polynomial.

Pb. When does such an F exist over $\mathbb{K}$?

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B) Atkin and Elkies

Consider $\varphi : (X, Y) \mapsto (X^q, Y^q)$ and its restriction φ_ℓ to $E[\ell]$:

$$\varphi_\ell^2 - c\varphi_\ell + q = 0,$$
$$\Delta = c^2 - 4q.$$

If $(\Delta/\ell) = +1$, then over $\mathbb{F}_\ell$,

$$\mathrm{Mat}(\varphi_\ell) \simeq \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Leftrightarrow \exists F, \varphi(F) = F \Leftrightarrow F$$

is a cyclic subgroup of order ℓ, defined over $\mathbb{K}$.

Clon. If $(\Delta/\ell) = +1$, f_ℓ has a factor of degree $(\ell - 1)/2$.

Pb. How do we know that $(\Delta/\ell) = +1$?

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Modular polynomials

Thm. $\exists \Phi_\ell(X, Y) \in \mathbb{Z}[X, Y]$ s.t. E and E^* are ℓ-isogenous over $\mathbb{K}$ only if $\Phi_\ell(j(E), j(E^*)) = 0$.

Rem. This polynomial comes from the theory of elliptic curves over $\mathbb{C}$.

Ex.

$$\Phi_2(X, Y) = X^3 + X^2\left(-Y^2 + 1488\,Y - 162000\right)$$
$$+X\left(1488\,Y^2 + 40773375\,Y + 8748000000\right)$$
$$+Y^3 - 162000\,Y^2 + 8748000000\,Y - 157464000000000.$$

Thm. $E/\mathbb{F}_q$:

$$\Phi_\ell(X, j(E)) = \begin{cases} (1)(1)(s)\cdots(s) & \text{if } (\Delta/\ell) = +1, \\ (s)\cdots(s) & \text{if } (\Delta/\ell) = -1 \end{cases}$$

and s is the order of λ_1/λ_2.

Atkin's idea: use the splitting of Φ_ℓ to deduce information on t and combine it via a clever match and sort algorithm.

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Clon. $(\Delta/\ell) = +1$ iff $\Phi_\ell(X, j(E))$ has two distinct roots over $\mathbb{K}$.

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Elkies's algorithm

repeat

1. factor $\Phi_\ell(X, j(E))$ over $\mathbb{K}$.
2. if type = $(1)(1)(r)\cdots(r)$:
   - 2.1 build E^*;
   - 2.2 build I;
   - 2.3 find $c \bmod \ell$;

until $\prod_{\ell \text{ good}} \ell > 4\sqrt{q}$.

Thm. $O(\log^6 q)$ probabilistic.

Rem. One can build Φ_ℓ using series operations: Φ^c, Φ^*, Φ^Θ (Atkin), Φ^M (Müller), for $\ell \leq 1000$.

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More details

- $p \gg \ell$**:** (Elkies, Atkin, 1988–1992)
  - use the theory of elliptic curves and lattices over $\mathbb{C}$ (Weierstrass $\wp$ function); rational formulas for E^*;
  - computing I takes $O(\ell)$ steps and uses $O(\ell)$ memory;
  - very efficient in practice: p with 500 decimal digits (M. 1995).
- p **small:** Couveignes I (first efficient method, 1994), II.
- $p = 2$**:** Lercier (1996), most efficient method until Satoh (1999-2000).

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C) Using formal groups

$$t = -X/Y, s = -1/Y$$

$$F(t,s) := t^3 + a_1 ts + a_2 t^2 s + a_3 s^2 + a_4 t s^2 + a_6 s^3 - s = 0.$$

$$s = \mathcal{S}(t) = t^3 + a_1 t^4 + (a_1^2 + a_2)t^5 + O(t^6).$$

$$Y = -\frac{1}{s} = -t^{-3} + a_1 t^{-2} + a_2 t^{-1} + a_3 + (a_1 a_3 + a_4)t + O(t^2)$$

$$X = \frac{t}{s} = -tY = t^{-2} - a_1 t^{-1} - a_2 - a_3 t - (a_1 a_3 + a_4)t^2 + O(t^3)$$

$$t_3 = F_a(t_1, t_2) = t_1 + t_2 - a_1 t_1 t_2 - a_2(t_1^2 t_2 + t_1 t_2^2)$$
$$-(2a_3 t_1^3 t_2 - (a_1 a_2 - 3a_3)t_1^2 t_2^2 + 2a_3 t_1 t_2^3) + \ldots$$

$$[2]t_1 = 2\,t_1 - a_1\,t_1^2 - 2\,a_2\,t_1^3 + (a_1\,a_2 - 7\,a_3)\,t_1^4 + \cdots$$

$$[n]t_1 = n\,t_1 + \cdots$$

Thm. $[p](t) = \Psi_{p,E}(t)^p = c_p(E)t^p + O(t^p)$.

