# PRIME VALUES OF PARTITION NUMBERS AND THE PRIMALITY OF $p_{1840926}$

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**Abstract**

We describe the computation of all partition numbers $p_n$ for $n \leq 10^6$ which are probable primes. We also give some probable prime values in the range $n \approx 1,800,000$ and in particular, the number $p_{1840926}$ is a 1505-digit prime the primality of which was established using ECPP. This number is the largest number tested by a general purpose primality proving algorithm so far.

## 1 Introduction

Let $n$ be a positive integer. The number of ways of writing $n$ as a sum of positive integers is called the partition number of $n$ and is noted $p_n$. For example, from

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$$

we deduce that $p_5 = 7$. The theory of partition numbers is very rich. Standard references on this topic include [1] and [6].

In 1991, Rivest introduced the *Challenge Partition List* as a benchmark for integer factorization algorithms (see [9]). This is a list of partition numbers with prime indices having at least 100 digits. The indices are of the order of $10^4$.

In search for ordinary numbers for testing his implementation of the ECPP algorithm (see [4]), the author wanted to use suitable values of $p_n$. Since the current primality record was 1226 digits [11], it was desirable to compute values of $p_n$ near 1500 digits. Due to the growth of $p_n$ (see section 2), such $n$'s are in the range $1,800,000$.

The paper is organized as follows. First, we recall some known facts about partition numbers. Then, we explain how we have computed tables of prime values of $p_n$ for $n \leq 10^6$. We compare the actual number of such prime values with a heuristic formula that we establish. We conclude with a short description of the new primality record, namely the primality proof of $p_{1840926}$ using ECPP, which is the largest known "ordinary" prime found so far. For the theory of ECPP, we refer to [4, 11].

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# 2 Properties of $p_n$

Among the many properties of $p_n$, we will use the following ones (we refer to [1] for these).

## 2.1 Arithmetical properties

The numbers $p_n$ have many arithmetical properties. In particular, if $24n - 1 \equiv 0 \bmod (5^a 7^b 11^c)$, then $p_n \equiv 0 \bmod (5^a 7^{[(b+2)/2]} 11^c)$. For this and other congruences as well, we refer to [2, 3].

## 2.2 Computing $p_n$

There are two ways for computing $p_n$. The first one uses Rademacher's exact formula, the second one uses the generating function of $p_n$.

### 2.2.1 Rademacher's formula

Letting $C = \pi\sqrt{2/3}$ and $\lambda_n = \sqrt{n - 1/24}$, Rademacher's exact formula for $p_n$ is

$$p_n = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \left( \frac{d}{dn} \frac{\sinh((C/k)\lambda_n)}{\lambda_n} \right)_{x=n},$$

where

$$A_k(n) = \sum_{\substack{h \bmod k \\ (h,k)=1}} \omega_{h,k} \exp(-2\pi i n h/k)$$

with $\omega_{h,k}$ a well defined $24k$-th root of unity. In particular, $A_1(n) = 1$ and $A_2(n) = (-1)^n$. It turns out that the functions $A_k$ have nice multiplicative properties that were worked out by Lehmer in [8]. Using this wonderful paper, we can write a very nice and short program for evaluating $p_n$.

From this formula, one deduces that, as $n$ goes to infinity,

$$p_n \sim \frac{1}{4n\sqrt{3}} \exp\left(C\sqrt{n}\right).$$

More precisely, let

$$f(n) = \frac{1}{2\pi\sqrt{2}} \frac{d}{dn} \left( \frac{e^{C\lambda_n}}{\lambda_n} \right).$$

Then, for $n \geq 26$, one has [13]:

$$p_n = f(n) + \theta \frac{e^{C\lambda_n/2}}{\lambda_n^2} \tag{1}$$

with $|\theta| < 1$.

### 2.2.2 The generating function for $p_n$

Let $P(x)$ be the generating function of the partition numbers

$$P(x) = \sum_{n=0}^{\infty} p_n x^n = \frac{1}{\prod_{i \geq 1}(1 - x^i)}.$$

Using Euler's identity, namely

$$\prod_{i \geq 1}(1-x^i) = 1 + \sum_{k=1}^{\infty}(-1)^k(x^{k(3k-1)/2} + x^{k(3k+1)/2}),$$

we deduce the recurrence

$$p_n = p_{n-1} + p_{n-2} - p_{n-5} - p_{n-7} + \cdots + (-1)^{m-1}p_{n-m(3m-1)/2} + (-1)^{m-1}p_{n-m(3m+1)/2} + \cdots$$

This formula is used in two ways. The first one, for $n$ small, enables one to compute $p_n$ rapidly. When $n$ is large, this formula is used modulo small primes $q_i$ and the value of $p_n$ is reconstructed from that of $p_n \bmod q_i$ using the Chinese remainder theorem. We determine the number of $q_i$'s to be used by estimating the value of $p_n$ with (1).