Thm. $\mathrm{Norm}_{\mathbb{K}/\mathbb{F}_p}(c_p(E)) \equiv \mathrm{Tr}_{\mathbb{K}/\mathbb{F}_p}(\varphi) \bmod p$.

Ex. $c_2([a_1, a_2, a_3, a_4, a_6]) = a_1$, $c_3([a_2, a_6]) = a_2$.

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CouveignesI

Looking for I is equivalent to searching for a morphism of formal groups $\mathcal{M} : \mathcal{E} \to \mathcal{E}^*$

$$\mathcal{M}(t(\tau), s(\tau)) = (\mathcal{U}(t(\tau)), \mathcal{S}^*(\mathcal{U}(t(\tau))))$$

s.t.

$$\mathcal{M}(P_1 \oplus P_2) = \mathcal{M}(P_1) \oplus \mathcal{M}(P_2)$$

or

$$\mathcal{U}(t_1 \oplus t_2(\tau)) = \mathcal{U}(t_1(\tau)) \oplus \mathcal{U}(t_2(\tau))$$

where $\mathcal{U}(t) = t + \sum_{i \geq 2} u_i t^i$. $\mathcal{U}$ is found incrementally using inderterminate coefficients.

Thm. time $O(\ell^3)$, space $O(\ell^2)$.

Rem. In real life, difficult to implement (huge series computations).

For more details, see Lercier & Morain, *Math. Comp.* 69, 2000.

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D) Lercier's algorithm

Motivation: CouveignesI too slow, too much memory involved (huge series computations).

$$p = 2, E: Y^2 + XY = X^3 + a, E^*: Y^2 + XY = X^3 + b$$

Prop. $I(X) = XL(X)^2/H(X)^2$.

Ideas:

- send $P = (0, \sqrt{a})$ to $P^* = (0, \sqrt{b})$ and use:
$$P \oplus M = (0, \sqrt{a}) \oplus (X, Y) = (\sqrt{a}/X, \ldots);$$
- use
$$I(P \oplus M) = P^* \oplus I(M);$$
- use
$$I \circ [2] = [2] \circ I;$$
- translate all these equations in a (non linear) system of equations in coefficients of $L(X)$ and $H(X)$ and solve it.

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Analysis: heuristically, time is $O(\ell^3)$ with a much better constant; space is $O(\ell)$.

In practice: considerably faster than Couveignes's first algorithm: 400 times faster for computing isogenies for $\mathbb{K} = \mathbb{F}_{2^{300}}$, resulting in a **speed-up of** 5 on the total running time for the complete SEA algorithm.

$\mathbb{F}_{2^{155}}$	86 sec
$\mathbb{F}_{2^{196}}$	308 sec
$\mathbb{F}_{2^{300}}$	2851 sec
$\mathbb{F}_{2^{1663}}$	330 days

(Alpha 266 MHz)

Rem. Vercauteren (October 1999) $\mathbb{F}_{2^{1999}}$ after 65 days of CPU time on 400 MHz PCs.

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E) CouveignesII

Idea: $E[p]$ seems to be of first importance.

Prop.

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[dr, "{[p]}"'] & E^p \arrow[d, "\hat{\varphi}"] \\ & E \end{tikzcd}$$

$$[p](X, Y) = (F_p(X)^p, G_p(X, Y)^p)$$

Coro.

$$f_{p^i}(X) = \tilde{f}_{p^i}^{p^i}, \deg \tilde{f}_{p^i} = d_i = p^{i-1}(p-1)/2 \text{ or } 2^{i-1}$$

Idea: I maps $E[p^k]$ to $E^*[p^k]$:

$$I([m]P) = [m]P^*, \quad I(x_m) = x_m^*$$

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Use interpolation: let $A(X)$ s.t. $A(x_m) = x_m^*$ for all $m, 1 \leq m \leq d_k$. Then

$$I(X) \equiv A(X) \bmod \tilde{f}_{p^k}(X)$$

Prop. If $d_k > 2\ell$, recover H via the euclidean algorithm.

Rem. In general, the p^k-division is not rational and the interpolation part must be done with some care.

Prop. Complexity should be time $O(\ell^{2+\epsilon})$, space $O(\ell)$.

Prop. Artin-Schreier approach of Couveignes: $O(n\mathcal{M}(\ell))$, see *Math. Comp.* 69, 2000.

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III. Satoh's algorithm

Def. $\mathbb{Z}_p$ ring of p-adic integers $(x_1, x_2, \ldots, x_n, \ldots)$ s.t. $x_n \in \mathbb{Z}/p^n\mathbb{Z}$ and $x_{n+1} \equiv x_n \bmod p^n$. Denote by $\pi : \mathbb{Z}_p \to \mathbb{F}_p$ sending x to x_1.

Def. Let $q = p^r$ and $f(t) \in \mathbb{Z}_p[t]$ s.t. $\pi(f)$ is irreducible in $\mathbb{F}_p[t]$. Then $\mathbb{Z}_q = \mathbb{Z}_p[t]/(f(t))$.

An element of $\mathbb{Z}_q$ is $\mathcal{A} = a_{r-1}t^{r-1} + \cdots + a_0$ with $a_i \in \mathbb{Z}_p$; $\mathbb{Z}_q$ contains $\mathbb{Z}_p$ as a subring.

$$\pi(\mathcal{A}) = \sum_i \pi(a_i)t^i.$$

Prop. Let σ be the *little* Frobenius sending x in $\mathbb{F}_q$ to x^p. There is a canonical way to lift σ to $\Sigma : \mathbb{Z}_q \to \mathbb{Z}_q$.

Extend σ to points $\sigma(x, y) = (\sigma(x), \sigma(y))$ and to curves: $\sigma(E) = [\sigma(a_i)]$, so that if $P \in E(\mathbb{K})$, then $\sigma(P) \in \sigma(E)(\mathbb{K})$.

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Thm (Lubin-Serre-Tate) Let $E/\mathbb{F}_q$ with $j = j(E) \in \mathbb{F}_q - \mathbb{F}_{p^2}$. There is a unique $\mathcal{J}$ in $\mathbb{Z}_q$ s.t.

$$\Phi_p(\mathcal{J}, \Sigma(\mathcal{J})) = 0,$$

$\pi(\mathcal{J}) = j$; $\mathcal{J}$ is the invariant of the *canonical lift* $\mathcal{E}$ of E and $\mathrm{End}(\mathcal{E}) = \mathrm{End}(E)$.

Isogeny cycles:

$$\begin{array}{ccccccccc} \mathcal{E}_0 & \xrightarrow{\Sigma_{r-1}} & \mathcal{E}_{r-1} & \xrightarrow{\Sigma_{r-2}} & \cdots & \xrightarrow{\Sigma_1} & \mathcal{E}_1 & \xrightarrow{\Sigma_0} & \mathcal{E}_0 \\ \downarrow \pi & & \downarrow \pi & & & & \downarrow \pi & & \\ E_0 & \xrightarrow{\sigma_{r-1}} & E_{r-1} & \xrightarrow{\sigma_{r-2}} & \cdots & \xrightarrow{\sigma_1} & E_1 & \xrightarrow{\sigma_0} & E_0 \end{array}$$

Prop. $\varphi = \sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_{r-1}$,
$\mathcal{F} = \Sigma_0 \circ \Sigma_1 \circ \cdots \circ \Sigma_{r-1}$.

Thm. $\mathrm{Tr}(\varphi) = \mathrm{Tr}(\mathcal{F})$.

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Computing $\mathrm{Tr}(\mathcal{F})$

Use the dual of Frobenius to get another isogeny cycle amenable to computations:

$$\begin{array}{ccccccccc} \mathcal{E}_0 & \xrightarrow{\hat{\Sigma}_0} & \mathcal{E}_1 & \xrightarrow{\hat{\Sigma}_1} & \cdots & \xrightarrow{\hat{\Sigma}_{r-2}} & \mathcal{E}_{r-1} & \xrightarrow{\hat{\Sigma}_{r-1}} & \mathcal{E}_0 \\ \downarrow \pi & & \downarrow \pi & & & & \downarrow \pi & & \\ E_0 & \xrightarrow{\hat{\sigma}_0} & E_1 & \xrightarrow{\hat{\sigma}_1} & \cdots & \xrightarrow{\hat{\sigma}_{r-2}} & E_{r-1} & \xrightarrow{\hat{\sigma}_{r-1}} & E_0 \end{array}$$

Prop. $\hat{\varphi} = \hat{\sigma}_{r-1} \circ \hat{\sigma}_{r-2} \circ \cdots \circ \hat{\sigma}_0$ (idem for $\hat{\mathcal{F}}$) and also $\mathrm{Tr}(\hat{\mathcal{F}}) = \mathrm{Tr}(\mathcal{F}) = \mathrm{Tr}(\varphi)$.