# 3 Looking for primes

We decided the following. In order to find all probable prime values of $p_n$, we first computed those $p_n$ for which $(24n-1, 5\times 7\times 11) = 1$, using the Chinese approach. Then, for each $p_n$ with no small factors and pseudoprime to two bases, we recomputed its value using Rademacher's formula. The first program was written is C (using the `BigNum` package [7]) and the second one in Maple.

The computation of $p_n \bmod q$ was distributed over a network of workstations. The values of $p_n \bmod q$ for $n$ in $[nmin, nmax]$ were put in a file $F_q$. Then, another program was used to build $p_n$ from the $F_q$'s. Once again, this was done in a distributed way with a "server" of indices to be checked.

Maple always confirmed the results of the C program. For the range $n \in [900,000; 1,000,000]$, computing $p_n$ modulo a single prime requires approximately half an hour on a DS3100; the search for probable primes took 110 hours of DS3100 on five machines. The Maple time for recomputing the value of $p_n$ for $n$ in that range is about one hour on a DS3100.

# 4 Data

We give at the end of the article the probable prime values of $p_n$ we found in the range $[0; 10^6]$. Numbers marked with a $*$ were later proven prime using ECPP. Also, we put $d(x) = \lfloor \log_{10} x \rfloor + 1$, the number of decimal digits of the integer $x$. We note that this table agrees with that of Dubner for $n \leq 2000$ (see [5]). We add a table containing the value of

$$\mathcal{P}(x) = \#\{n \leq x, p_n \text{ is prime}\}$$

for $x = 10^k$, $1 \leq k \leq 6$.

With these data, we can try to estimate $\mathcal{P}(x)$. The remainder of this section develops a heuristic model for this. We expect that the probability that $p_n$ is prime is roughly $1/\log p_n$, for those numbers for which $(24\times n - 1, 5\times 7\times 11) = 1$ and the properties described in [3] are not satisfied. More precisely, using the notations of [3], let $q$ be a prime equal to 5, 7 or 13 and

$$\delta(q) = \frac{q-1}{2q}\left\{1 - \prod_{p\equiv -2 \bmod q}(1-\frac{p-1}{2p^3})\prod_{p\equiv -1 \bmod q}(1-\frac{p-1}{p^4})\right\}$$

where all products run through prime values of $p$ (with the exception of $p = 3$ when $q = 5$). Then, one expects that the proportion of $n$'s for which $p_n$ are not prime is about

$$K = (1 - (1/5 + \delta(5)))(1 - (1/7 + \delta(7)))(1 - \delta(13))(1 - 1/11) \approx 0.615\ldots$$

This leads to

$$\mathcal{P}(x) \approx K \int_2^x \frac{dn}{\log p_n}.$$

Using the fact that $p_n \approx \exp(C\sqrt{n})/(4\sqrt{3}n)$ with $C = \pi\sqrt{2/3}$, we deduce that

$$\mathcal{P}(x) \approx K(\frac{2}{C}\sqrt{x} + \frac{1}{2C^2}(\log x)^2 + \frac{C_1}{C^2}\log x) + O((\log x)^3/\sqrt{x})$$

where $C_1 = \log(4\sqrt{3})$. We define $\hat{P}(x)$ as the right hand side of this relation, but for the $O()$ term. We have plotted the real values of $\mathcal{P}(x)$ and those of the approximation $\hat{P}$ in Figure 1, for $x \in [0; 10^6]$. It appears that this heuristic is not too bad for $x$ in this range.

# 5 A new record for ECPP

ECPP has had the primality record since 1988 (see [10, 12, 11]). For the first time, the author was able at that time to prove the primality of a 1000-digit number using a general purpose primality proving algorithm.

After a complete rewriting of the program, the author decided to try this new program on a 1505-digit number, namely $p_{1840926}$. This number was selected as the first probable prime value of $p_n$ in the range $[1,800,000; 1,9000,000]$ with more than 1500 digits. Some other probable prime values are $n \in \{1802231, 1802703, 1806587, 1811718, 1849288\}$; no attempt has been made to find *all* probable primes in that range. It took $12,000$ seconds of CPU to compute $p_{1840926}$ in Maple, on a DS3100.