Let τ (resp. τ_i) denote the local parameter of $\mathcal{E}$ (resp. $\mathcal{E}_i$).

$$\mathcal{F}(\tau) = \sum_{k \geq 1} c_k \tau^k$$

Prop. (Satoh) $\mathrm{Tr}(\mathcal{F}) = c_1 + q/c_1$.

$$c_1 = \prod_{i=0}^{d-1} g_i$$

where (Vélu's formulas again)

$$\hat{\Sigma}_i(\tau_i) = g_i \tau_i + O(\tau_i^2)$$

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Satoh's algorithm in brief

1. Compute the curves E_0, E_1, E_{r-1} and their invariants j_i.
2. Lift all the j_i's simultaneously by a Newton iteration to get $\mathcal{J}_i$:

   $\Theta((x_i)) = (\Phi_p(x_0, x_1), \Phi_p(x_1, x_2), \ldots, \Phi_p(x_{r-1}, x_0))$

   as

   $$(x_i) \leftarrow (x_i) - ((D\Theta)^{-1}\Theta)((x_i)).$$
3. Lift each E_i coefficient by coefficient.
4. Lift the p-torsion subgroup of E_i.
5. Compute the $\hat{\Sigma}_i$'s.
6. Compute the trace.

Thm. (Satoh-FGH) For fixed p, Satoh-FGH requires $O(r^3)$ memory and $O(r^{3+\varepsilon})$ bit-operations.

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Results

Implementation details: too many of them, see [FGH] and the URL

`http://www.xent.com/~harley`

for a demo version of the ECPC program.

Record: $\mathbb{F}_{2^{8009}}$ in 313 hours on an Alpha EV6, 750 MHz, 16.9 Gb (in Cornell).

More timings: on Alpha 750 MHz

Degree	Time	Degree	Time
160	$2.5s$	1000	$22m$
200	$4.6s$	2000	$3h$
240	$7.1s$	4000	$29h$

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IV. Generalization to genus 2

- **Division polynomials:** Cantor.
- **Schoof:** Pila; Kampkötter (Gröbner bases).
- **Modular equations:**
  - Siegel's modular forms; large output. On-going work by Gaudry and Harley.
  - Alternative equations by Gaudry and Schost (similar to Charlap, Coley and Robbins for elliptic curves): computed for $g = 2$, $\ell = 3$ (see PG's web page for the equation).
  - Factorization patterns: exist; cf. PG for $g = 2$.
- **Isogenies:** Vélu's formulas?
- **Satoh's algorithm:** LST valid. Need modular equation.

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Current status

- **Genus $g > 2$:** Stein-Teske (1999): $q = 10^8 + 7$, $g = 3$, $\#\mathrm{Jac} \approx 10^{24}$ in 10 hours on a Pentium Pro/200 with SIMATH.
- **Genus 2:** Gaudry & Harley (ANTS IV)
  - find $\#\mathrm{Jac} \bmod \ell$ using Cartier-Manin ($\ell = p$) or Cantor's division polynomials;
  - find $\#\mathrm{Jac} \bmod 2^k$ by looking for solutions of $[2]D = D_0$;
  - combine all this to reduce the birthday paradox.
  - record: $p = 10^{19} + 51$;

    $$y^2 = x^5 + \ 3141592653589793238\, x^4 + \cdots$$

    $$\#\mathrm{Jac} = 99999999982871020671452277000281660080,$$

ℓ^n	2^{16}	3	5	7	11	13
time	$10h$	$1200s$	$300s$	$12h$	$19h$	$205h$

Width 2.5×10^{29}, 50 days on a 500 MHz Alpha.

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V. Generating cryptographically strong curves

$\mathbb{F}_p$ with large p or $\mathbb{F}_{2^n}$ with n prime (Weil descent, see Menezes & Qu); subgroups of large prime order.

- **Supersingular curves:** too much structure (?).
- **CM curves:** quite efficient for $g = 1$ or $g = 2$, but who knows?
- **Fixed curves:** The NIST curves (?).
- **Random curves:**
  - $g > 1$ not efficient yet.
  - $g = 1$: use SEA for large p, Satoh for $p = 2$. Very efficient when combined to the early-abort approach in Lercier's EUROCRYPT'97 article. Experiments conducted by FGH combining SEA and Satoh show that it takes **5 min** on Alpha 750 MHz to build a good curve over $\mathbb{F}_{2^{233}}$.

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2nd MAGC Workshop

November 19, 2000

III. Discrete log algorithms

F. Morain

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Plan

I. The problem.

II. Generic methods.

III. Index calculus.

IV. Cryptographic implications.

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I. The problem

$\mathbb{K} = \mathbb{F}_q = \mathbb{F}_{p^r}$

Input: $P \in \mathrm{Jac}(C)$ of order $n, Q \in < P >$.

Output: k s.t. $Q = [k]P$.

Ex. elliptic case

$C/\mathbb{K} : Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6$

or hyperelliptic case:

$$C/\mathbb{K} : Y^2 + H(X)Y = X^{2g+1} + \cdots$$

Benchmark CERTICOM exercises (cf. `http://www.certicom.com/chal`).