Using a star network *à la* Silverman [14], the process was started on February 4, 1992 and ended on April 19, 1992. The total time is estimated to 4 years of CPU time of a SUN 3/60, distributed among 21 workstations at INRIA, but also at the Ecole Polytechnique (Computer Science Laboratory, LIX). The program produced a certificate of 800 kbytes. T. Valente checked this certificate independently and agreed with its correctness (private communication, May 26, 1992).

# 6 Conclusion

We have described the computation of a long table of pseudoprime and prime values of partition numbers. We hope this will serve as a list of useful ordinary numbers that can be used as benchmarks for future primality proving programs.

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| $n$ | $d(p_n)$ | $n$ | $d(p_n)$ | $n$ | $d(p_n)$ | $n$ | $d(p_n)$ | $n$ | $d(p_n)$ |
|---|---|---|---|---|---|---|---|---|---|
| *2 | 1 | *2502 | 52 | *18290 | 146 | 42118 | 224 | 84017 | 318 |
| *3 | 1 | *2508 | 52 | *18470 | 147 | 44255 | 229 | 87266 | 324 |
| *4 | 1 | *2568 | 53 | *20511 | 155 | 44508 | 230 | 88722 | 327 |
| *5 | 1 | *2633 | 53 | *20877 | 156 | 47105 | 237 | 92101 | 333 |
| *6 | 1 | *2727 | 54 | *21091 | 157 | 47666 | 238 | 96565 | 341 |
| *13 | 3 | *2732 | 54 | *21645 | 159 | 51855 | 249 | 96966 | 342 |
| *36 | 5 | *2871 | 56 | *21765 | 160 | 53372 | 252 | 97330 | 342 |
| *77 | 8 | *2912 | 56 | *22235 | 161 | 54391 | 255 | 98666 | 345 |
| *132 | 10 | *3027 | 57 | *22317 | 162 | 55597 | 258 | 98892 | 345 |
| *157 | 11 | *3098 | 58 | *22670 | 163 | 56382 | 259 | 99932 | 347 |
| *168 | 12 | *3168 | 59 | *23202 | 165 | 58852 | 265 | *101770 | 350 |
| *186 | 13 | *3342 | 61 | *23558 | 166 | 59937 | 268 | *101803 | 350 |
| *188 | 13 | *3542 | 62 | *24193 | 169 | 61750 | 272 | *102742 | 352 |
| *212 | 14 | *3641 | 63 | *24293 | 169 | 61768 | 272 | *103905 | 354 |
| *216 | 14 | *4118 | 68 | *24556 | 170 | 61795 | 272 | 106231 | 358 |
| *302 | 17 | *4262 | 69 | *25370 | 173 | 61887 | 272 | 106485 | 358 |
| *366 | 18 | *4456 | 70 | *25838 | 174 | 61945 | 272 | 106995 | 359 |
| *417 | 20 | *4662 | 72 | *26281 | 176 | 62311 | 273 | 108132 | 361 |
| *440 | 20 | *5002 | 75 | *26677 | 177 | 62528 | 273 | 108866 | 362 |
| *491 | 22 | *5347 | 77 | *28965 | 185 | 62986 | 274 | 109727 | 364 |
| *498 | 22 | *6798 | 88 | *30600 | 190 | 65050 | 279 | 110211 | 364 |
| *525 | 22 | *7405 | 92 | *31756 | 194 | 65432 | 280 | 114195 | 371 |
| *546 | 23 | *7458 | 92 | *32552 | 196 | 66725 | 283 | 114830 | 372 |
| *658 | 25 | *8117 | 96 | *32890 | 197 | 67395 | 284 | 116586 | 375 |
| *735 | 27 | *8448 | 98 | *33240 | 198 | 67616 | 285 | 118002 | 377 |
| *753 | 27 | *8912 | 101 | *33972 | 200 | 68427 | 286 | 118822 | 379 |
| *825 | 29 | *9292 | 103 | *34083 | 201 | 72660 | 295 | 123897 | 387 |
| *841 | 29 | *9998 | 107 | 34666 | 203 | 72698 | 295 | 124657 | 388 |
| *863 | 29 | *10186 | 108 | 34696 | 203 | 73635 | 297 | 126098 | 390 |
| *1085 | 33 | *11080 | 113 | 35227 | 204 | 76367 | 303 | 127080 | 392 |
| *1086 | 33 | *11901 | 117 | 35402 | 205 | 76782 | 303 | 128810 | 394 |
| *1296 | 37 | *12028 | 118 | 36488 | 208 | 77098 | 304 | 129116 | 395 |
| *1477 | 39 | *12895 | 122 | 37522 | 211 | 77212 | 304 | 129495 | 395 |
| *1578 | 41 | *13020 | 123 | 37628 | 211 | 78322 | 307 | 129511 | 395 |
| *1586 | 41 | *13472 | 125 | 38457 | 214 | 79696 | 309 | 132066 | 399 |
| *1621 | 41 | *13513 | 125 | 39305 | 216 | 81698 | 313 | *134968 | 404 |
| *1793 | 44 | *15040 | 132 | 39970 | 218 | 82305 | 314 | *136192 | 406 |
| *2051 | 47 | *16248 | 137 | 40113 | 218 | 82453 | 315 | *136320 | 406 |
| *2136 | 48 | *17593 | 143 | 41380 | 222 | 83121 | 316 | 138985 | 410 |
| *2493 | 52 | *18276 | 146 | 42031 | 223 | 83156 | 316 | 140515 | 412 |

Table 1: Pseudoprime and prime values of $p_n$ for $2 \le n \le 140515$

| $n$ | $d(p_n)$ | $n$ | $d(p_n)$ | $n$ | $d(p_n)$ | $n$ | $d(p_n)$ | $n$ | $d(p_n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 141360 | 413 | *216617 | 513 | 288203 | 592 | 384122 | 685 | 520396 | 798 |
| 141637 | 414 | *216908 | 513 | 291972 | 596 | 385522 | 686 | *528923 | 804 |
| 144140 | 417 | *219660 | 516 | 293621 | 598 | 386378 | 687 | *531065 | 806 |
| 144377 | 418 | *221323 | 518 | 294177 | 598 | 387823 | 688 | *536585 | 810 |
| 157045 | 436 | *224518 | 522 | *304492 | 609 | 387881 | 688 | *537130 | 810 |
| 157833 | 437 | *227625 | 526 | *305396 | 610 | 389220 | 689 | *547571 | 818 |
| 158817 | 438 | *227867 | 526 | *305650 | 610 | 391533 | 691 | 548038 | 819 |
| 159170 | 439 | *229670 | 528 | 307068 | 611 | 397685 | 697 | 556643 | 825 |
| 161353 | 442 | 232242 | 531 | 307440 | 612 | 399147 | 698 | 561212 | 828 |
| 161795 | 443 | 233697 | 533 | 310858 | 615 | *404378 | 702 | 564147 | 831 |
| 163123 | 444 | 234213 | 533 | 314367 | 619 | *405446 | 703 | 565287 | 831 |
| 163401 | 445 | 235563 | 535 | 315466 | 620 | *406615 | 704 | 566657 | 832 |
| 163737 | 445 | 238286 | 538 | 316526 | 621 | 407998 | 706 | 566750 | 833 |
| 164090 | 446 | 240082 | 540 | 316550 | 621 | 417410 | 714 | 568215 | 834 |
| 164452 | 446 | 243716 | 544 | 320222 | 625 | 418436 | 715 | 570031 | 835 |
| 166827 | 449 | 245666 | 546 | 320562 | 625 | 423793 | 719 | 571141 | 836 |
| 167418 | 450 | 249031 | 550 | 320895 | 625 | 430538 | 725 | 574273 | 838 |
| 170271 | 454 | 250456 | 552 | 335647 | 640 | 435550 | 729 | 578448 | 841 |
| 170428 | 454 | 251677 | 553 | 337830 | 642 | 438343 | 732 | 579505 | 842 |
| 170537 | 454 | 253625 | 555 | 338456 | 642 | 441703 | 734 | 580461 | 843 |
| 172131 | 457 | 255005 | 557 | 345922 | 649 | 443320 | 736 | 581836 | 844 |
| 173391 | 458 | 255422 | 557 | 348973 | 652 | 444167 | 736 | 582371 | 844 |
| 176140 | 462 | 259626 | 562 | 350255 | 653 | 444387 | 737 | 586418 | 847 |
| 177488 | 464 | 261361 | 564 | 350368 | 654 | 449318 | 741 | 590417 | 850 |
| 184296 | 473 | 261758 | 564 | 353807 | 657 | 459028 | 749 | 593495 | 852 |
| 187966 | 477 | 262375 | 565 | 354832 | 658 | 459328 | 749 | 594610 | 853 |
| 188842 | 478 | 264382 | 567 | 356168 | 659 | 478285 | 764 | 601226 | 858 |
| 189353 | 479 | 267186 | 570 | 357685 | 660 | 478571 | 765 | 602868 | 859 |
| 191330 | 482 | 267975 | 571 | 358322 | 661 | 479901 | 766 | 606416 | 861 |
| 192667 | 483 | 268095 | 571 | 359612 | 662 | 481401 | 767 | 609630 | 864 |
| 193810 | 485 | 269257 | 572 | 362217 | 665 | 481590 | 767 | 611766 | 865 |
| 194078 | 485 | 271077 | 574 | 366103 | 668 | 482628 | 768 | 614813 | 867 |
| 197595 | 490 | 271182 | 574 | 366855 | 669 | 482836 | 768 | 616263 | 868 |
| 199067 | 491 | 274585 | 578 | 367621 | 670 | 490316 | 774 | 617838 | 870 |
| 201821 | 495 | 275031 | 578 | 367907 | 670 | 492627 | 776 | 620763 | 872 |
| 202462 | 496 | 277666 | 581 | 368120 | 670 | 495032 | 778 | 624260 | 874 |
| 202628 | 496 | 281631 | 585 | 368248 | 670 | 496476 | 779 | 624837 | 874 |
| 204277 | 498 | 282912 | 587 | 369411 | 671 | 499047 | 781 | 626333 | 876 |
| 204737 | 498 | 283536 | 587 | 372023 | 674 | 518360 | 796 | 627442 | 876 |
| *214256 | 510 | 288001 | 592 | 381525 | 682 | 518657 | 796 | 628828 | 877 |