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General results

- **No (known) subexponential method for small** g (including $g = 1$); recover a subexp method when g increases.
- **Reduction** $\mathrm{Jac}(C)/\mathbb{F}_q \hookrightarrow \mathbb{F}_{q^k}$ with k small:
  - Supersingular curves: *MOV* (Menezes, Okamoto, Vanstone using the Weil pairing); Frey & Rück (using the Tate pairing); Galbraith. See G. Frey's talks.
  - other cases: elliptic curves with $t = 2$ with the Tate pairing.
- Discrete logs in subgroups of order p^e of $\mathrm{Jac}(C)/\mathbb{F}_{p^r}$ can be found in **polynomial time**: $g = 1$ (anomalous curves) done by Satoh-Araki, Semaev, Smart; $g > 1$ by Rück.
- **Elliptic curves:** lifting points or *XEDNI*: Silverman *et al.*, Cheon *et al.*. Do not work in the present form.

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II. Generic methods

A) Shanks

$$k \in [-(n-1)/2, (n-1)/2], \quad w = \sqrt{(n-1)/2}$$
$$k = uw + v, 0 \leq v < w, \quad |u| \leq w.$$
$$[k]P = Q \Longleftrightarrow [u]([w]P) = Q \ominus [v]P.$$

baby steps: compute the points $Q \ominus [v]P$ for all v's in $O(w)$ steps.

giant steps: find u in $O(w)$ steps.

Cost: $O(w) = O(\sqrt{n})$ in time and space, **deterministic**.

B) Pohlig-Hellman

$$n = \prod_i p_i^{\alpha_i}$$

Idea: finding $[x]P = Q$ is equivalent to finding $x \bmod n$, i.e., $x \bmod p_i^{\alpha_i}$ for all i (CRT). It reduces the discrete log $\bmod n$ to that $\bmod p_i^{\alpha_i}$ for all i.

Cost: $O(\max(\alpha_i \mathrm{DL}(p_i)))$.

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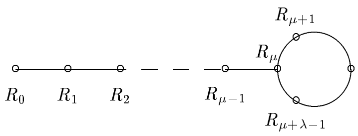
C) Pollard's ρ

Idea: use the properties of functional graphs.

Let f be a random function: $< P > \rightarrow < P >$, $R_0 \in < P >_R$ and build $R_{i+1} = f(R_i)$:



More precisely, one starts from $R_0 = [u_0]P + [v_0]Q$ with u_0 and v_0 in $\mathbb{Z}/n\mathbb{Z}_R$ and one builds the sequence $R_{i+1} = f(R_i)$, which also builds (u_i) and (v_i) modulo n s.t. $R_i = [u_i]P + [v_i]Q$.

Random mapping statistics; (Flajolet & Odlyzko, EUROCRYPT'89) a functional graph for a random f with n vertices should have $0.5 \log n$ components, $\overline{\mu}_n = \overline{\lambda}_n = \sqrt{\pi n/8}$, $\overline{\rho}_n = \sqrt{\pi n/2}$.

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Clon: after $\approx \sqrt{\pi n/2}$ iterations, we should find a collision, that is i and j s.t. $R_i = R_j$:

$$[u_i - u_j]P + [v_i - v_j]Q = 0$$

hence k if $v_i \neq v_j$ (which happens with probability $1/n$).

Floyd's algorithm: there is an $e = O(\sqrt{\mu})$ s.t. $R_e = R_{2e}$:

- $R := R_0$, $S := R_0$;
- **repeat** $R := f(R)$, $S := f(f(S))$ **until** $R = S$.

Cost: time $O(\sqrt{n})$ and space $O(1)$.

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D) Parallel collision search

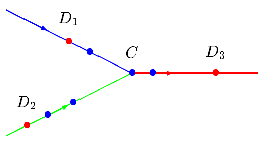
1) Distinguished points

(Quisquater & Delescaille, EUROCRYPT'89)

Idea: store points with a distinguishing property (e.g., $P = (x, y)$ and $x \equiv 0 \bmod 2^b$). If θ is the proportion of distinguished points, one should find a collision after having computed $\theta\sqrt{\pi n/2}$ distinguished points.

2) The algorithm

(van Oorschot & Wiener, *J. Cryptology*, 1999)



M processors run through the graph at the same time: one should find a collision after each processor has computed $\sqrt{\pi n/2}/M$ points, or $\theta\sqrt{\pi n/2}/M$ distinguished points.

Rem. All solved exercises of Certicom done this way.