Table 2: Pseudoprime and prime values of $p_n$ for $141360 \leq n \leq 628828$

| $n$ | $d(p_n)$ | $n$ | $d(p_n)$ | $n$ | $d(p_n)$ | $n$ | $d(p_n)$ | $n$ | $d(p_n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 630766 | 879 | 697360 | 924 | 769443 | 971 | 836542 | 1013 | 907658 | 1055 |
| 633027 | 880 | 701327 | 927 | 778350 | 977 | 838125 | 1014 | 914190 | 1059 |
| 634600 | 881 | 706652 | 930 | 780707 | 978 | 840847 | 1015 | 922966 | 1064 |
| 638088 | 884 | 708763 | 932 | 784826 | 981 | 851843 | 1022 | 930057 | 1068 |
| 639808 | 885 | 718232 | 938 | 796007 | 988 | 852478 | 1022 | 939522 | 1073 |
| 640952 | 886 | 732453 | 947 | 801717 | 991 | 855470 | 1024 | 945972 | 1077 |
| 641006 | 886 | 736162 | 950 | 809266 | 996 | 856111 | 1024 | 950403 | 1080 |
| 641237 | 886 | 736785 | 950 | 810953 | 997 | 867230 | 1031 | 953010 | 1081 |
| 641418 | 886 | 737702 | 951 | 812912 | 998 | 871611 | 1034 | 953417 | 1081 |
| 642891 | 887 | 742995 | 954 | 814850 | 999 | 872741 | 1034 | 956188 | 1083 |
| 645281 | 889 | 743763 | 955 | 816521 | 1000 | 874846 | 1036 | 958118 | 1084 |
| 647188 | 890 | 743926 | 955 | 816826 | 1001 | 875261 | 1036 | 960028 | 1085 |
| *662108 | 900 | 749505 | 958 | 818888 | 1002 | 879063 | 1038 | 961402 | 1086 |
| 662603 | 901 | 752301 | 960 | 823918 | 1005 | 881910 | 1040 | 964268 | 1088 |
| 671295 | 907 | 752473 | 960 | 825717 | 1006 | 884552 | 1041 | 977746 | 1095 |
| 671808 | 907 | 754663 | 962 | 829551 | 1008 | 888460 | 1044 | 986581 | 1100 |
| 677200 | 911 | 757505 | 963 | 831097 | 1009 | 890850 | 1045 | 992681 | 1104 |
| 681517 | 913 | 762772 | 967 | 833238 | 1011 | 894887 | 1048 | 993110 | 1104 |
| 682362 | 914 | 766023 | 969 | 833456 | 1011 | 897332 | 1049 | 997213 | 1106 |
| 690252 | 919 | 768350 | 970 | 835502 | 1012 | 900172 | 1051 | | |