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Choosing f

- **additive random walk:**
  - precompute r points $(T^{(j)})_{0 \leq j < r}$ in $< P >$, e.g. $T^{(j)} = [u^{(j)}]P + [v^{(j)}]Q$.
  - $f(R) = R + T^{(j)}$, with $j = \mathcal{H}(R)$ where $\mathcal{H}$ sends $< P >$ to $\{0, 1, \ldots, r-1\}$. In other words: $R_i = [u_i]P + [v_i]Q$, with $u_{i+1} \equiv u_i + u^{(j)} \bmod n, \quad v_{i+1} \equiv v_i + v^{(j)} \bmod n$.
  - Satler & Schnorr: $r \geq 8$; Teske: $r \geq 20$ (+ Hildebrand's result on random walks $\bmod n$).
- **multiplicative random walk:** (Gallant *et al.*)
  - choose multipliers $(\mu_j)_{0 \leq j < r}$;
  - $f(R) = [\mu_j]R$, $j = \mathcal{H}(R)$, i.e.:

$$f(R) = \left[\prod_{j=0}^{r-1} \mu_j^{f_j}\right] R_0;$$

  - requires at least two processors;
  - computing $[\mu_j]R$ must be done quickly.

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E) Adaptation to algebraic curves

Using equivalence classes

Idea: $\sim$ equivalence relation s.t. $\#(< P > / \sim) = n/m$. A random walk should find a collision after $\sqrt{\pi n/(2m)}$ iterations.

Finding $\sim$:

- Wiener, Zuccherato: $(x, y) \sim (x, -y)$.
- Gallant, Lambert et Vanstone; Harley: use little Frobenius on ABC curves.
- Duursma, Gaudry, M.: automorphism α of order m of C: $S \sim T \iff \exists i, S = \alpha^i T$.
  Ex1. $Y^2 = X^3 + aX$ with $[i](x, y) = (-x, iy)$ where $i^2 = -1$ (+ Frobenius).
  Ex2. On $Y^2 = X^5 - 1$, $\alpha(x, y) = (\zeta_5 x, y)$, where $\zeta_5^5 = 1$, $-\alpha$ has order 10; action on Jac as:
  $$[z^2 + u_1 z + u_0, v_1 z + v_0] \mapsto [z^2 + \zeta_5 u_1 z + \zeta_5^2 u_0, -\zeta_5^{-1} v_1 z - v_0].$$

Technical details: well defined f; short cycles.

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ECC2K-108

(Harley *et al.*, taken from `http://cristal.inria.fr/~harley/`)

- 1300 individuals, 9500 machines, dec 1999 until april 2000.
- **200,000 days on a 450 MHz PC with MMX**, i.e. more than 500 years. For comparison, cracking a 56-bit DES key by exhaustive search would take about 110000 days.
- 2.8×10^{15} elliptic-curve operations of which 2.3×10^{15} led to distinguished points recorded at INRIA; 2.05 million distinguished points in 1.3 Gigabytes of email.

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III. Index calculus methods

A) The basic scheme

Western and Miller; Pollard, Adleman, Merkle, etc. Methods in

$$L_N[\alpha, c] = \exp((c+o(1))(\log N)^\alpha (\log\log N)^{1-\alpha}).$$

Input: $G = \mathbb{F}_q^*, \mathrm{Jac}(C)$ of large genus; $G =< g >$, $a = g^z$.

Step 1: find the logarithms of the primes (resp. irreducible polynomials, prime divisors) of a factor basis $\mathcal{F} = \{p_1, p_2, \ldots, p_k\}$.

Step 2: look for b s.t. ag^b factors over $\mathcal{F}$:

$$ag^b = \prod_{j=1}^{k} p_j^{\alpha_j}$$

yields

$$z + b \equiv \sum_{j=1}^{k} \alpha_j \log_g p_j \bmod \#G.$$

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How do we find the $\log_g p_j$?

Choose random integers b_i for which

$$g^{b_i} = \prod_{j=1}^{k} p_j^{\alpha_{i,j}}$$

or:

$$b_i \equiv \sum_{i=1}^{k} \alpha_{i,j} \log_g p_j \bmod \#G.$$

When enough relations have been gathered, solve the system.

Cost: $O(L_{\#G}[1/2, c])$ where c depends on G and/or $\#G$.

Best algorithm so far: *à la* NFS $O(L_p[1/3, c'])$ (Gordon, Schirokauer).

Record: Joux & Lercier (**100 digits**, 1999).

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B) Discrete log on hyperelliptic curves

- Algorithm ADH from *Adleman, DeMarrais, Huang* (ANTS I):

$$\mathbf{L_{p^{2g+1}}[1/2, c]}$$

with $c \leq 2.181$ if $\log p \leq (2g+1)^{0.98}$ (heuristic using Lovorn's theorem on smooth polynomials); SNF.

- *Flassenberg & Paulus:* using sieving techniques; experiments with $y^2 = x^{2g+1} + 2x + 1$, faster than Shanks for $g \geq 6$.
- $y^2 = x^{2g+2} + \cdots$ (Müller-Stein-Thiel): proved $L_{p^{2g+2}}[1/2, 1.44]$.
- Extensions, proved analysis and optimizations by *Enge*:

$$\mathbf{L_{q^g}[1/2, c(\theta)]}$$

if $\theta \log q \leq g$, with $\lim_{\theta \to 0} c(\theta) = +\infty$; easier SNF. Smaller $c = \sqrt{2}$ by Enge and Gaudry.