Table 3: Pseudoprime and prime values of $p_n$ for $630766 \leq n \leq 997213$

| $x$ | $\mathcal{P}(x)$ |
|---|---|
| $10^1$ | 5 |
| $10^2$ | 8 |
| $10^3$ | 29 |
| $10^4$ | 68 |
| $10^5$ | 170 |
| $10^6$ | 499 |

Table 4: Values of $\mathcal{P}(x)$ for $10 \leq x \leq 10^6$

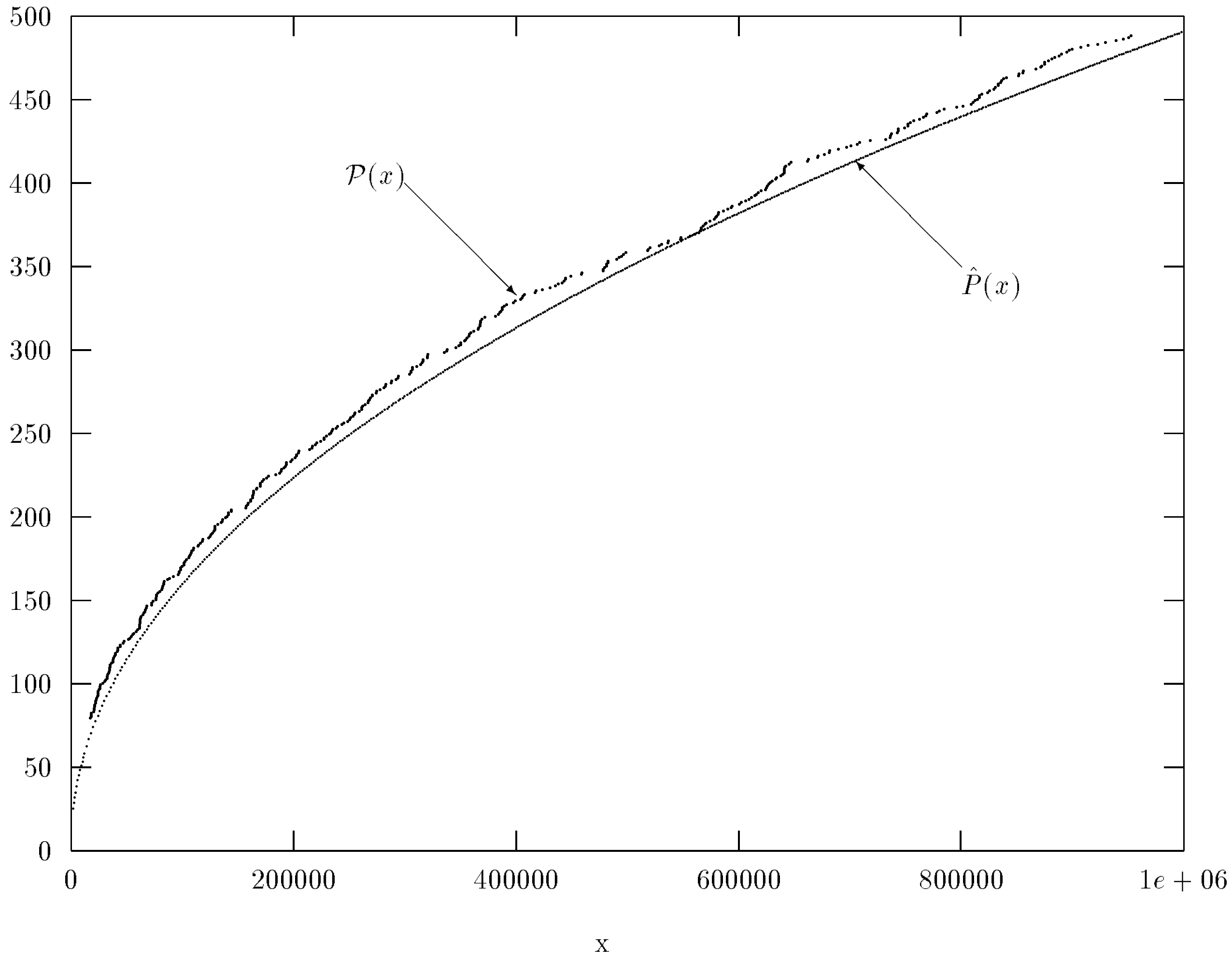


Figure 1: Comparison of $\mathcal{P}(x)$ and $\hat{P}(x)$