- *Gaudry*'s variant: **fast, practical, implemented, breaks a lot of proposed systems**.

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C) Gaudry's variant

Main features:

- *easy* to describe and to analyze.
- *index calculus* using a basis of S-smooth divisors, i.e. $\mathcal{F} = \{< u(z), v(z) > \in \mathrm{Jac}(C), u(z)$ is irreducible of degree $\leq S\}$, with $S = 1$ for $g \leq 8$.
- Finding relations using a *random walk*.
- Easier linear algebra.
- **exponential asymptotic complexity** but **very practical** and **breaks a lot of systems** proposed by various authors.

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The algorithm

Input: $D_1, D_2 \in \mathrm{Jac}(C), n = \mathrm{ord}(D_1)$ **known**.

Output: λ s.t. $D_2 = [\lambda]D_1$.

1. [Initialisations:] Fix a smoothness bound S and build the basis $\mathcal{F} = \{g_1, g_2, \ldots, g_w\}$. Compute $T^{(j)} := [\alpha^{(j)}]D_1 + [\beta^{(j)}]D_2$ for $j \in [1..r]$ and set $R_0 := [\alpha_0]D_1 + [\beta_0]D_2$. Put $k := 1$.
2. [Main loop:]

   (a) Look for a smooth divisor:

   repeat

   $j := \mathcal{H}(R_0), R_0 := R_0 + T^{(j)}$,

   $\alpha_0 += \alpha^{(j)} \bmod n, \beta_0 += \beta^{(j)} \bmod n$.

   until $R_k := R_0 = [u_0(z), v_0(z)]$ is S-smooth.

   (b) [Write R_0 on $\mathcal{F}$:] factor $u_0(z)$ over $\mathbb{F}_q$, store $R_k = \sum m_{ik} g_i$ and $\alpha_k = \alpha_0, \beta_k = \beta_0$.

   (c) If $k < \#\mathcal{F} + 1$, then $k := k + 1$, go to 4.
3. [Linear algebra:] find (γ_k) in $\mathrm{Ker}({}^tM)$ on $\mathbb{Z}/n\mathbb{Z}$.
4. Return $\lambda = -(\sum \alpha_k \gamma_k)/(\sum \beta_k \gamma_k) \bmod n$.

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Analysis for $S = 1$

(enough for $g \leq 8$)

Prop. The probability for $u(z) \in \mathbb{F}_q[z]$ of degree g to be 1-smooth is $1/g!$ ($\#\{g-$ sets of $P \in C\} \approx q^g/g!$).

- **building** $\mathcal{F}$**:** $\#\mathcal{F} = O(q) \Rightarrow O(q)$ computations of squareroots in $\mathbb{F}_q$.
- **random walk:** $O(g!q)$ additions on the jacobian and smoothness test of a degree g polynomial, followed by $O(q)$ EDF's of degree g polynomials (total $\leq O(gq)$).
- **linear algebra:** with Lanczos on a $q \times q$ matrix over $\mathbb{Z}/n\mathbb{Z}$, each row having $\leq g$ entries $\Rightarrow O(q(qg)(\log n)^2) = O(g^3 q^2 (\log q)^2)$ (since $n \approx q^g$).

Thm. Gaudry's variant takes time $O(q^2 \log^c q)$ for fixed g.

Fine points: smoothness test (use discriminant *à la* Swan), EDF, SGE for fast linear algebra ($\neq$ factorization, $\neq$ discrete log over $\mathbb{F}_{p^n}$).

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More properties

Rem. The algorithm has **asymptotic exponential** complexity, but works very well in practice (see below), since it must factor polynomials of smaller degree than ADH. Provably (and practically) better than Pollard's ρ for $g > 4$.

Size of Jac **breakable**: $\#\mathrm{Jac} = q^g = (2\#\mathcal{F})^g$:

$\#\mathcal{F}\backslash g$	4	5	6	7	8
10^5	2^{70}	2^{88}	2^{105}	2^{123}	2^{140}
10^6	2^{83}	2^{104}	2^{125}	2^{146}	2^{167}

Crucial rem. An automorphism of order m reduces $\#\mathcal{F}$ by a factor m and the linear algebra by m^2.

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Hyperelliptic cryptosystems

Author	equation	q	auto
Koblitz	$Y^2+Y=X^5+X^3$ $(*)$	2^ℓ	4ℓ
	$Y^2+Y=X^5+X^3+X$ $(*)$	2^ℓ	2ℓ
	$Y^2+Y=X^{2g+1}+X$	2^ℓ	2ℓ
	$Y^2+Y=X^{2g+1}$	2^ℓ	2ℓ
Spallek	$Y^2=X^5-140X^3-240X^2+3810X+6928$	p	2
Buhler Koblitz Chao et al.	$Y^2+Y=X^{2g+1}$ (+ twists)	p with $p\equiv 1\ (2g+1)$	$2(2g+1)$
Sakai Sakurai Smart	$Y^2+Y=X^{13}+X^{11}+X^9+X^5+1$	2^{29}	4×29
Duursma Sakurai	$Y^2=X^p-X+1$	p^ℓ	$2\ell p$

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Timings

Magma 2.3 on Pentium II 450 MHz, 128 Mb.

Example 1:

field	$\mathbb{F}_{5026243}$
equation	$y^2+y=x^{13}$
genus	6
auto	13
$\#\mathrm{Jac}$	$\approx 10^{40}$
basis	$193,485$
matrix	$165,778\times 165,779$

construct. basis	1638 s
relations	6 days
SGE	1218
Lanczos	9 days

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Example 2:

field	$\mathbb{F}_{2^{23}}$
equation	$y^2+(x+1)y=x^{13}+x^{11}+x^8+x^7+x^5+x^4+x+1$
genus	6
auto	2×23
$\#\mathrm{Jac}$	$\approx 10^{41}$
basis	$182,462$
matrix	$162,873\times 162,874$

construct. basis	6575 s
relations	9 days
SGE	880 s
Lanczos	12 days

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Perspectives

Proven analyses: (Enge & Gaudry) unification of the methods used for finite fields and class groups.

Limiting factor: linear algebra stage, not the collection of relations ($\approx$ factorization). Wait for progress in distributed linear algebra.

Ex. (Sakai Sakurai Smart)
$y^2 + y = x^{13} + x^{11} + x^9 + x^5 + 1$ (genus 6) on $\mathbb{F}_{2^{29}}$ with an automorphism of order 4×29, $\#\mathrm{Jac} \approx 2^{174}$. Requires $\#\mathcal{F} \approx 4,500,000$.

Genus 4**:** rho and PG's are $O(q^2)$; dividing the size of the basis by $q^{1/5}$ yields a $q^{8/5}$ method.

Genus 2**:** (Gaudry) using a union-find type method for performing the linear algebra phase, same complexity as rho.

Genus 3**:** linear time linear algebra?

Important application: use this as a tool in the Weil descent.

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Weil descent: an example

$$\mathbb{F}_{q^n} = \mathbb{F}_q[t]/(f(t))$$
$$E : Y^2 = X^3 + \alpha X + \beta$$

Write $\alpha = \sum_i a_i t^i$, $\beta = \sum_i b_i t^i$,

$$X = \sum_i x_i t^i, \quad Y = \sum_i y_i t^i,$$

$$0 = Y^2 - (X^3 + \alpha X + \beta) = \sum \varphi_j(a_i, b_i, x_i, y_i) t^i$$

Thm. $\{\varphi_i\}$ defines an **abelian variety of dimension** n, with law (and points) inherited from E. Moreover, $\chi_A(T) = \chi_E(T^n)$.

A nice case: $2 \mid q$, $\mathbb{F}_{q^n}$ with basis $\{\theta, \theta^2, \ldots, \theta^{2^{n-1}}\}$ and $\theta + \theta^2 + \ldots + \theta^{2^{n-1}} = 1$. On $E : Y^2 + XY = X^3 + \beta$, write $X = \sum_i x_i \theta^{2^i}, \ldots$ and find the equation of C on A s.t. $x_0 = x_1 = x_2 = \cdots = x_{n-1} = x$.

$$C : y^{2^n} + x^{2^n - 1} y + \sum_{i=0}^{n-1} x^{2^n + 2^i} + g(x) = 0$$

with $\deg(g) \le 2^n$.

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Thm. $\exists m \in [1..n]$ s.t. C has an irreducible component which is hyperelliptic of genus 2^{m-1} or $2^{m-1} - 1$. Moreover, we have an homomorphism $E \to \mathrm{Jac}(C)$.

If n is small, solve DL on Jac by Gaudry's algorithm.

Ex. $\mathbb{F}_q = \mathbb{F}_{2^{21}}$, $E/\mathbb{F}_{q^4}$, $g = 4$.

Rem. m further analyzed by Menezes & Wu, $\mathbb{F}_{2^p}$ **not breakable**.

Rem. Recent computations of Smart: can break $E/\mathbb{F}_{q^4}$, $g = 8$, faster than ρ for $q > 2^{17}$.

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IV. Cryptographic implications

- Be very **careful** when using curves with an automorphism of large order.
- Find curves with cardinality easy to compute but without automorphisms: use **random** curves or maybe $X_0(N)$ (which has only involutions).
- Low genus is 5 or maybe 4 or 3 (surely not 1)? Implies make the base field larger (bad news for smart cards).
- **buy random elliptic curves!**

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